On rational transformations of linear functionals: direct problem

Manuel Alfaro $^{a,1}$, Francisco Marcellán $^{b,*2}$, Ana Peña $^{a,3}$, M. Luisa Rezola $^{a,1}$

$^a$ Departamento de Matemáticas, Universidad de Zaragoza, Spain
$^b$ Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain

Received 10 February 2004
Available online 4 August 2004
Submitted by K. Jarosz

Abstract

Let $u$ be a quasi-definite linear functional. We find necessary and sufficient conditions in order to the linear functional $v$ satisfying $(x - \tilde{a})u = \lambda(x - a)v$ be a quasi-definite one. Also we analyze some linear relations linking the polynomials orthogonal with respect to $u$ and $v$.

© 2004 Elsevier Inc. All rights reserved.

Keywords: Orthogonal polynomials; Recurrence relations; Spectral transformations; Linear functionals

Corresponding author. Fax: 34-9162-49151.
E-mail address: pacomarc@ing.uc3m.es (F. Marcellán).

1 Partially supported by MCYT Grant BFM 2003-06335-C03-03 (Spain), FEDER funds (EU), and DGA E-12/25 (Spain).
2 Partially supported by MCYT Grant BFM 2003-06335-C03-02 (Spain) and INTAS Research Network NeCCA INTAS 03-51-66378.
3 Partially supported by MCYT Grant BFM 2001-1793 (Spain), FEDER funds (EU), and DGA E-12/25 (Spain).

0022-247X/5 – see front matter © 2004 Elsevier Inc. All rights reserved.
1. Introduction

Let \( u \) be a linear functional in the linear space \( \mathbb{P} \) of polynomials with complex coefficients and denote by \( \{u_n\}_{n \geq 0} \) the sequence of the moments associated with \( u, u_n = \langle u, x^n \rangle \), \( n \geq 0 \), where \( \langle \cdot, \cdot \rangle \) means the duality bracket.

The linear functional \( u \) is said to be quasi-definite if the Hankel matrix \( H = (u_{i+j})_{i,j=0}^{\infty} \) is quasi-definite, i.e., the principal submatrices \( H_n = (u_{i+j})_{i,j=0}^{n} \), \( n \in \mathbb{N} \cup \{0\} \), are non-singular.

The linear functional \( \delta_a \) given by \( \langle \delta_a, P \rangle = P(a), \) for every \( P \in \mathbb{P} \), is not a quasi-definite linear functional since rank \( H_n = 1 \) for every \( n \geq 0 \). This linear functional is said to be either the Dirac linear functional or the Dirac mass at the point \( a \).

To the linear functional \( u \) we can associate a formal power series \( S_u(z) = \sum_{n=0}^{\infty} u_n z^n + 1 \) which is related with the \( z \)-transform of the sequence \( \{u_n\} \) of moments of \( u \). \( S_u \) is said to be the Stieltjes function of \( u \). For the Dirac linear functional \( u = \delta_a \) given as above, we have \( S_u(z) = 1/(z-a) \) in a neighborhood of infinite.

Assuming \( u \) quasi-definite, there exists a sequence of monic polynomials \( \{P_n\}_{n \geq 0} \) such that (see [2])

(i) \( \deg P_n = n, n \geq 0 \),
(ii) \( \langle u, P_n P_m \rangle = k_n \delta_{n,m} \) with \( k_n \neq 0 \).

The sequence \( \{P_n\}_{n \geq 0} \) is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional \( u \).

If \( \{P_n\}_{n \geq 0} \) is an SMOP with respect to the quasi-definite linear functional \( u \), then it is well known (see [2]) that it satisfies a three-term recurrence relation

\[
P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0,
\]

with \( \gamma_n \neq 0 \) and \( P_{-1}(x) = 0, \) \( P_0(x) = 1 \).

Conversely, given a sequence of monic polynomials generated by a recurrence relation as above, there exists a unique quasi-definite linear functional \( u \) such that the family \( \{P_n\}_{n \geq 0} \) is the corresponding SMOP. Such a result is known as the Favard theorem (see [2]).

For an SMOP \( \{P_n\}_{n \geq 0} \) relative to \( u \), let \( \{P_n^{(1)}\}_{n \geq 0} \) be the sequence of monic polynomials such that

\[
P_{n+1}^{(1)}(x) = (x - \beta_{n+1}) P_n^{(1)}(x) - \gamma_n P_{n-1}^{(1)}(x), \quad n \geq 0,
\]

\[
P_{-1}^{(1)}(x) = 0, \quad P_0^{(1)}(x) = 1.
\]

According to the Favard theorem there exists a quasi-definite linear functional \( u^{(1)} \) such that \( \{P_n^{(1)}\}_{n \geq 0} \) is the corresponding SMOP. The family \( \{P_n^{(1)}\}_{n \geq 0} \) is said to be the sequence of polynomials of first kind associated with the linear functional \( u \).

