Local and global solutions of well-posed integrated Cauchy problems

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Abstract

In this paper we study the local well-posed integrated Cauchy problem,

\[ v'(t) = Av(t) + \frac{t^\alpha}{\Gamma(\alpha + 1)} x, \quad v(0) = 0, \quad t \in [0, \kappa), \]

with \( \kappa > 0, \alpha \geq 0, \) and \( x \in X \) where \( X \) is a Banach space and \( A \) a closed operator on \( X \). We extend solutions increasing the regularity in \( \alpha \). The global case \( (\kappa = \infty) \) is also treated in detail. Growths of solutions are given in both cases.

Key words: Abstract Cauchy problems, integrated semigroups, distribution semigroups.

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1 Introduction

Let \( X \) be a Banach space, \( A \) a closed operator on \( X \) with domain \( D(A) \) and \( f : [0, \kappa) \to \mathbb{C} \) a continuous function, \( f \in C([0, \kappa); \mathbb{C}), (0 < \kappa \leq \infty) \). Evolution equations,

\[ v'(t) = Av(t) + f(t)x, \quad t \in [0, \kappa), \]

\[ \star \]

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have a long history. Many ordinary and partial differential equations may be written in this form. Different ideas and techniques have been developed to deal with this problem.

Recently local convoluted semigroups have been investigated deeply to express the solution of this equation (see for example [12] and reference in them). $\alpha$-Times integrated semigroups are examples of convoluted semigroups obtained for $f(t) = t^\alpha/\Gamma(\alpha + 1)$ with $\alpha \in \mathbb{R}^+$, (in this case, the equation (1) defines the local integrated Cauchy problem). $\alpha$-Times integrated semigroups were introduced first for $\alpha \in \mathbb{N}$ ([3]) and later for $\alpha \in \mathbb{R}^+$, ([10]). In fact, to interchange $n$ to $\alpha$ may or not be complicated. Sometimes, it may be straightforward (see Theorem 1); other times it is necessary some integral expressions (see equality (6)) or some estimates about special functions (Example 1). The background theory which is behind is the fractional calculus (see section 3 and [18]).

In the second section, we consider solutions of the well-posed integrated Cauchy problem. We show that they are in fact, local $\alpha$-times integrated semigroups or local mild $\alpha$-times integrated existence families (Theorem 1). Moreover, we show that every local $\alpha$-times integrated semigroup may be extended if one is ready to give up the regularity (Theorem 2). This interesting extension property appeared in [2] for local $n$-times integrated semigroups and in [23] for local $C$-semigroups.

On the other hand, Lions introduced in 1960 the so-called (vector-valued) distribution semigroups, in connection with Cauchy problems ([15]). Particular classes of distribution semigroups have been considered since then; for example quasi-distribution semigroups were introduced and studied by Wang in [22]. A brief description of distribution semigroups as well as references on the subject can be found in [3] and [17].

Local $\alpha$-times integrated semigroups are equivalent to quasi-distribution semigroups, see the third section. We show that quasi-distribution semigroups of fractional order are equivalent to global solutions of the integrated Cauchy problem with fairly general growth. We present an approach which allows us to extend some known results ([1, Theorem 4.4], [7, Theorem 3.6], [22, Theorem 4.13]) in quite large way.
2 Extending solutions of well-posed local integrated Cauchy problems

Let $X$ be a Banach space, $B(X)$ the set of linear and bounded operators on $X$, $(A, D(A))$ a closed linear operator on $X$, $x \in X$ and $\kappa > 0$. The local $\alpha$-times integrated Cauchy problem

$$C_{\alpha}(\kappa) \equiv \begin{cases} v \in C([0, \kappa); D(A)) \cap C^{(1)}([0, \kappa); X) \\
v'(t) = Av(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}x, \quad t \in [0, \kappa), \\
v(0) = 0, \end{cases}$$

has been studied in detail for $\alpha \in \mathbb{N} ([2], [22])$ and later for $\alpha \in \mathbb{R}^+ ([16])$.

The Cauchy problem $C_{\alpha}(\kappa)$ is well-posed if for all $x \in X$ there exists a unique solution of $C_{\alpha}(\kappa)$. If $E(a, b) \subset \rho(A)$ where $E(a, b) := \{ \lambda \in \mathbb{C} ; \Re \lambda \geq b, |\Im \lambda| \leq e^{a\Re \lambda} \}$, for some $a, b > 0$ ($\rho(A)$ is the resolvent set of $A$) and

$$\| (\lambda - A)^{-1} \| \leq M|\lambda|^{\alpha-1}, \quad \lambda \in E(a, b),$$

for $\alpha \in \mathbb{R}^+$, then the Cauchy problem $C_{\beta}(\kappa)$ is well-posed with $\beta > \alpha$ and $\kappa = a(\beta - \alpha)$ ([16, Theorem 2.2]). Using some ideas from special functions, a converse result holds ([16, Theorem 2.1]).

