Dual mixed volumes and extremal positions of convex bodies

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From John to Gauss-John ellipsoids via dual mixed volumes
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$0 \in K \subset \mathbb{R}^n$ convex body (compact, $Int(K) \neq \emptyset$)

**Example.** Ellipsoid of minimal volume containing $K$, (F. John, 1948).

- $\min\{|T(D_n)|; \quad K \subseteq T(D_n), \quad T \in GL(n)\}$.

or equivalently,

- $\max\{|T(K)|; \quad T(K) \subseteq D_n, \quad T \in GL(n)\}$.

**Theorem.** $D_n$ is the extremal ellipsoid if and only if there are vectors $w_1, \ldots, w_N \in \partial K \cap S^{n-1}$ and numbers $\lambda_1, \ldots, \lambda_N > 0 \ (\sum \lambda_j = 1)$ such that

$$I_n = n \sum_{j=1}^{N} \lambda_j w_j \otimes w_j \quad \text{(decomposition of $I_n$)}$$

where $w_j \otimes w_j(\cdot) = \langle \cdot, w_j \rangle w_j$.

**Fact:** Relevant positions of convex bodies appear as solutions of optimization problems.
Examples: \( \min / \max \left\{ f(T(K)) ; \text{det}(T) = 1 \right\} \)

- Minimal mean width position ([GM], 2000).
- Minimal surface area ([P], 1961. [GP], 1999).
- Isotropic position ([MP], 1989).
- \( MM^* \)-position ([BR], 2004).
- \( L_p \) ellipsoids ([LYZ], 2004).

Characterizations given in terms of "isotropic properties of a certain measure".

\[
\begin{align*}
\min / \max \left\{ f(T(K)) ; T(K) \subseteq D_n , T \in GL(n) \right\}
\end{align*}
\]

- John's position.

\[
\begin{align*}
\max\{|T(K)| ; T(K) \subseteq D_n , T \in GL(n) \}
\end{align*}
\]

- Gauss-John position.

\[
\begin{align*}
\min \left\{ \int_{S^{n-1}} \|u\|_{TK} d\sigma(u) ; TK \subseteq D_n , T \in GL(n) \right\},
\end{align*}
\]

In [GMR] 2000, necessary conditions were obtained for a symmetric convex body to be in Gauss-John position.
The John position and Gauss-John position are particular cases of a more general phenomenon concerning dual mixed volumes.

Let $0 \in K \subseteq \mathbb{R}^n$ be a star shaped body and $i \in \mathbb{R}$. The $i$th-dual mixed volume $\tilde{W}_i(K)$ is defined as

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) d\sigma(u),$$

where

$$\rho_K(x) = \max\{\lambda \geq 0; \lambda x \in K\} \quad x \in \mathbb{R}^n \setminus \{0\}.$$

**Problem:** If $i \in \mathbb{R}$ and $K \subseteq D_n$ is a convex body with $0 \in K$, can we characterize the solution of the following extremal problems?

$$\max \{\tilde{W}_i(TK); \quad T \in GL(n), \quad T(K) \subseteq D_n\}.$$ 

- If $i = 0$, the extremal problem yields to John position since

$$\tilde{W}_0(TK) = \int_{S^{n-1}} \rho_{TK}^n(u) d\sigma(u) = |TK|.$$ 

- If $i = n + 1$, the extremal problem yields to Gauss-John position since

$$\tilde{W}_{n+1}(TK) = \int_{S^{n-1}} \rho_{TK}^{-1}(u) d\sigma(u)$$

$$= \frac{1}{n} \int_{S^{n-1}} \|x\|_{TK} d\sigma(x) = c_n M(TK).$$
If $0 \in K \subseteq \mathbb{R}^n$ is a convex body, the extremal problems we are dealing with are

$$\max \{ \tilde{W}_i(TK); \ TK \subseteq D_n; \ T \in GL(n) \} \quad (i < n)$$
$$\min \{ \tilde{W}_i(TK); \ TK \subseteq D_n; \ T \in GL(n) \} \quad (i > n)$$

**Theorem 1. (Necessary condition).** Let $0 \in K \subseteq D_n$ be a convex body and let $i \in \mathbb{R}$.

If $\tilde{W}_i(K) = \max \{ \tilde{W}_i(TK); \ TK \subseteq D_n \}$ $(i < n)$
or $\tilde{W}_i(K) = \min \{ \tilde{W}_i(TK); \ TK \subseteq D_n \}$ $(i > n)$,
then there exist $\lambda_1, \ldots, \lambda_N > 0$ ($\sum \lambda_j = 1$), and vectors $w_1, \ldots, w_N \in \partial K \cap S^{n-1}$ such that

$$I_n = i \int_{S^{n-1}} u \otimes u \, d\mu_i(u) + (n - i) \sum_{j=1}^N \lambda_j w_j \otimes w_j,$$

where $d\mu_i(u) = \frac{\rho_{K}^{n-i}(u)d\sigma(u)}{\int_{S^{n-1}} \rho_{K}^{n-i}(u)d\sigma(u)}$.