Another representation of \( \{P_n^{(1)}\}_{n \geq 0} \) is given by

\[
P_n^{(1)}(y) = \frac{1}{u_0} \left[ u_0, \frac{P_{n+1}(y) - P_{n+1}(x)}{y-x} \right],
\]
Notice that \( P_1(z)/P_{n+1}(z) \) is the \((n+1)\)-convergent of the continued fraction
\[
\frac{1}{z - \beta_0 - \frac{1}{\gamma_1 - \frac{1}{z - \beta_1 - \ldots}}}
\]
Thus
\[
S_u(z) = \frac{u_0}{z - \beta_0 - \frac{1}{\gamma_1 - \frac{1}{z - \beta_1 - \ldots}}}
\]
from a formal point of view (see [2]).

For simplicity we will assume \( u_0 = 1 \).

Let \( \{P_n(x, \alpha)\}_{n \geq 0} \) be the sequence of monic polynomials satisfying (1.1) with initial conditions \( P_0(x, \alpha) = 1 \), \( P_1(x, \alpha) = P_1(x) - \alpha \). Taking into account the Favard theorem, there exists a quasi-definite linear functional \( u_\alpha \) such that \( \{P_n(x, \alpha)\}_{n \geq 0} \) is the corresponding SMOP. This sequence is said to be the co-recursive SMOP of parameter \( \alpha \) associated with the linear functional \( u \). It is known see [2,7] that
\[
P_n(x, \alpha) = P_n(x) - \alpha P_{n-1}(x).
\]
From (1.2) we get
\[
S_u^{(1)}(z) = \frac{1}{\gamma_1} \left[ z - \beta_0 - \frac{1}{S_u(z)} \right],
\]
\[
S_{u_\alpha}(z) = \left[ \frac{1}{S_u(z) - \alpha} \right]^{-1} = \frac{S_u(z)}{1 - \alpha S_u(z)}.
\]
These two bilinear rational transforms are related to self-similar reductions and spectral transformations in the theory of nonlinear integrable systems (see [12]).

For a linear functional \( u \), a polynomial \( \pi \), and a complex number \( a \), let \( \pi u \), \( (x - a)^{-1}u \), and \( Du \) be the linear functionals defined on \( P \) by
\[
\langle \pi u, P \rangle = \langle u, \pi P \rangle,
\]
\[
\langle (x - a)^{-1}u, P \rangle = \left\langle u, \frac{P(x) - P(a)}{x - a} \right\rangle,
\]
\[
\langle Du, P \rangle = -\langle u, P' \rangle,
\]
where \( P \in P \).

A Cauchy product of two linear functionals \( u, v \) can be defined as the linear functional \( uv \) such that \( \langle uv, x^n \rangle = \sum_{h=0}^{n} u(x) v_{n-h} \), \( n \geq 0 \). Obviously, \( uv = vu \) and \( \delta_0 u = u \delta_0 = u \).

Since \( u_0 = 1 \), there exists a unique linear functional \( v \) such that \( uv = vu = \delta_0 \). This linear functional \( v \) is said to be the inverse linear functional of \( u \) and it will be denoted by \( u^{-1} \).

Notice that \( (u^{-1})_0 = 1 \) and \( (u^{-1})_n = -\sum_{h=0}^{n-1} u_{n-h} (u^{-1})_h \), \( n \geq 1 \) (see [10]).

Since \( z^2 S_{u^{-1}}(z) S_u(z) = 1 \), we have
\[
S_{u^{(1)}}(z) = \frac{1}{S_u(z)} \left[ z - \beta_0 - z^2 S_{u^{-1}}(z) \right].
\]
Taking into account \( (u^{-1})_0 = 1 \) and \( (u^{-1})_1 = -\beta_0 \), we get
\[
u^{(1)} = -\frac{1}{\gamma_1} x^2 u^{-1}.
\]
Concerning the linear functional \( u_\alpha \), it is easy to check that \( u_\alpha = (a^{-1} + \alpha \delta_0)^{-1} \). This is an alternative proof of the result of [10] but notice that there the Stieltjes function has an opposite sign.
In the constructive theory of orthogonal polynomials the so-called direct problem is considered. A direct problem for linear functionals can be stated as follows: given two linear functionals \( u, v \) such that \( v = F(u) \), where \( F \) is a function defined in \( P' \), the dual space of \( P \), to find necessary and sufficient conditions in order to \( F \) preserves quasi-definiteness.

As a subsequent question, to find the explicit relations between the corresponding SMOP \( \{ P_n \} \) and \( \{ Q_n \} \) associated with \( u \) and \( v \), respectively.

If \( u \) is a linear functional defined by a nonnegative measure \( \mu \) on some interval \( I \) of the real line, with an infinite set of increasing points such that the moments exist, i.e., \( \langle u, x^n \rangle = \int_I x^n d\mu < \infty \) then we can introduce the linear functional \( v \) such that

\[
\langle v, x^n \rangle = \int_I x^n \frac{p(x)}{q(x)} d\mu,
\]

(1.3)

where \( p, q \) are two polynomials with pairwise distinct zeros that has constant sign on \( I \). If we assume (1.3) is finite for every \( n \), the generalized Christoffel theorem gives the SMOP with respect to \( v \) in terms of polynomials of the SMOP with respect to \( u \) (see [4,11]). In terms of linear functionals, the above transform reads \( qv = pu \). Notice that \( pu = qv \) is a more general transform because of Dirac measures and derivatives of Dirac measures at the zeros of \( q(x) \) can be considered for \( v \) in addition in such a general problem.