Given $\alpha \in \mathbb{R}^+$, the solution of the well-posed abstract Cauchy problem $C_{\alpha+1}(\kappa)$, $(v_x(t))_{t \in [0, \kappa)}$ defines a family of linear and bounded operators, $(S_{\alpha}(t))_{t \in [0, \kappa)} \subset B(X)$ by

$$S_{\alpha}(t)x := v'_x(t), \quad x \in X, t \in [0, \kappa).$$

In fact, $(S_{\alpha}(t))_{t \in [0, \kappa)}$ is a nondegenerate local $\alpha$-times integrated semigroup, i.e.,

$$S_{\alpha}(t)S_{\alpha}(s)x = \frac{1}{\Gamma(\alpha)} \left( \int_{s}^{t+s} - \int_{0}^{t}(t + s - r)^{\alpha-1}S_{\alpha}(r)xdr \right),$$

holds for $\kappa > t + s \geq t, s \geq 0$, and $x \in X$ (nondegenerate in the usual sense, if $S_{\alpha}(t)x = 0$ for all $t \in [0, \kappa)$ then $x = 0$) see [1], [16], [22]. It is straightforward
to check \((S_\beta(t))_{t \in [0, \kappa]}\) is a local-\(\beta\) times integrated semigroup defined by
\[
S_\beta(t)x = \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t - s)^{\beta - \alpha - 1} S_\alpha(s)xds, \quad x \in X, t \in [0, \kappa),
\]
and \(\beta > \alpha\). For a nondegenerate local \(\alpha\)-times integrated semigroup we may define its generator in the following way. Let \(D(A)\) be the set of all \(x \in X\) for which there exists \(y \in X\) such that
\[
S_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha + 1)}x = \int_0^t S_\alpha(s)yds, \quad t \in [0, \kappa);
\]
and \(Ax := y\). It is easy to check \((A, D(A))\) is a closed operator on \(X\). Moreover \(S_\alpha(t)x\) is differentiable for \(t \in [0, \kappa)\) and \(x \in X\) if and only if \(S_\alpha(t)x \in D(A)\) and in this case
\[
\frac{d}{dt}S_\alpha(t)x = AS_\alpha(t)x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}x, \quad \kappa > t > 0.
\]
In the case \(\kappa = \infty\), the growth of \(\|S_\alpha(t)\|\) when \(t \to \infty\) can be bigger than exponential, see for example [17, Example 1.2.5]. If \(\|S_\alpha(t)\| \leq Ce^{\lambda t}\) with \(C, \lambda_0 \geq 0\), the condition (3) is equivalent (via Laplace transform) to
\[
R(\lambda, A) := \lambda^\alpha \int_0^\infty e^{-\lambda t} S_\alpha(t)dt, \quad \Re \lambda > \lambda_0,
\]
is a pseudo-resolvent operator, i.e., \(R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)\) for any \(\Re \lambda, \Re \mu > \lambda_0\) ([10]). In the nondegenerate case, \(\lambda \in \rho(A)\) (the resolvent set) and \(R(\lambda, A) = (\lambda - A)^{-1}\) for \(\Re \lambda > \lambda_0\).

R. deLaubenfelds introduced the concept of mild \(n\)-times integrated existence family in [6]. Suppose \(A\) is a closed operator on \(X\) and \(\alpha \geq 0\). A strongly continuous family \((W(t))_{t \in [0, \kappa]} \subset \mathcal{B}(X)\) is a local mild \(\alpha\)-times integrated existence family for \(A\) if for any \(x \in X\), \(t \in [0, \kappa)\), \(\int_0^t W(s)xds \in D(A)\) and
\[
A \int_0^t W(s)xds = W(t)x - \frac{t^\alpha}{\Gamma(\alpha + 1)}x.
\]
see [22, Definition 2.3] for \(\alpha \in \mathbb{N}\). The following theorem is well-known in the case \(\alpha \in \mathbb{N} \cup \{0\}\), see for example [2, Proposition 2.3] and [22, Theorem 2.4].

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The special case \( \alpha = 0 \) appears in [2, Theorem 1.2]. The proof in the case \( \alpha \in \mathbb{R}^+ \) is similar to the case \( \alpha \in \mathbb{N} \) and we omit it.

**Theorem 1** Let \( \alpha \geq 0 \) and \( 0 < \kappa \leq \infty \). The following are equivalent.

(i) \( C_{\alpha+1}(\kappa) \) is well-posed.

(ii) \( A \) generates a nondegenerate \( \alpha \)-times integrated semigroup.

(iii) All solutions of \( C_{\alpha+1}(\kappa) \) are unique and there exists a local mild \( \alpha \)-times integrated existence family for \( A \).

**Example 1.** This example appears in [21] for the case \( \alpha \in \mathbb{N} \), see also [17, Example 1.2.6]. Let \( \ell^2 \) be the Hilbert space of all sequences \( x = (x_m)_{m=1}^\infty \) such that

\[
\sum_{m=1}^\infty |x_m|^2 < \infty,
\]

with the usual norm \( \|x\| := (\sum_{m=1}^\infty |x_m|^2)^{\frac{1}{2}} \). Take \( T > 0 \) and define

\[
am_m = \frac{m}{T} + i \left( \left( \frac{m}{T} \right)^2 - \left( \frac{m}{T} \right)^2 \right)^{\frac{1}{2}}, \quad m \in \mathbb{N},
\]

where \( i^2 = -1 \). For any \( \alpha \in \mathbb{R}^+ \), let \( (U_\alpha(t))_{t \geq 0} \) be defined by

\[
U_\alpha(t)x = \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{a_m s} x_m ds \right)_{m=1},
\]

for \( x \in D(U_\alpha(t)) \) where \( D(U_\alpha(t)) = \{ x \in \ell^2 : U_\alpha(t)x \in \ell^2 \} \). Then \( (U_\alpha(t))_{t \in [0,\alpha T)} \) is a local \( \alpha \)-times integrated semigroup on \( \ell^2 \):