**Key fact in the proof:** Consider $T \in \mathbb{R}^{n^2}$ and use a general variational result by John.

The necessary condition implies that $\forall \theta \in S^{n-1}$,

$$1 = i \int_{S^{n-1}} \langle u, \theta \rangle^2 \, d\mu_i(u) + (n - i) \sum_{j=1}^N \lambda_j \langle \omega_j, \theta \rangle^2$$
**Theorem 2. (Sufficient conditions)** Let $0 \in K \subseteq D_n$ be a convex body and let $i \in \mathbb{R}$ such that

$$I_n = i \int_{S^{n-1}} u \otimes u \, d\mu_i(u) + (n - i) \sum_{j=1}^{N} \lambda_j w_j \otimes w_j,$$

for some numbers $\lambda_1, \ldots, \lambda_N > 0$ ($\sum \lambda_j = 1$), and some vectors $w_1, \ldots, w_N \in \partial K \cap S^{n-1}$. Then,

(i) if $i \in [-2, 0] \cup [n + 1, +\infty)$, $K$ is in extremal position.

(ii) if $i \in (-\infty, -2) \cup (0, n)$ and the measure $d\mu_i$ is isotropic, $K$ is in extremal position.

Moreover, the extremal position of $K$ is unique up to orthogonal transformations.

In particular the necessary conditions in [GMR] (for symmetric convex bodies) are also *sufficient* for any convex body (containing the origin).

$d\mu_i$ isotropic means that $\int_{S^{n-1}} \langle u, \theta \rangle^2 \, d\mu_i(u)$ does not depend on $\theta$. 
Proof. \( i = n + 1 \)

If \( T \) is linear transformation, there exist a diagonal transformation \( D = (d_{ij}) \) \((d_{ij} > 0)\) and \( U, V \in O(n) \) such that \( T = V^* D U \).

If we denote \( K_1 = U K \), it is enough to prove that

\[
\tilde{W}_{n+1}(DK_1) \geq \tilde{W}_{n+1}(K_1)
\]

\[
\tilde{W}_{n+1}(DK_1) = \frac{1}{n} \int_{S^{n-1}} \rho^{-1}_{DK_1}(u) \, d\sigma(u)
= \frac{1}{n} \int_{S^{n-1}} h((DK_1)^{(u)}(u) \, d\sigma(u).
\]

Note that \( h((DK_1)^{(u)}(u) \geq \langle \nabla h_{K_1^{(u)}}, D^{-1} u \rangle \), and hence by using Laplace-Beltrami operator techniques

\[
\tilde{W}_{n+1}(DK_1) \geq \frac{1}{n} \int_{S^{n-1}} \langle \nabla h_{K_1^{(u)}}, D^{-1} u \rangle \, d\sigma(u)
= -\text{tr}(D^{-1}) \tilde{W}_{n+1}(K_1)
+ \frac{n+1}{n} \int_{S^{n-1}} \rho^{-1}_{K_1}(u) \langle u, D^{-1} u \rangle \, d\sigma(u).
\]
Since

\[ I_n = (n + 1) \int_{S^{n-1}} u \otimes u \, d\mu_{n+1}(u) - \sum_{j=1}^{N} \lambda_j \, w_j \otimes w_j, \]

it is easy to check that

\[ \text{tr}(D^{-1}) = (n + 1) \int_{S^{n-1}} \langle u, D^{-1}u \rangle \, d\mu_{n+1}(u) - \sum_{j=1}^{N} \lambda_j \langle w_j, D^{-1}w_j \rangle \]

Hence,

\[ \tilde{W}_{n+1}(DK_1) \geq \tilde{W}_{n+1}(K_1) \sum_{j=1}^{N} \lambda_j \langle w_j, D^{-1}w_j \rangle, \]

but

\[ 1 = \langle w_j, w_j \rangle^2 \leq \langle w_j, Dw_j \rangle \langle w_j, D^{-1}w_j \rangle \leq \langle w_j, D^{-1}w_j \rangle \]

and therefore we conclude that

\[ \tilde{W}_{n+1}(DK_1) \geq \tilde{W}_{n+1}(K_1) \sum_{j=1}^{N} \lambda_j = \tilde{W}_{n+1}(K_1). \]
Proof. \(i \in (0, n)\)

As before, we can restrict ourselves to \(D\) diagonal matrices such that \(DK \subseteq D_n\).

If \(DK \subseteq D_n\) then \(\int_{S^{n-1}} |D\omega|^2 d\nu(\omega) \leq 1\) and so, by the necessary condition and since \(d\mu_i\) is isotropic, it is enough to show

\[
\frac{\tilde{W}_i(DK)}{\tilde{W}_i(K)} = \left(\prod_{j=1}^n d_j\right) \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \leq 1
\]

under the constraint \(\sum_{j=1}^n d_j^2 \leq n\).

This function has a maximum value but, by differentiating, it cannot be attained at interior point.

It remains to study \(\max\{\frac{\tilde{W}_i(DK)}{\tilde{W}_i(K)}; \sum_{j=1}^n d_j^2 = n\}\).

Since, \(\left(\prod_{j=1}^n d_j^2\right)^{1/n} \leq \frac{1}{n} \left(\sum_{j=1}^n d_j^2\right) = 1\), it suffices to show \(\int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \leq 1\) whenever \(\sum_{j=1}^n d_j^2 = n\).
By using Lagrange multipliers we get that,

\[
\int_{S^{n-1}} |Du|^{-i} d\mu_i(u) = \int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u)
\]

If \( i \in (0, n) \) we take \( p = (-i - 2)/(-i) > 1 \) and we use Hölder inequality to obtain that

\[
\int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \leq \left( \int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u) \right)^{-i/(-i-2)} = \left( \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \right)^{-i/(-i-2)},
\]

which implies \( \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \leq 1. \)