When \( q(x) = 1 \) and \( p(x) = x - \tilde{a} \), the transform for linear functionals is said to be a Christoffel transform (see [12]). Using the Jacobi matrix \( J \) associated with the linear functional \( u \), the shifted Darboux transform of \( J \) without free parameter yields the Jacobi matrix of \( v \) (see [6]).

It is known that \( v \) is quasi-definite if and only if \( P_n(\tilde{a}) \neq 0 \), \( n \geq 1 \), and

\[
(x - \tilde{a})Q_n(x) = P_{n+1}(x) - \frac{P_{n+1}(\tilde{a})}{P_n(\tilde{a})}P_n(x)
\]
as well as

\[
\frac{Q_n(x)P_n(\tilde{a})}{\langle u, P_n^2 \rangle} = \sum_{k=0}^{n} \frac{P_k(x)P_k(\tilde{a})}{\langle u, P_k^2 \rangle}.
\]

The polynomials \( \{ Q_n \}_{n \geq 0} \) are said to be the monic kernel polynomials of parameter \( \tilde{a} \) associated with the linear functional \( u \) (see [2]).

If \( p(x) = 1 \) and \( q(x) = \lambda(x - a) \) then the transform is said to be the Geronimus transform of the linear functional \( u \) (see [10,12]). The Jacobi matrix of \( v \) is the shifted Darboux transform with free parameter of the Jacobi matrix of \( u \) (see [6]).

Notice that in such a case, \( v = \lambda^{-1}(x - a)^{-1}u + \delta_a \) is a quasi-definite linear functional if and only if \( P_n(a, -\lambda^{-1}) \neq 0 \), \( n \geq 1 \), and then

\[
Q_n(x) = P_n(x) - \frac{P_n(a, -\lambda^{-1})}{P_{n-1}(a, -\lambda^{-1})}P_{n-1}(x)
\]

(see [9]).

In our contribution, we analyze the direct problem stated as above for the case \( p(x) = (x - \tilde{a}) \) and \( q(x) = \lambda(x - a) \). For \( a \neq \tilde{a} \) this situation has not been studied in the literature as far as we know up to in the so-called positive definite case (see [4]).
In Section 2, given a quasi-definite linear functional $u$ and complex numbers $a$, $\tilde{a}$, and $\lambda$ with $a \neq \tilde{a}$ and $\lambda \neq 0$, we characterize the quasi-definiteness of the linear functional $v = \frac{1}{\lambda}(x - a)^{-1}(x - \tilde{a})u + (1 - \frac{1}{\lambda})\delta_a$. Instead of the analysis of the quasi-definiteness of the linear functional $v$ in two steps (first, the rational perturbation and, second, the addition of the Dirac linear functional) we consider the whole transformation taking into account the first one cannot preserve the quasi-definiteness of the linear functional $u$. Indeed in [4] this constraint must be emphasized when polynomial perturbations are introduced. Further, we show that $(x - \tilde{a})Q_n(x)$ is a linear combination of three consecutive polynomials of the SMOP $\{P_n\}_{n \geq 0}$.

Notice that the confluent case $a = \tilde{a}$ yields a perturbation of $u$ via the addition of a Dirac mass at the point $x = a$. This corresponds to the Uvarov transform of the linear functional $u$ (see [12]). The direct problem has been solved in [8]. We point out that the results for $a \neq \tilde{a}$ extend in a natural way those already known for $a = \tilde{a}$.

In Section 3, under the thesis of Section 2 we characterize when the relation between $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, obtained there, can be reduced to a relation $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$ for every $n \geq 1$, and $s_1 \neq t_1$. This last type of relation, as an inverse problem, has been analyzed in [1]. The motivation for such a kind of problems is reflected in [3] when an extension of the concept of coherent pairs of measures associated with Sobolev inner products is considered.

We also observe that there is an important difference for the cases $a = \tilde{a}$ and $a \neq \tilde{a}$. Namely, if $a = \tilde{a}$ then $s_n \neq t_n$ for every $n \geq 1$ while if $a \neq \tilde{a}$ both situations, i.e., either $s_n \neq t_n$ for every $n \geq 1$ or $s_n = t_n$ for some values of $n$, can appear as we show in some examples.

2. Direct problem

In this section, we study the direct problem for $v = \frac{1}{\lambda}(x - a)^{-1}(x - \tilde{a})u + (1 - \frac{1}{\lambda})\delta_a$ where $u$ is a given quasi-definite linear functional, and $a$, $\tilde{a}$, $\lambda \in \mathbb{C}$ with $a \neq \tilde{a}$, $\lambda \neq 0$.