We consider the case \( \alpha \not\in \mathbb{N} \). Then \( 0 < \alpha - [\alpha] < 1 \) and

\[
\int_0^t (t-s)^{\alpha-1} e^{a_m s} ds = \frac{e^{a_m t}}{\Gamma(\alpha)} \int_0^t \int_0^t (\alpha - [\alpha]) e^{a_m s} ds + \sum_{j=1}^{[\alpha]} \frac{t^{\alpha-j}}{\Gamma(\alpha + 1 - j) a_m^j},
\]

for \( t \geq 0 \). Moreover, we have that

\[
\int_0^t \frac{s^{\alpha-[\alpha]-1}}{\Gamma(\alpha-[\alpha])} e^{-a_m s} ds = \frac{t^{\alpha-[\alpha]+1}}{\Gamma(\alpha)} \left( \frac{e^{-a_m t}}{a_m} O(1) + \frac{1}{a_m^{\alpha-[\alpha]}} O(1) \right), \quad t \geq 0,
\]

when \( |a_m| \to \infty \), see for example [16, Theorem 2.1]. Since \( |a_m| = m^{-1} e^{m} \) and \( |e^{a_m t}| = e^{m} \), we obtain that

\[
\|U_\alpha(t)\| = \sup_{m \in \mathbb{N}} \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{a_m s} ds \right| < \infty,
\]

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if and only if \( 0 \leq t < \alpha T \). It is easily shown that \((U_{\alpha}(t))_{t \in [0,\alpha T]}\) verifies (3) and \( t \to U_{\alpha}(t)x \) is strongly continuous, see these ideas in the case \( \alpha = n \) in [21]. Note that \((U_{\alpha}(t))_{t \in [0,\alpha T]}\) cannot be extended to \( t \geq \alpha T \).

Now we prove that solutions in the local well-posed case may be extended. A loss of regularity appears in the same way as in the integer case [2, Theorem 4.1]. Note that the extension given in [2, formula (4.2)] for \( \alpha \in \mathbb{N} \) is not possible in the case \( \alpha \in \mathbb{R}^{+} \).

**Theorem 2** Let \( \kappa_0 > 0 \) and \( \alpha \in \mathbb{R}^{+} \). Suppose that \( C_{\alpha+1}(\kappa_0) \) is well-posed. Then \( C_{2\alpha+1}(2\kappa_0) \) is also well-posed. In particular for all \( \kappa' > 0 \) there exists \( \beta > 0 \) such that \( C_{\beta}(\kappa') \) is well posed.

**Proof:** Take \( \kappa < \kappa_0 \). We will prove that \( C_{2\alpha+1}(2\kappa) \) is well-posed. By Theorem 1, there exists a nondegenerate local \( \alpha \)-times integrated semigroup \((S_\alpha(t))_{t \in [0,\kappa\alpha)}\) generated by \((A,D(A))\). Then we define \((S_{2\alpha}(t))_{t \in [0,2\kappa]}\) by (4) if \( 0 \leq t \leq \kappa \) and

\[
S_{2\alpha}(t)x := S_\alpha(\kappa)S_\alpha(t-\kappa)x + \frac{1}{\Gamma(\alpha)} \int_{0}^{\kappa} (t-s)^{\alpha-1}S_\alpha(s)x\,ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{0}^{t-\kappa} (t-s)^{\alpha-1}S_\alpha(s)x\,ds,
\]

(5) if \( \kappa \leq t \leq 2\kappa \) and \( x \in X \). It is clear \( S_{2\alpha} : [0,2\kappa] \to \mathcal{B}(X) \) is strongly continuous. To show \( C_{2\alpha+1}(2\kappa) \) is well-posed, we prove \((S_{2\alpha}(t))_{t \in [0,2\kappa]}\) is a local mild \( 2\alpha \)-times integrated existence family for \( A \) and apply the Theorem 1.

If \( 0 \leq t \leq \kappa \), it is clear that

\[
A \int_{0}^{t} S_{2\alpha}(s)\,ds = S_{2\alpha}(t)x - \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}x, \quad x \in X.
\]

Take \( \kappa \leq t \leq 2\kappa \). Then

\[
A \int_{0}^{t} S_{2\alpha}(s)\,ds = S_{2\alpha}(\kappa)x - \frac{\kappa^{2\alpha}}{\Gamma(2\alpha + 1)}x + A \int_{\kappa}^{t} S_{2\alpha}(s)\,ds.
\]

We use (6) and the Fubini theorem to obtain that

\[
\int_{\kappa}^{t} S_{2\alpha}(s)\,ds = S_{\alpha}(\kappa) \int_{0}^{t} S_{\alpha}(u)\,du + \int_{0}^{\kappa} S_{\alpha}(r)x \int_{r}^{t} \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\,ds\,dr.
\]

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\[ + \int_0^{t-\kappa} S_\alpha(r) x \int_{r+\kappa}^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)} ds dr. \]

