Computer proofs of new identities in the Catalan triangle

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Abstract

In this paper we prove new polynomial identities in the Catalan triangle using the WZ theory. In fact, we check new polynomial identities in the general expression

\[ P(n, i, k) := \sum_{p=1}^{i} B_{n,p}B_{n,n+i}(n+2p-i)^k, \quad i \leq n, \]

for some \( i \leq n \) and \( k \in \mathbb{N} \cup \{0\} \), being

\[ B_{n,p} := \frac{p}{n} \binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \ p \leq n, \]

the \((n, p)\) entry in the Catalan triangle.

We show that \( P(n, i, k) \) involves the well-known Catalan numbers, combinatorial numbers and polynomials which depend on two variables, \( i \) and \( n \).

1. Introduction

The generalized Catalan numbers \( B_{n,p} \) where

\[ B_{n,p} := \frac{p}{n} \binom{2n}{n-p}, \quad n, p \in \mathbb{N}, \ p \leq n. \]

2000 Mathematics Subject Classification: 05A19; 11B65; 05A10

Keywords: Catalan numbers; Combinatorial identities; Binomial coefficients; WZ theory
were introduced by L. Shapiro in [8]. There he presented them in the following triangle of numbers

\[
\begin{array}{cccccc}
 n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
 1 & 1 & & & & & & \\
 2 & 2 & 1 & & & & & \\
 3 & 5 & 4 & 1 & & & & \\
 4 & 14 & 14 & 6 & 1 & & & \\
 5 & 42 & 48 & 27 & 8 & 1 & & \\
 6 & 132 & 165 & 110 & 44 & 10 & 1 & \\
 \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

Note that the entries in the first column are the well known Catalan numbers \( C_n \), i.e. \( B_{n,1} = C_n \),

\[
C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 1,
\]

and the above triangle is the called Catalan triangle. Catalan numbers may be also defined recursively by the law

\[
C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad n \geq 1,
\]

and \( C_0 = 1 \).

Catalan numbers appear in a wide range of problems. Between them, they give the solution of the famous Euler’s Polygon Division Problem (how many ways a plane convex polygon of \( n+2 \) sides can be divided into triangles by diagonals [3]). They also give the number of binary brackets of \( n+1 \) letters (Catalan’s problem); the solution to the ballot problem [6]; the number of trivalent planted planar trees [2] and so many others, see more in [10].

The numbers \( B_{n,p} \) given by (1.1) are not as known as Catalan numbers. However, they have also several applications: \( B_{n,p} \) is the number of leaves at level \( p+1 \) in all ordered trees with \( n+1 \) edges; \( B_{n,p} \) is also the number of walks of \( n \) steps, each in direction \( N, S, W \) or \( E \), starting at the origin, remaining in the upper half-plane and ending at height \( p \); or \( B_{n,p} \) denotes the number of pairs of non-intersecting paths of length \( n \) and distance \( p \). In addition, the recurrence relation

\[
B_{n,p} = B_{n-1,p-1} + 2B_{n-1,p} + B_{n-1,p+1}, \quad p \geq 2
\]

is satisfied by the numbers \( B_{n,p} \). See [1, 8] and [9] for more information. Other generalized Catalan numbers are considered in [6].
In this paper we establish new identities for the numbers $B_{n,p}$. In fact we check the value of $P(n, i, k)$ where

\begin{equation}
P(n, i, k) := \sum_{p=1}^{i} B_{n,p}B_{n,n+p-i}(n + 2p - i)^k, \quad i \leq n,
\end{equation}

for some values of $k \in \mathbb{N}$, note that $P(n, 1, k) = (n+1)^kC_n$ for $n, k \in \mathbb{N}$. The case $P(n, n, 0)$ is found in the original paper of Shapiro [8]; $P(n, i, 1)$ (with $1 \leq i \leq n$) and $P(n, n, 2)$ have been given in [4]. The case $P(n, i, 1)$ appears in a problem related with the dynamical behavior of a family of iterative processes, see more details in [4, 5].

Here we give the value of $P(n, i, 3)$ with $1 \leq i \leq n$ (Theorem 2.3); and $P(n, n, k)$ for $0 \leq k \leq 7$ (Theorems 2.1 and 2.2). In this case the equation (1.2) takes the form

$$P(n, n, k) = 2^k \sum_{p=1}^{n} p^k(B_{n,p})^2.$$ 

Furthermore, we derive new relations that involve the Catalan numbers. For instance, in the case $k = 3$ and $i = n$ we obtain that

$$\sum_{p=1}^{n} p^3(B_{n,p})^2 = \frac{n^2(n + 1)}{2}C_nC_{n-1},$$

with $n \in \mathbb{N}$, see Theorem 2.2 (ii). Our results answer partially the second open question which was posed in [4].

Proofs of these identities are made using the WZ theory and the EKHAD package of the MAPLE program [7, 11]. Alternative proofs may be found with a hard skilful handling of Vandermonde’s convolution product, see ideas and techniques in [4]. In the last section we give some comments and open questions about other polynomial identities in the Catalan triangle.

Authors thank Prof. L. W. Shapiro and Prof. D. Zeilberger for valuable suggestions and references that lead to these new results.

2. Results and computer proofs

The WZ theory is a powerful tool to show hypergeometric identities. To check the following combinatorial identities, we have used the Maple program, and the EKHAD package as software for the WZ method, see [7, Example 7.5.3].