**Theorem 2.1.** Let $u$, $v$ be two linear functionals related by

$$(x - \tilde{a})u = \lambda(x - a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}. \quad (2.1)$$

Assume $u_0 = 1 = v_0$ and $a \neq \tilde{a}$. If $u$ is a quasi-definite linear functional with corresponding SMOP $\{P_n\}_{n \geq 0}$ then, the linear functional $v$ is quasi-definite if and only if

$$\Delta_n \equiv \left| \begin{array}{cc} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{array} \right| \neq 0, \quad n \geq 1,$$

where $R_n(x) = (\lambda - 1)P_n(x) + (a - \tilde{a})P_n^{(1)}(x)$. Furthermore, if $\{Q_n\}_{n \geq 0}$ is the SMOP associated with $v$ then

$$(x - \tilde{a})Q_n(x) = \Delta_n^{-1} \left| \begin{array}{ccc} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_n(\tilde{a}) & P_{n-1}(\tilde{a}) & P_{n-2}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{array} \right|, \quad n \geq 1. \quad (2.2)$$
Proof. Assume \( v \) is a quasi-definite linear functional and \( \{Q_n\}_{n \geq 0} \) is its corresponding SMOP.

Consider the Fourier expansion of \((x - \tilde{a})Q_n\) in terms of the polynomials \( P_n \), that is

\[
(x - \tilde{a})Q_n(x) = P_{n+1}(x) + \sum_{j=0}^{n} \alpha_{n,j} P_j(x), \quad n \geq 1,
\]

where \( \alpha_{n,j} = \langle u, P_j^2 \rangle^{-1} \langle u, (x - \tilde{a})Q_nP_j \rangle \). From formula (2.1) we get

\[
(x - \tilde{a})Q_n(x) = P_{n+1}(x) + \alpha_{n,n} P_n(x) + \alpha_{n,n-1} P_{n-1}(x)
\]

with \( \alpha_{n,n-1} = \frac{\lambda}{\langle v, P_n^2 \rangle} \neq 0 \).

For \( x = \tilde{a} \)

\[
0 = P_{n+1}(\tilde{a}) + \alpha_{n,n} P_n(\tilde{a}) + \alpha_{n,n-1} P_{n-1}(\tilde{a}). \tag{2.4}
\]

On the other hand,

\[
(a - \tilde{a})Q_n(a) = P_{n+1}(a) + \alpha_{n,n} P_n(a) + \alpha_{n,n-1} P_{n-1}(a). \tag{2.5}
\]

Subtracting (2.5) to (2.3) and dividing by \( x - a \), we can apply \( u \) in order to get

\[
\left\langle u, \frac{(x - \tilde{a})Q_n(x) - (a - \tilde{a})Q_n(a)}{x - a} \right\rangle
= P_n^{(1)}(a) + \alpha_{n,n} P_{n-1}^{(1)}(a) + \alpha_{n,n-1} P_{n-2}^{(1)}(a). \tag{2.6}
\]

The left-hand side becomes

\[
\left\langle u, \frac{(x - \tilde{a})Q_n(x) - Q_n(a)}{x - a} \right\rangle + Q_n(a) = \lambda \langle v, Q_n(x) - Q_n(a) \rangle + Q_n(a)
= (1 - \lambda) Q_n(a)
\]

and therefore

\[
(1 - \lambda) Q_n(a) = P_n^{(1)}(a) + \alpha_{n,n} P_{n-1}^{(1)}(a) + \alpha_{n,n-1} P_{n-2}^{(1)}(a). \tag{2.7}
\]

Thus, (2.5) and (2.7) yield

\[
0 = R_{n+1}(a) + \alpha_{n,n} R_n(a) + \alpha_{n,n-1} R_{n-1}(a). \tag{2.8}
\]

Since the system of Eqs. (2.4) and (2.8) in \( \alpha_{n,n} \) and \( \alpha_{n,n-1} \) has a non-zero solution, then we get \( \Delta_n \neq 0 \) for every \( n \geq 1 \).

Besides, from (2.3), (2.4), and (2.8) we obtain (2.2).

Conversely, if \( \Delta_n \neq 0 \) for every \( n \geq 1 \) we will prove that the polynomials \( Q_n \) defined by

\[
(x - \tilde{a})Q_n(x) = \Delta_n^{-1} \begin{vmatrix} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geq 1,
\]

are orthogonal with respect to \( v \). Indeed, for \( 0 \leq j \leq n - 2 \),

\[
\lambda \langle v, Q_n(x)(x - a)P_j(x) \rangle = \langle u, (x - \tilde{a})Q_n(x)P_j(x) \rangle = 0
\]
and for \( j = n - 1 \),
\[
\lambda \langle v, Q_n(x)(x - a)P_{n-1}(x) \rangle = \langle u, (x - \tilde{a})Q_n(x)P_{n-1}(x) \rangle = \Delta_n \Delta_n^{-1} \langle u, P_{n-1}^2 \rangle \neq 0.
\]

Thus, we only need to prove that \( \langle v, Q_n \rangle = 0 \) for every \( n \geq 1 \). In order to do this, observe that
\[
\lambda \langle v, Q_n \rangle = \lambda \left[ \langle v, (x - a)Q_n(x) \rangle + Q_n(a) \right]
\]
\[
= \left\{ (x - \tilde{a})u, \frac{Q_n(x) - Q_n(a)}{x - a} \right\} + \lambda Q_n(a)
\]
\[
= \left\{ u, \frac{(x - \tilde{a})Q_n(x) - (a - \tilde{a})Q_n(a)}{x - a} \right\} + (\lambda - 1)Q_n(a).
\]