Note that
\[ S_\alpha(\kappa) A \int_0^{t-\kappa} S_\alpha(u) x du = S_\alpha(\kappa) S_\alpha(t-\kappa) x - S_\alpha(\kappa) \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}. \]

As \[ \int_{\kappa}^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)} ds = \frac{(t-r)^\alpha}{\Gamma(\alpha+1)} - \frac{(\kappa-r)^\alpha}{\Gamma(\alpha+1)}, \]
we now check easily
\[ A \int_0^\kappa S_\alpha(r) x \frac{(\kappa-r)^\alpha}{\Gamma(\alpha+1)} dr = S_\alpha(\kappa) x - \frac{\kappa^2\alpha}{\Gamma(2\alpha+1)} x. \]

and also we conclude that
\[ A \int_0^\kappa S_\alpha(r) x \frac{(t-r)^\alpha}{\Gamma(\alpha+1)} dr = \int_0^\kappa S_\alpha(s) x \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + S_\alpha(\kappa) x \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)} \]
\[ - \frac{\kappa^\alpha}{\Gamma(\alpha+1)} \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)} x - \int_0^\kappa \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^\alpha ds x. \]

For \( r > t-\kappa \), we use similar ideas and \[ \int_{\kappa+r}^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)} ds = \frac{(t-r)^\alpha}{\Gamma(\alpha+1)} - \frac{\kappa^\alpha}{\Gamma(\alpha+1)} \]
to check easily that
\[ A \int_0^{t-\kappa} S_\alpha(r) x \frac{\kappa^\alpha}{\Gamma(\alpha+1)} dr = \frac{\kappa^\alpha}{\Gamma(\alpha+1)} S_\alpha(t-\kappa) x - \frac{\kappa^\alpha}{\Gamma(\alpha+1)} \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)} x, \]

and we may obtain that
\[ A \int_0^{t-\kappa} S_\alpha(r) x \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)} dr = \int_0^{t-\kappa} S_\alpha(s) x \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\kappa^\alpha}{\Gamma(\alpha+1)} S_\alpha(t-\kappa) x \]
\[ - \int_0^{t-\kappa} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^\alpha ds - \frac{\kappa^\alpha}{\Gamma(\alpha+1)} \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)} x. \]

Note that
\[ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\kappa^\alpha}{\Gamma(\alpha+1)} \frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)} + \int_0^{t-\kappa} S_\alpha(r) x \int_{r+\kappa}^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)} ds dr. \]
\[
\int_0^\kappa (t-s)^{\alpha-1} \frac{s^\alpha}{\Gamma(\alpha)} ds + \int_0^{t-\kappa} (t-s)^{\alpha-1} \frac{s^\alpha}{\Gamma(\alpha+1)} ds,
\]
see for example [13, Lemma 3.1]. To finish the proof we join together all summands to obtain
\[
A \int_0^\kappa S_{2\alpha}(s)x ds = S_\alpha(\kappa)S_\alpha(t-\kappa)x + \int_0^\kappa S_\alpha(s)x \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds
\]
\[
- \int_0^{t-\kappa} S_\alpha(s)x \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} x = S_{2\alpha}(t)x - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} x,
\]
and this proves the claim.

\[\square\]

**Remark 3** In the case \( \alpha = k \), we recover the extension given in [2, Theorem 4.1], due to
\[
\frac{1}{\Gamma(\alpha)} \int_0^\kappa (t-s)^{\alpha-1} S_\alpha(s)x ds + \frac{1}{\Gamma(\alpha)} \int_0^{t-\kappa} (t-s)^{\alpha-1} S_\alpha(s)x ds
\]
\[
= \sum_{m=0}^{k-1} \frac{1}{m!} (\kappa^m S_{2k-m}(t-\kappa) + (t-\kappa)^m S_{2k-m}(\kappa)).
\]

Due to uniqueness of the solutions and the proof of Theorem 2, the functional equation
\[
S_{2\alpha}(t+s)x = S_\alpha(t)S_\alpha(s)x + \frac{1}{\Gamma(\alpha)} \left( \int_0^t + \int_0^s (t+s-u)^{\alpha-1} S_\alpha(u)x du \right)
\]
holds for \( t, s \in [0, \kappa) \) and \( x \in X \).

Take \( \alpha \geq 0 \) and \( \kappa \in \mathbb{R}^+ \cup \{\infty\} \). We denote by \( \Omega_{\alpha,\kappa} \) the set of non-decreasing and continuous functions \( \tau_\alpha \) on \([0, \kappa)\) such that \( \inf_{\kappa > t > 0} t^{-\alpha} \tau_\alpha(t) > 0 \) and there exists a constant \( C_\alpha > 0 \) with
\[
\int_{[0,\tau] \cup [s, s+\tau]} t^{\alpha-1} \tau_\alpha(r+s-t) dt \leq C_\alpha \tau_\alpha(r) \tau_\alpha(s), \quad 0 \leq r \leq s \leq s + \tau < \kappa.
\]
If \( \kappa' > \kappa \) then \( \Omega_{\alpha,\kappa'} \subset \Omega_{\alpha,\kappa} \). Functions \( \tau_\alpha(t) = t^\alpha \); \( t^\beta(1+t)^\nu \) with \( \beta \in [0, \alpha] \) and \( \nu \geq \alpha - \beta \); \( t^\beta e^{\tau t} \), with \( \tau > 0 \) and \( \beta \in [0, \alpha] \), belong to \( \Omega_{\alpha,\infty} \). If \( \tau_\alpha \in \Omega_{\alpha,\kappa} \) then \( \tau_\nu \in \Omega_{\nu,\kappa} \), where \( \tau_\nu(t) := t^{\nu-\alpha} \tau_\alpha(t) \) for \( t \geq 0 \) and \( \nu \geq \alpha \). The subset of functions \( \tau_\alpha(t) = t^\alpha w(t) \) (where \( w \) is a continuous and non-decreasing weight.
function on $[0, \kappa)$, $w(t+s) \leq Cw(t)w(s)$ for $0 \leq t, s \leq t+s < \kappa$ is denoted by $\Omega^h_{\alpha, \kappa}$, see [9] for more details.