Although items (i) and (ii) of the next result appears in [4] we have decided to include them to give a complete list of identities in this field.
Theorem 2.1. Let $n \in \mathbb{N}$. Then

(i) $\sum_{p=1}^{n} (B_{n,p})^2 = C_{2n-1}.$

(ii) $\sum_{p=1}^{n} p^2 (B_{n,p})^2 = \frac{(3n - 2)n}{4n - 3} C_{2n-1}.$

(iii) $\sum_{p=1}^{n} p^4 (B_{n,p})^2 = \frac{(15n^3 - 30n^2 + 16n - 2)n}{(4n - 3)(4n - 5)} C_{2n-1}.$

(iv) $\sum_{p=1}^{n} p^6 (B_{n,p})^2 = \frac{(105n^5 - 420n^4 + 588n^3 - 356n^2 + 96n - 10)n}{(4n - 3)(4n - 5)(4n - 7)} C_{2n-1}.$

Proof. We use the WZ theory and the Maple program to show (i), (ii) (iii) and (iv). We detail the proof of (i). We define the function

$$f(n, p) := \frac{(B_{n,p})^2}{C_{2n-1}} = \frac{2p^2 (\binom{2n}{n-p})^2}{n(\binom{2n}{2n-1})};$$

and we execute the instruction $ct(f(n,p), 1, p, n, N)$ and the program $ct$ returns the pair, $1 - N$ (a telescoping recurrence) and the rational function (self-certifying)

$$(p - 1) \frac{-8np^3 + 16n^3p + 12np^2 - 2p^3 - 8n^3 + 18n^2p - p + 3p^2 - 9n^2 - 2n}{2p(16n^2 - 1)(n + 1 - p)^2}.$$

Using this function we construct the WZ mate of $f$ and we apply the WZ theory to show the identity, see more details in [7, Example 7.5.3].

Now we consider similar results in the case of odd exponent. Item (i) of the next theorem was proved in [4] and item (ii) was conjectured there.

Theorem 2.2. Let $n \in \mathbb{N}$. Then

(i) $\sum_{p=1}^{n} p (B_{n,p})^2 = (n + 1)C_n(2n - 3)C_{n-2}.$

(ii) $\sum_{p=1}^{n} p^3 (B_{n,p})^2 = (n + 1)C_n n(2n - 3)C_{n-2}.$
(iii) \[ \sum_{p=1}^{n} p^5 (B_{n,p})^2 = (n + 1)C_n n(3n^2 - 5n + 1)C_{n-2}. \]

(iv) \[ \sum_{p=1}^{n} p^7 (B_{n,p})^2 = (n + 1)C_n n(6n(n - 1)^2 - 1)C_{n-2}. \]

**Proof.** Proofs follow the same lines than in the previous result.

In fact, item (ii) of Theorem 2.2 is a particular case of the next result.

**Theorem 2.3.** Let \( n, i \in \mathbb{N} \) where \( i \leq n \). Then,

\[ \sum_{p=1}^{i} B_{n,p}B_{n,n+p-i}(n + 2p - i)^3 = (n + 1)C_n \left( \binom{2(n-1)}{i-1} \right) \left( n^2 + 4n - 2ni + i^2 \right). \]

**Proof.** We define the function

\[ F(n, i, p) := \frac{p(n + 2p - i)^3(n + p - i)\binom{2n}{n-p}\binom{2n}{2n-2n}}{n^2(n^2 + 4n - 2ni + i^2)\binom{2n}{2n-2n}}, \]

where \( 1 \leq p \leq i \leq n \). We execute the instruction \( ct(F(n,i,p),1,p,n,N) \) and the program \( ct \) returns the pair

\[ 1 + N, \quad R(n, i, p) \]

where \( R(n, i, p) \) is a rational function. Following similar steps than in previous results, the proof is finished.

\[ \square \]

3. Open questions and remarks

After results of the second section, we may pose the following open questions.

(1) Exist there polynomials \( P_k \) of natural coefficients such that

\[ \sum_{p=1}^{n} p^{2k}(B_{n,p})^2 = \frac{P_k(n)}{\prod_{l=1}^{k} (4n - (2l + 1))} C_{2n-1}, \]

for \( k \in \mathbb{N} \)?

(2) Exist there polynomials \( P_l \) of natural coefficients and degree \( l \) such that

\[ \sum_{p=1}^{n} p^{2l-1}(B_{n,p})^2 = (n + 1)C_n P_l(n)C_{n-2}, \]
for \( l \in \mathbb{N} \)?

(3) Exist there polynomials \( P_r(n, i) \) and \( Q_r(n, i) \) of natural coefficients such that

\[
P(n, i, 2r - 1) = (n + 1)C_n \frac{P_r(n, i)}{Q_r(n, i)} \binom{2n - 2}{i - 1},
\]

for \( r \in \mathbb{N} \) and \( 1 \leq i \leq n \). For \( 2r - 1 = 5 \) we have checked that \((n + 1)C_n\) divides to \( P(n, i, 2r - 1) \) for a great amount of \( n \).

References


First author is partly supported by DGI-FEDER (MTM 2004-03036) and the DGA project (E-64) and the second author by the Ministry Education and Science (MTM 2005-03091) and the University of La Rioja (ATUR-05/43).
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