Applying the expression of \( (x - \tilde{a})Q_n(x) \) in terms of the polynomials \( P_n(x) \) and (2.7) we get
\[
\left\{ u, \frac{(x - \tilde{a})Q_n(x) - (a - \tilde{a})Q_n(a)}{x - a} \right\}
\]
\[
= \Delta_n^{-1} \begin{vmatrix} P_n^{(1)}(a) & P_n^{(1)}(a) & P_n^{(1)}(a) \\ P_{n+1}(\tilde{a}) & P_{n+1}(\tilde{a}) & P_{n+1}(\tilde{a}) \\ R_n(a) & R_n(a) & R_n(a) \end{vmatrix}
\]
\[
= (1 - \lambda)Q_n(a).
\]

So \( \langle v, Q_n \rangle = 0 \) for every \( n \geq 1 \).

As a conclusion, \( \langle v, Q_n^2 \rangle = \langle v, Q_n(x - a)P_{n-1} \rangle \neq 0 \), and \( \langle v, Q_n p \rangle = 0 \) for every polynomial \( p \) of degree less than \( n \).

**Corollary 2.2.** Under the conditions of Theorem 2.1 the linear functional \( v \) is quasi-definite if and only if \( 1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})R_j(a)}{\langle v, f_j \rangle} \neq 0 \), for every \( n \geq 1 \).

Furthermore, we have
\[
(x - \tilde{a})Q_n(x) = P_{n+1}(x) + a_n(a, \tilde{a})P_n(x) + b_n(a, \tilde{a})P_{n-1}(x), \quad n \geq 1 \tag{2.9}
\]
with
\[
a_n(a, \tilde{a}) = \beta_n - \tilde{a} + (a - \tilde{a})\Delta_n^{-1}P_{n-1}(\tilde{a})R_n(a) \tag{2.10}
\]
and
\[
b_n(a, \tilde{a}) = \gamma_n + (\tilde{a} - a)\Delta_n^{-1}P_{n}(\tilde{a})R_n(a). \tag{2.11}
\]

**Proof.** From the expression of \( \Delta_n \), using the Christoffel–Darboux formula (see [2]), we have for \( n \geq 1 \)
\[
\Delta_n = (a - \tilde{a}) \left[ (1 - \lambda)K_{n-1}(a, \tilde{a}; u) + B_n(a, \tilde{a}) \right],
\]
where \( K_n(x, y; u) \) denotes the reproducing kernel of degree \( n \) associated with \( u \) and
\[
B_n(a, \tilde{a}) = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ P_n^{(1)}(a) & P_{n-1}^{(1)}(a) \end{vmatrix}.
\]
Inserting the three-term recurrence relation for both polynomials \( P_n \) and \( P_{n-1}^{(1)} \), we get

\[
\frac{B_n(a, \tilde{a})}{\langle u, P_n^2 \rangle} = (\tilde{a} - a) \frac{P_{n-1}(\tilde{a}) P_{n-1}^{(1)}(a)}{\langle u, P_{n-1}^2 \rangle} + \frac{B_{n-1}(a, \tilde{a})}{\langle u, P_{n-2}^2 \rangle}, \quad n \geq 2.
\]

Iteration yields

\[
\frac{B_n(a, \tilde{a})}{\langle u, P_n^2 \rangle} = (\tilde{a} - a) \sum_{j=0}^{n-1} \frac{P_j(\tilde{a}) P_{j-1}^{(1)}(a)}{\langle u, P_j^2 \rangle} - 1, \quad n \geq 1.
\] (2.12)

Therefore

\[
\Delta_n = (\tilde{a} - a) \sum_{j=0}^{n-1} \frac{P_j(\tilde{a}) P_{j-1}^{(1)}(a)}{\langle u, P_j^2 \rangle} = (\tilde{a} - a) \sum_{j=0}^{n-1} \frac{P_j(\tilde{a}) R_{j-1}(a)}{\langle u, P_j^2 \rangle},
\] (2.13)

and the first part of the corollary follows from Theorem 2.1.

On the other hand, we can write formula (2.2) as follows

\[
(x - \tilde{a}) Q_n(x) = P_{n+1}(x) + a_n(a, \tilde{a}) P_n(x) + b_n(a, \tilde{a}) P_{n-1}(x), \quad n \geq 1.
\]

Using the three-term recurrence relation for \( P_{n+1}(\tilde{a}) \) and \( R_{n+1}(a) \) we get

\[
a_n(a, \tilde{a}) = \beta_n - \Delta_n^{-1} \left[ \tilde{a} P_n(\tilde{a}) R_{n-1}(a) - a P_{n-1}(\tilde{a}) R_n(a) \right]
\]

\[
= \beta_n - \tilde{a} + (a - \tilde{a}) \Delta_n^{-1} P_{n-1}(\tilde{a}) R_n(a).
\]