We use the equality (2) and the proof of Theorem 2 to obtain the following corollary.

**Corollary 4** Take $\alpha \geq 0$ and $\tau_\alpha \in \Omega_{\alpha, \kappa}$. Let $(v_x(t))_{t \in [0, \kappa)}$ be the solution of the local well-posed integrated Cauchy problem $C_{\alpha+1}(\kappa)$ such that

$$\|v_x'(t)\| \leq C\tau_\alpha(t), \quad t \in [0, \kappa).$$

Then the solution of the local integrated Cauchy problem $C_{2\alpha+1}(2\kappa)$, $(u_x(t))_{t \in [0, 2\kappa)}$, verifies

$$\|u_x'(t)\| \leq \begin{cases} Ct^\alpha \tau_\alpha(t), & t \in [0, \kappa), \\ C\tau_\alpha(\kappa)(\kappa^\alpha + \tau_\alpha(t-\kappa)), & t \in [\kappa, 2\kappa). \end{cases}$$

3 Quasi-distribution semigroups and global solutions of abstract Cauchy problem

In this section we start considering quasi-distribution semigroups introduced in [22]. Let $D_+$ be the class of $C^\infty$ functions of compact support on $[0, \infty)$; $\mathcal{D}$ be the class of $C^\infty$ functions of compact support on $\mathbb{R}$, and $D_0$ be the subspace of those $\phi$’s of $\mathcal{D}$ with $\text{supp}(\phi) \subset [0, \infty)$. Note that if $\phi, \psi \in D_+$ then $\phi \ast \psi \in D_+$ where

$$\phi \ast \psi(t) = \int_0^t \phi(t-s)\psi(s)ds, \quad t \geq 0.$$ 

We consider the usual topology defined in $D_+, D$ and $D_0$. A quasi-distribution semigroup on $X$ is a linear and continuous map, $\mathcal{G} : D_+ \rightarrow \mathcal{B}(X)$ (QDSG) such that verifies:

(i) $\mathcal{G}(\phi \ast \psi) = \mathcal{G}(\phi)\mathcal{G}(\psi)$ for $\phi, \psi \in D_+$,

(ii) $\bigcap\{\ker(\mathcal{G}(\phi)) ; \phi \in D_0\} = \{0\},$

see [22, Definition 3.3]. Although Wang considered maps from $\mathcal{D}$ to $\mathcal{B}(X)$, both approaches are equivalent [22, Remark 3.4(ii)]. Quasi-distribution semigroup extend distribution semigroups in the sense of Lions ([22, Corollary 3.11]).

For a given QDSG $\mathcal{G}$, the operator $A_1$ is defined by

$$D(A_1) = \bigcup\{\text{Im}(\mathcal{G}(\phi)) ; \phi \in D_+\};$$
\[ A_1 \mathcal{G}(\phi)(x) := -\mathcal{G}(\phi')(x) - \phi(0)x, \quad x \in X, \phi \in \mathcal{D}_+. \]

It is not difficult to check that \((A_1, D(A_1))\) is well-defined and closable (see [22, Proposition 3.5]. The closure of \(A_1\), denoted by \(A\) is called the generator of the QDSG \(\mathcal{G}\) and given \(\phi \in \mathcal{D}_+, \mathcal{G}(\phi) \subset AG(\phi)\) hold ([22, Proposition 3.7]). An alternative definition of a generator is given in [14, Definition 3.3].

It is well-known that QDSGs are equivalent to well-posed Cauchy problems \(C_{\alpha+1}(\kappa)\).

**Theorem 5** [22, Theorem 2.4, Theorem 3.8] Let \(A\) be a closed operator. Then the following are equivalent.

(i) There exists \(\alpha \geq 0\) and \(\kappa > 0\) such that \(C_{\alpha+1}(\kappa)\) is well posed.

(ii) \(A\) generates a QDSG.

A direct proof of \((i) \Rightarrow (ii)\) involves Weyl fractional derivatives. For a function \(f \in \mathcal{D}_+\) and \(\alpha \in \mathbb{R}^+\), the Weyl fractional integral \(W_+^{-\alpha}f\) of order \(\alpha\) is defined by

\[ W_+^{-\alpha}f(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s)ds, \]

for \(t \geq 0\), and the Weyl fractional derivative \(W_+^\alpha f\) of order \(\alpha\) is defined by

\[ W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)ds, \]

with \(n = [\alpha] + 1\) and \(t \geq 0\). It can be seen that \(W_+^{\alpha+\beta} = W_+^\beta(W_+^\alpha)\) for any \(\alpha, \beta \in \mathbb{R}\), where \(W_+^0 = Id\) is the identity operator ([19]). The Weyl fractional calculus can be applied to more functions than those belonging to \(\mathcal{D}_+\), see [19, p. 248]. In this sense, for example, let \(f\) and \(g\) be measurable functions on \([0, \infty)\) such that \(W_+^{-\alpha}f\) exists and \(g = W_+^{-\alpha}f\) a.e. Then we consider \(W_+^\alpha g = f\) and we follow the same notation.

To show that \((i) \Rightarrow (ii)\) Take \((S_\alpha(t))_{t \in [0, \kappa]}\) the non-degenerate \(\alpha\)-times integrated semigroup which gives the solution of the problem \(C_{\alpha+1}(\kappa)\) (see (2)) and \(\phi \in \mathcal{D}_+, \text{ with supp}(\phi) \subset [0, R]\). Then there exists \(n \in \mathbb{N} \cup \{0\}\) such that
$2^n \alpha > R$, and we define $\mathcal{G}(\phi)$ by

$$
\mathcal{G}(\phi)x := \int_0^{2^n \alpha} W_+^{2^n \alpha} \phi(t) S_{2^n \alpha}(t) x dt, \quad x \in X,
$$

where $(S_{2^n \alpha}(t))_t$ is defined recursively from the extension given in Theorem 2. To prove that

$$
\mathcal{G}(\phi \ast \psi) = \mathcal{G}(\phi) \mathcal{G}(\psi)
$$

for $\phi, \psi \in \mathcal{D}_+$, see similar ideas in [18, Theorem 3.1]. We may conclude that $\mathcal{G}$ is a $QDSG$.

Now we consider the case $\kappa = \infty$. Although there are local $\alpha$-times integrated semigroups which cannot be extended (see Example 1), differential operators in Euclidean spaces are important examples of global $\alpha$-times integrated semigroups ([11]).

Let $\omega : [0, \infty) \to (0, \infty)$ be a weight function and $L^1(\mathbb{R}^+, \omega)$ the usual convolution Banach algebra of measurable functions on $[0, \infty)$ such that $\|f\|_\omega := \int_0^{\infty} |f(t)| \omega(t) dt < \infty$. Riesz functions $(R^\theta_t)_t > 0$ are defined by

$$
R^\theta_t(s) := \frac{(t-s)^\theta}{\Gamma(\theta+1)} \chi_{(0,t)}(s), \quad s \geq 0,
$$

$t \geq 0$, $R^\theta_0 = 0$ and $\theta > -1$. We denote by $Mul(A)$ the set of multipliers of a Banach algebra $A$.

**Theorem 6** [9, Proposition 1.4, Proposition 1.5] Let $\alpha \in \mathbb{R}^+$ and $\tau_\alpha \in \Omega_{\alpha, \infty}$. The expression

$$
q_{\tau_\alpha}(f) := \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \tau_\alpha(t)|W_+^\alpha f(t)| dt,
$$

defines a norm on $\mathcal{D}_+$. Moreover, $q_{\tau_\alpha}(f \ast g) \leq C_\alpha q_{\tau_\alpha}(f) q_{\tau_\alpha}(g)$ for $f, g \in \mathcal{D}_+$, and $C_\alpha > 0$ is independent of $f$ and $g$. We denote by $T_{+}^{(\alpha)}(\tau_\alpha)$ the Banach algebra obtained as the completion of $\mathcal{D}_+$ in the norm $q_{\tau_\alpha}$ with $\tau_\alpha \in \Omega_{\alpha, \infty}$. Moreover, we have the following continuous embeddings.

(i) $T_{+}^{(\alpha)}(\tau_\alpha) \hookrightarrow T_{+}^{(\alpha)}(t^\alpha) \hookrightarrow L^1(\mathbb{R}^+)$. 

(ii) If $\beta > \alpha \in \mathbb{R}^+$, and $\tau_\beta \in \Omega_{\beta, \infty}$ such that

$$
\frac{1}{\Gamma(\beta - \alpha) \Gamma(\alpha + 1)} \int_0^t (t-s)^{\beta-\alpha-1} \tau_\alpha(s) ds \leq \frac{1}{\Gamma(\beta + 1)} \tau_\beta(t), \quad t \geq 0,
$$

then $T_{+}^{(\beta)}(\tau_\beta) \hookrightarrow T_{+}^{(\alpha)}(\tau_\alpha)$ holds; and in particular $T_{+}^{(1)}(t^\beta) \hookrightarrow T_{+}^{(\alpha)}(t^\alpha)$. 11
Let $\alpha > 0$ and $\nu > \alpha$; 

$$q_{\tau}(R^\nu t) \leq C_{\nu,\alpha} t^{\nu - \alpha} \tau(t),$$

for any $t > 0$ where $C_{\nu,\alpha} > 0$ is independent of $t$.

(ii) $R^\nu t = T^\nu(t)\tau(t)$ with $t > 0$ and $\nu > \alpha$.

(iii) $R^\nu t^{-1} \in T^\nu_+(\tau_\alpha)$ with $t > 0$ and $\nu > \alpha$; 

$$q_{\tau}(R^\nu t^{-1}) \leq C_{\nu,\alpha} t^{\nu - \alpha} \tau(t),$$

for any $t > 0$ where $C_{\nu,\alpha} > 0$ is independent of $t$.