Besides, from (2.13) we obtain

\[
\frac{\Delta_{n+1}}{\langle u, P_{n+1}^2 \rangle} = \frac{\Delta_n}{\langle u, P_n^2 \rangle} + (\tilde{a} - a) \frac{P_n(\tilde{a}) R_n(a)}{\langle u, P_n^2 \rangle}
\]

and, since \( b_n(a, \tilde{a}) = \Delta_{n+1}/\Delta_n \) and \( \gamma_n = \langle u, P_n^2 \rangle / \langle u, P_{n-1}^2 \rangle \), then

\[
b_n(a, \tilde{a}) = \gamma_n + (\tilde{a} - a) \Delta_n^{-1} P_n(\tilde{a}) R_n(a).
\]

In Theorem 2.1 and Corollary 2.2 we have assumed \( a \neq \tilde{a} \). Notice that if \( a = \tilde{a} \) the relation (2.1) between the linear functionals \( u \) and \( v \) becomes \( u = \lambda v + (1 - \lambda) \delta_n \). In this situation it is well known (see [8]) that \( v \) is quasi-definite if and only if for every \( n \geq 1 \)

\[
1 + (\lambda - 1) K_n(a, a; u) \neq 0
\]

and then

\[
(x - a) Q_n(x) = P_{n+1}(x) + a_n(a) P_n(x) + b_n(a) P_{n-1}(x), \quad n \geq 1,
\] (2.14)

holds, where

\[
a_n(a) = \beta_n - a - \frac{(\lambda - 1) P_{n-1}(a) P_n(a)}{\langle u, P_{n-1}^2 \rangle [1 + (\lambda - 1) K_{n-1}(a, a; u)]}
\]
and
\[ b_n(a) = \gamma_n \frac{1 + (\lambda - 1)K_n(a, a; u)}{1 + (\lambda - 1)K_{n-1}(a, a; u)}. \]

Notice that, these results can be recovered from Corollary 2.2, when \( \tilde{a} \) tends to \( a \).

3. Linear relations between the polynomials \( \{ P_n \} \) and \( \{ Q_n \} \)

Let \( u \) and \( v \) be quasi-definite linear functionals with corresponding SMOP \( \{ P_n \}_{n \geq 0} \) and \( \{ Q_n \}_{n \geq 0} \), respectively. In Section 2, we have obtained that if \( u \) and \( v \) satisfy the relation \((x - \tilde{a})u = \lambda(x - a)v\) with \( a, \tilde{a}, \lambda \in \mathbb{C} \) then an expression of the form
\[(x - \tilde{a})Q_n(x) = P_n(x) + \frac{1}{\lambda}R_n(a)K_{n-1}(x, a; v), \quad n \geq 1, \quad (3.1)\]
holds (see formulas (2.9) and (2.14)). That is, a linear combination of three consecutive polynomials \( P_n \) coincides with a linear combination of three consecutive polynomials \( Q_n \).

On the other hand, in [1], it was proved that if the linear functionals \( u \) and \( v \) are quasi-definite and they are related as above, then there exists a relation
\[ P_n(x) + s_nP_{n-1}(x) = Q_n(x) + t_nQ_{n-1}(x) \]
with \( s_n t_n \neq 0 \) for every \( n \geq 1 \) and \( s_1 \neq t_1 \) if and only if for every \( n \geq 1, P_n \neq Q_n \).

Thus, at the present, we have two expressions linking the polynomials \( P_n \) and \( Q_n \), the last quoted and the one given in formula (3.1).

We see below that if \( P_n \neq Q_n, n \geq 1 \), then both formulas are not independent. In fact, one of them can be reduced to the other.

**Theorem 3.1.** Let \( u, v \) be two different quasi-definite linear functionals normalized by \( u_0 = 1 = v_0 \) and related by
\[(x - \tilde{a})u = \lambda(x - a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}. \]
Let \( \{ P_n \}_{n \geq 0} \) and \( \{ Q_n \}_{n \geq 0} \) be their corresponding SMOP. The following conditions are equivalent:

(i) Formula (3.1) can be reduced to an expression
\[ P_n(x) + s_nP_{n-1}(x) = Q_n(x) + t_nQ_{n-1}(x) \]
with \( s_n t_n \neq 0 \) for every \( n \geq 1 \) and \( s_1 \neq t_1 \).

(ii) For all \( n \geq 1 \), \( R_n(a) = (\lambda - 1)P_n(a) + (a - \tilde{a})P_{n-1}(a) \neq 0 \).

**Proof.** Suppose that (i) holds. In [1, Theorem 2.4] it has been proved that whenever such a relation (3.2) is satisfied then \( P_n \neq Q_n \), for every \( n \), and besides \( P_n(x) = Q_n(x) + \lambda^{-1}R_n(a)K_{n-1}(x, a; v), n \geq 1 \) (see formula (2.24) in [1]). So, (ii) follows.