(iv) $R^\nu t^{-1} \in Mul(T^\nu_+(\tau_\alpha))$ and 

$$\|R^\nu t^{-1}\|_{Mul(T^\nu_+(\tau_\alpha))} \leq C \tau(t)$$

with $t > 0$.

In the case $\alpha = 0$, we identify $T_+^{(0)}(\tau_0)$ and $L^1(\mathbb{R}^+, \tau_0)$. If $\alpha = n$ and $\tau_n(t) = t^n$ for any $t \geq 0$, the algebra $T_+^{(n)}(t^n)$ is $T_+^{n+}$ as defined in [1] and considered in [4] and [5]. If $\alpha = n$ and $\tau_n(t) = e^{rt}$ $(r > 0, t \geq 0)$, then the algebra $T_+^{(n)}(e^{rt})$ is $D_+^{n+}$ defined as in [22].

If $\tau_\alpha \in \Omega^\alpha_{\alpha, \infty}$ with $\alpha \geq 0$, the algebra $T_+^{(\alpha)}(\tau_\alpha)$ has bounded approximate identities (take $\phi \in T_+^{(\alpha)}(\tau_\alpha)$ such that $\int_0^\infty \phi(t)dt = 1$ and let consider $(\phi_s = 1_s \phi(\cdot))_{0 < s < 1}$. In general, algebras $T_+^{(\alpha)}(\tau_\alpha)$ do not have any bounded approximate identity ([9]).

**Definition 7** We say that a quasi-distribution semigroup $G : D_+ \rightarrow B(X)$ is of order $\alpha \in \mathbb{R}^+$ and growth $\tau_\alpha \in \Omega_{\alpha, \infty}$ if $G$ can be extended to an algebra continuous homomorphism from $T_+^{(\alpha)}(\tau_\alpha)$ into $B(X)$, i.e., 

$$\mathcal{G} : T_+^{(\alpha)}(\tau_\alpha) \rightarrow B(X).$$

This definition contains [22, Definition 4.3] and [22, Definition 4.7].

**Lemma 8** Let $\alpha \geq 0$, $\tau_\alpha \in \Omega^\alpha_{\alpha, \infty}$ and let $(A, D(A))$ be a closed and densely defined operator which generates a quasi-distribution semigroup $G$ on $X$ of order $\alpha \in \mathbb{R}^+$ and growth $\tau_\alpha$. Then $\mathcal{G}(T_+^{(\alpha)}(\tau_\alpha))X$ is dense in $X$.

**Proof:** As $\tau_\alpha(t) = t^\alpha \omega(t)$ for some continuous and non-decreasing weight 

$$\omega : [0, \infty) \rightarrow [0, \infty)$$

and $\omega(t) \leq Ce^{\alpha t}$ $(t \geq 0, C, \kappa > 0)$, then we get that $(\kappa, \infty) \subset \rho(A)$ and sup$_{\lambda > \kappa + 1} \|\lambda R(\lambda, A)\| < \infty$. By [3, Lemma 3.3.12],

$$\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)x = x$$

holds for $x \in X$. Since $\mathcal{G}(e^{-\lambda t}) = R(\lambda, A)$ for $\lambda \geq \kappa$ we conclude that $\mathcal{G}(T_+^{(\alpha)}(\tau_\alpha))X$ is dense in $X$. \[\square\]

The following theorem extends results from [1], [7] and [18] given in terms of integrated semigroups.

**Theorem 9** Suppose $(v_x(t))_{t \geq 0}$ is the solution of the $C_{\alpha+1}(\infty)$ such that

$$\|v_x^\prime(t)\| \leq C_\tau(t)\|x\|, \quad x \in X,$$

with $\tau_\alpha \in \Omega_{\alpha, \infty}$. Then there exists a quasi-distribution semigroup of order $\alpha$.
and growth \( \tau_\alpha \), \( \mathcal{G} : T_+^{(\alpha)}(\tau_\alpha) \rightarrow \mathcal{B}(X) \) given by

\[
\mathcal{G}(f)x = \int_0^\infty W^\alpha_x f(t)v'_x(t)dt, \quad x \in X,
\]

with \( f \in T_+^{(\alpha)}(\tau_\alpha) \) generated by \((A, D(A))\).

**Proof:** By Theorem 5, \((A, D(A))\) is the generator of a quasi-distribution semigroup, \( \tilde{\mathcal{G}} : \mathcal{D} \rightarrow \mathcal{B}(X) \). Since \( \|v'_x(t)\| \leq C\tau_\alpha(t)\|x\| \) for any \( t \geq 0 \), the expression

\[
\mathcal{G}(f)x := \int_0^\infty W^\alpha_x f(t)v'_x(t)dt, \quad f \in T_+^{(\alpha)}(\tau_\alpha), \quad x \in X,
\]

defines a continuous and linear homomorphism, \( \mathcal{G} : T_+^{(\alpha)}(\tau_\alpha) \rightarrow \mathcal{B}(X) \). There are several ways to conclude that \( \mathcal{G}|_{\mathcal{D}} = \tilde{\mathcal{G}} \). We do this by checking that \( \mathcal{G} \) is a quasi-distribution semigroup generated by \((A, D(A))\) and \((v'_x(t))_{t \geq 0}\) defines an \( \alpha \)-times integrated semigroup, see similar proof in [18, Theorem 3.1]. \( \Box \)