In order to derive the converse result we will first consider the case \( a \neq \tilde{a} \). Inserting the three-term recurrence relation in (3.1) successively for \( P_{n+1} \) and \( P_n \) we get, for \( n \geq 2 \),
\[(x - \tilde{a})Q_n(x) = (x - \tilde{a})P_n(x) + (\tilde{a} - a)P_{n-1}(x) + (b_n - \gamma_n)P_{n-1}(x) \]
The first part of the formula (3.3) for \( n = 1 \) reads:

\[
(x - \tilde{a}) Q_{n-1}(x) = (x - \tilde{a}) P_{n-1}(x) + (\tilde{a} - \beta_{n-1} + a_{n-1}) P_{n-1}(x)
\]

\[
+ (b_{n-1} - \gamma_{n-1}) P_{n-2}(x).
\]

Taking into account (2.10) and (2.11), the above two formulas can be written

\[
(x - \tilde{a}) Q_n(x) = (x - \tilde{a}) P_n(x) + \frac{(a - \tilde{a})}{\Delta_n} R_n(a) P_{n-1}(\tilde{a}) P_n(x)
\]

\[
+ \frac{(a - \tilde{a})}{\Delta_n} R_n(a) \gamma_n \left[ P_{n-2}(\tilde{a}) P_n(x) - P_{n-1}(\tilde{a}) P_{n-2}(x) \right].
\]

Thus, for any \( t_n \in \mathbb{R}, n \geq 2 \)

\[
(x - \tilde{a}) \left[ Q_n(x) + t_n Q_{n-1}(x) \right]
\]

\[
= (x - \tilde{a}) \left[ P_n(x) + \frac{(a - \tilde{a})}{\Delta_n} R_n(a) P_{n-1}(\tilde{a}) + t_n \right] P_{n-1}(x)
\]

\[
+ (a - \tilde{a}) \left[ \frac{R_n(a)}{\Delta_n} \gamma_n + \frac{R_{n-1}(a)}{\Delta_{n-1}} t_n \right] \left[ P_{n-2}(\tilde{a}) P_n(x) - P_{n-1}(\tilde{a}) P_{n-2}(x) \right].
\]

Now, since by hypothesis \( R_n(a) \neq 0 \) for all \( n, \) if we take

\[
t_n = - \frac{R_n(a)}{R_{n-1}(a)} \frac{\Delta_{n-1}}{\Delta_n} \gamma_n - 1, \quad n \geq 2,
\]

we get \( t_n \neq 0 \) as well as

\[
Q_n(x) + t_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x),
\]

where \( s_n = (a - \tilde{a}) \Delta_n^{-1} R_n(a) P_{n-1}(\tilde{a}) + t_n. \)

Observe that, using (2.11), we can obtain

\[
s_n = - \frac{R_n(a)}{R_{n-1}(a)} \neq 0, \quad n \geq 2.
\]

For \( n = 1, \) from the values of \( a_1 \) and \( b_1, \) the first part of formula (3.3) becomes \( Q_1(x) = P_1(x) + \frac{(a - \tilde{a})}{\Delta_1} R_1(a). \) Then \( P_1(x) + s_1 = Q_1(x) + t_1 \) holds with \( s_1 t_1 \neq 0 \) and \( s_1 - t_1 \neq 0. \)

Finally, notice that the case \( a = \tilde{a} \) can be derived in a similar way. \( \square \)

**Remarks.** (1) In Section 2, we have seen that the linear functional \( v \) is quasi-definite if and only if \( 1 + \sum_{j=0}^{n} \frac{P_j(\tilde{a}) R_j(a)}{[\mu_j P_j]} \neq 0, n \geq 1. \) It is worth noticing that the parameters \( \{R_n(a)\}_{n \geq 0}, \) which appear in the above result, also characterize the existence of formula (3.2).
(2) In terms of the linear functionals, we have that \( R_n(a) \neq 0 \) \((n \geq 1)\) if and only if the linear functional \((x - a)w\) is quasi-definite, where \(w\) is either the linear functional \(u\) (case \(a = \tilde{a}, \lambda \neq 1\)), or the linear functional \(u^{(1)}\) (case \(a \neq \tilde{a}, \lambda = 1\)) or the linear functional associated with the co-recursive polynomials (case \(a \neq \tilde{a}, \lambda \neq 1\)).

(3) If \(a \neq \tilde{a}\) and \(\lambda \neq 1\) it was proved in \([9]\) that \(R_n(a) \neq 0\) for every \(n \geq 1\) if and only if the linear functional \(\frac{1}{\sqrt{\Delta_n}}(x - a)^{-1}u + \delta_a\) is quasi-definite. When \(u\) and \(v\) are related as in Theorem 3.1, this last condition is equivalent to the quasi-definiteness of the linear functional \(\lambda v - u\). Moreover, in this case the SMOP associated with \(\lambda v - u\) is \([P_n - R_n(a)P_{n-1}]_{n \geq 0}\).

Next, we want to point out that a difference appears between the cases \(a = \tilde{a}\) and \(a \neq \tilde{a}\) with respect to the parameters \(s_n\) and \(t_n\) in formula (3.2).

In Theorem 3.1, it has been shown that there exists a relation of the form

\[
P_n(x) + s_nP_{n-1}(x) = Q_n(x) + t_nQ_{n-1}(x) \tag{3.5}
\]

with \(s_n, t_n \neq 0, n \geq 1\), and \(s_1 \neq t_1\) if and only if \(R_n(a) \neq 0, n \geq 1\). Moreover, we get for every \(n \geq 1\)

\[
t_n - s_n = \frac{P_{n-1}(\tilde{a})R_n(a)}{\langle u, P_{n-1}(a) \rangle [1 + \sum_{j=0}^{n-1} \frac{P_j(a)R_j(a)}{\langle u, P_j(a) \rangle}]}.
\]

Then, whenever \(a = \tilde{a}\) and \(\lambda \neq 1\), (3.5) holds if and only if the linear functional \((x - \tilde{a})u\) is quasi-definite. Besides \(s_n \neq t_n\), for \(n \geq 1\).

However, if \(a \neq \tilde{a}\), even if the condition \(R_n(a) \neq 0\) is satisfied for all \(n \geq 1\) then both situations either \((x - \tilde{a})u\) is quasi-definite or \((x - \tilde{a})u\) is not quasi-definite can appear. In fact, an example of the first situation was given in [1] being \(u\) and \(v\) the Jacobi linear functionals with parameters \(\alpha - 1, \beta\) and \(\alpha, \beta - 1\) (\(\alpha, \beta > 0\), respectively, and \(\alpha = -1, \beta = 1, \lambda = -\alpha \beta^{-1}\)). In this case, also \(s_n \neq t_n\) for every \(n \geq 1\).

Next, we are going to show an example of the second situation, that is, when the linear functional \((x - \tilde{a})u\) is not quasi-definite and, as a consequence, the condition \(s_n \neq t_n\) is not satisfied for every \(n \geq 1\).

Let \(u\) be the Chebyshev linear functional of second kind, that is, the Jacobi linear functional with parameters \(\alpha = \beta = 1/2\), and take \(\alpha = 1, \tilde{a} = 0\), and \(\lambda = 3\). We denote by \(\{P_n\}\) the monic polynomials associated with \(u\) whose recurrence coefficients are \(b_n = 0\) and \(\gamma_n = 1/4\) (see [2]). Observe that the linear functional \(xu\) is not quasi-definite.

With these conditions the co-recursive polynomials \(R_n\) are given by

\[
R_n(x) = 2 \left[ P_n(x) + \frac{1}{2} P_{n-1}(x) \right]. \tag{3.6}
\]

Notice that \(\frac{1}{\Delta_n}R_n(x)\) are the monic Chebyshev polynomials of fourth kind, that is, the monic Jacobi polynomials with parameters \(\alpha = 1/2\) and \(\beta = -1/2\), see [5].

First, we check that the linear functional \(v\) defined by \(xu = 3(x - 1)v\) is quasi-definite. As we have introduced in Theorem 2.1

\[
\Delta_n = \begin{vmatrix}
P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\
R_n(a) & R_{n-1}(a)
\end{vmatrix}, \quad n \geq 1,
\]
and since \( P_{2n}(0) = (-1)^n/4^n \), \( P_{2n+1}(0) = 0 \), and \( R_n(1) = (2n + 1)/2^{n-1} \) we get
\[
\Delta_{2n} = (-1)^n \frac{4n + 1}{4^{2n-1}} \quad \text{and} \quad \Delta_{2n+1} = (-1)^n + 1 \frac{4n + 3}{4^{2n}}.
\]
Therefore, \( \Delta_n \neq 0 \) for every \( n \geq 1 \), and thus \( v \) is quasi-definite. Observe that \( v = -\frac{4}{\omega} + \delta_1 \) where \( \omega \) denotes the Chebyshev linear functional of third kind.

As \( R_n(1) \neq 0 \), for \( n \geq 1 \), from Theorem 3.1 a relation of the form (3.5) holds with
\[
s_n = -\frac{R_n(1)}{R_{n-1}(1)} = -\frac{2n + 1}{2(2n - 1)}, \quad n \geq 2,
\]
and
\[
t_n = \frac{\Delta_{n-1}}{4\Delta_n} s_n, \quad n \geq 2.
\]
Therefore, taking into account \( P_1(x) = Q_1(x) + 1 \), we deduce
\[
P_{2n}(x) - \frac{4n + 1}{2(4n - 1)} P_{2n-1}(x) = Q_{2n}(x) - \frac{4n + 1}{2(4n - 1)} Q_{2n-1}(x), \quad n \geq 1,
\]
\[
P_{2n+1}(x) - \frac{4n + 3}{2(4n + 1)} P_{2n}(x) = Q_{2n+1}(x) + \frac{4n - 1}{2(4n + 1)} Q_{2n}(x), \quad n \geq 0.
\]
Notice that in this case \( s_{2n} = t_{2n}, n \geq 1 \).

Eventually, from the values of the recurrence coefficients of \( \{P_n\} \) and Theorem 2.2 in [1], we can deduce that the recurrence parameters for \( \{Q_n\} \) are \( \bar{\beta}_n = (-1)^n, n \geq 0 \), and
\[
\bar{\gamma}_{2n+1} = -\frac{4n - 1}{4(4n + 3)}, \quad n \geq 0, \quad \text{and} \quad \bar{\gamma}_{2n} = -\frac{4n + 3}{4(4n - 1)}, \quad n \geq 1.
\]

References