**Remark 10** Note that in this theorem we do not assume that \( A \) is a densely defined operator.

**Example 2.** The function \( E_c : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), defined by \( E_c(t) := e^{ct^2} \) \((t > 0)\) does not belong to \( \Omega_{1,\infty} \) for any \( c > 0 \). We have

\[
\lim_{t \to \infty} \int_0^t e^{cs^2} ds = \lim_{t \to \infty} \frac{e^{ct^2} - e^{cr^2}}{2cte^{ct^2}} = \infty.
\]

Then there is no \( C > 0 \) such that \( \int_0^t e^{cs^2} ds \leq Ce^{ct^2}e^{cr^2} \) for \( 0 < r < t \). However, there are 1-integrated semigroups such that \( \|S_1(t)\| = e^{t^2} \), \((t \geq 0)\) ([17, Example 1.2.5]). In this case we cannot apply Theorem 9 with \( \tau_1(t) = e^{t^2} \).

We now prove the first converse to Theorem 9. Some precedents of this result are [1, Theorem 4.4], [7, Theorem 3.6] and [22, Theorem 4.16]. They consider integer derivation and particular growths \( \tau_\alpha \).

**Theorem 11** Given \( \alpha \geq 0 \), \( \tau_\alpha \in \Omega_{a,\infty} \), and \( \mathcal{G} : T_+^{(\alpha)}(\tau_\alpha) \rightarrow \mathcal{B}(X) \) a quasi-distribution semigroup generated by \((A, D(A))\). Then for any \( \nu > \alpha \), the abstract Cauchy problem \( C_{\nu+1}(\infty) \) is well-posed and the solution \((v_x(t))_{t \geq 0}\) verifies

\[
\|v'_x(t)\| \leq C_\nu t^{\nu-\alpha} \tau_\alpha(t)\|x\|, \quad t \geq 0.
\]
Proof: Take \( \nu > \alpha \). The family of Riesz functions \((R_t^{\nu-1})_{t \geq 0}\), is a \( \nu \)-times integrated semigroup in \( T^{(a)}_+(\tau_a) \) and \( g_{\tau_a}(R_t^{\nu-1}) \leq C_{\nu,\alpha} t^{\nu-\alpha} \tau_a(t) \), if \( \nu > \alpha \), \( t \geq 0 \), see the Theorem 6 (iii). Put \( S_\nu(t) := G(R_t^{\nu-1}) \) for any \( t \geq 0 \). It is clear that \((S_\nu(t))_{t \geq 0}\) verifies (3) and is a \( \nu \)-times integrated semigroup. From the continuity of \( G \) we have that \( \|S_\nu(t)\| \leq C_{\nu,\alpha} t^{\nu-\alpha} \tau_a(t) \), for any \( t \geq 0 \). It is straightforward to check \((S_\nu(t))_{t \geq 0}\) is generated by \((A, D(A))\). \( \square \)

By Theorem 6 (iv), the Riesz function \( R_t^{\alpha-1} \) is a multiplier of the algebra \( T^{(a)}_+(\tau_a) \) for every \( t > 0 \). If \( T^{(a)}_+(\tau_a) \) has a bounded approximate identity, we may calculate \((G(R_t^{\alpha-1}))_{t \geq 0}\) so getting the second converse to Theorem 9.

**Theorem 12** Let \( \alpha \geq 0 \), \( \tau_a \in \Omega^h_{\alpha,\infty} \) and let \((A, D(A))\) be a closed and densely defined operator on \( X \). The following conditions are equivalent.

(i) The abstract Cauchy problem \( C_{\alpha+1}(\infty) \) is well-posed and the solution \((v_x(t))_{t \geq 0}\) verifies
\[
\|v_x'(t)\| \leq C_\alpha \tau_a(t) \|x\|, \quad t \geq 0.
\]

(ii) \((A, D(A))\) generates a quasi-distribution semigroup \( G \) of order \( \alpha \in \mathbb{R}^+ \) and growth \( \tau_a \).

**Proof:** By Theorem 9 we define \( G \) and prove (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (i) As \( \tau_a \in \Omega^h_{\alpha,\infty} \), \( T^{(a)}_+(\tau_a) \) has a bounded approximate identity. Since \( G(T^{(a)}_+(\tau_a))X \) is dense in \( X \) (see Lemma 8) then \( X = G(T^{(a)}_+(\tau_a))X \) by Cohen’s factorization Theorem. The map \( G \) extends to a Banach algebra homomorphism \( \bar{G} : Mul(T^{(a)}_+(\tau_a)) \to \mathcal{B}(X) \). Then, if \( T \in Mul(T^{(a)}_+(\tau_a)) \) and \( x = \Theta(f)y \) \( (f \in T^{(a)}_+(\tau_a), y \in X) \) we define \( \bar{G}(T)x := G(T(f))y \), see [8, Proposition 5.2]. By the Theorem 6 (iv), \( R_t^{\alpha-1} \in Mul(T^{(a)}_+(\tau_a)) \) and we put \( S_\alpha(t) := \bar{G}(R_t^{\alpha-1}) \), for all \( t \geq 0 \). The proof is finished in the similar way as the proof in Theorem 11. \( \square \)

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**References**


