Triality, composition algebras, and gradings on $D_4$

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(joint work with Mikhail Kochetov)
1. Gradings

2. Gradings on simple classical Lie algebras

3. (Cyclic) composition algebras

4. Gradings on $D_4$
1 Gradings

2 Gradings on simple classical Lie algebras

3 (Cyclic) composition algebras

4 Gradings on $D_4$
**Gradings**

$G$ abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

**$G$-grading on $\mathcal{A}$:**

\[
\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,
\]

\[
\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \quad \forall g, h \in G.
\]
Cartan grading:

\[ g = h \oplus \left( \bigoplus_{\alpha \in \Phi} g_\alpha \right) \]

(root space decomposition of a semisimple complex Lie algebra).
Examples

Cartan grading:

\[ g = h \oplus (\oplus_{\alpha \in \Phi} g_{\alpha}) \]

(root space decomposition of a semisimple complex Lie algebra).

This is a grading by \( \mathbb{Z}^n \), \( n = \text{rank} \, g \).
Examples

Pauli matrices: \( \mathcal{A} = \text{Mat}_n(\mathbb{F}) \)

\[
X = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \epsilon & 0 & \ldots & 0 \\
0 & 0 & \epsilon^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \epsilon^{n-1}
\end{pmatrix}
\]

\[
Y = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

(\( \epsilon \) a primitive \( n \)th root of 1)

\[X^n = 1 = Y^n, \quad YX = \epsilon XY\]

\[\mathcal{A} = \bigoplus_{(\bar{i}, \bar{j}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}(\bar{i}, \bar{j}), \quad \mathcal{A}(\bar{i}, \bar{j}) = \mathbb{F}X^i Y^j.\]
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\( \mathcal{A} \) becomes a graded division algebra.
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\( \mathcal{A} \) becomes a **graded division algebra**.

This grading induces a grading on \( \mathfrak{sl}_n(\mathbb{F}) \).
Fine gradings

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}_{g'}, \]  gradings on \( \mathcal{A} \).
Fine gradings

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- \( \Gamma \) is a refinement of \( \Gamma' \) if for any \( g \in G \) there is a \( g' \in G' \) such that \( \mathcal{A}_g \subseteq \mathcal{A}_{g'} \).
- Then \( \Gamma' \) is a coarsening of \( \Gamma \).
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- \( \Gamma \) is fine if it admits no proper refinement.

Remark: Any grading is a coarsening of a fine grading.
Fine gradings

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}, \quad \text{gradings on } \mathcal{A}. \]

- \( \Gamma \) is a \textit{refinement} of \( \Gamma' \) if for any \( g \in G \) there is a \( g' \in G' \) such that \( \mathcal{A}_g \subseteq \mathcal{A}_{g'} \).
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- \( \Gamma \) is \textit{fine} if it admits no proper refinement.
Fine gradings

$Γ : \mathcal{A} = \bigoplus_{g \in G} A_g, \quad Γ' : \mathcal{A} = \bigoplus_{g' \in G'} A_{g'}$, gradings on $\mathcal{A}$.

- $Γ$ is a refinement of $Γ'$ if for any $g \in G$ there is a $g' \in G'$ such that $A_g \subseteq A_{g'}$.
  Then $Γ'$ is a coarsening of $Γ$.

- $Γ$ is fine if it admits no proper refinement.

Remark

Any grading is a coarsening of a fine grading.
Gradings and affine group schemes

\[ \Gamma : A = \bigoplus_{g \in G} A_g \iff \eta : G \mathcal{D} \to \text{Aut}(A) \] (morphism of affine group schemes)

where

\[ G \mathcal{D} : \text{Alg}_F \to \text{Grp}_R \to \text{Hom}_{\text{Alg}_F}(F \otimes R, R) \]

\[ \text{Aut}(A) : \text{Alg}_F \to \text{Grp}_R \to \text{Aut}_{R\text{-alg}}(A \otimes F R) \]
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where

\[ G^D : \text{Alg}_F \to \text{Grp} \]
\[ R \mapsto G^D(R) = \text{Hom}_{\text{Alg}_F}(FG, R) \left( \simeq \text{Hom}_{\text{Grp}}(G, R^\times) \right), \]

\[ \text{Aut}(\mathcal{A}) : \text{Alg}_F \to \text{Grp} \]
\[ R \mapsto \text{Aut}_{R\text{-alg}}(\mathcal{A} \otimes_F R). \]
\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \iff \eta : G^D \to \text{Aut}(\mathcal{A}) \]

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where

\[ \eta_R(f)(x_g \otimes r) = x_g \otimes f(g)r \]

for \( f \in G^D(R) = \text{Hom}_{\text{Alg}_F}(F G, R), x_g \in \mathcal{A}_g \) and \( r \in R \).
Gradings and affine group schemes

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \iff \quad \eta : G^D \to \text{Aut}(\mathcal{A}) \]

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Conversely,

\[ \eta : G^D \to \text{Aut}(\mathcal{A}) \quad \implies \quad \eta_{FG}(\text{id}) \in \text{Aut}(\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G) \]

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \text{with} \]

\[ \mathcal{A}_g = \{ a \in A : \eta_{FG}(\text{id})(a \otimes 1) = a \otimes g \} \quad \forall g \in G. \]
Given a morphism $\text{Aut}(\mathcal{A}) \to \text{Aut}(\mathcal{B})$, any grading on $\mathcal{A}$ induces a grading on $\mathcal{B}$.
Consequences

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Example

$Ad : \text{Aut}(\mathcal{A}) \to \text{Aut}(\text{Der}(\mathcal{A}))$. 
Consequences

Given a morphism $\text{Aut}(\mathcal{A}) \rightarrow \text{Aut}(\mathcal{B})$, any grading on $\mathcal{A}$ induces a grading on $\mathcal{B}$.

Example

$Ad : \text{Aut}(\mathcal{A}) \rightarrow \text{Aut}(\text{Der}(\mathcal{A}))$.

If $\text{Aut}(\mathcal{A}) \cong \text{Aut}(\mathcal{B})$, the problems of classifying fine gradings on $\mathcal{A}$ and on $\mathcal{B}$ up to equivalence (or the problem of classifying gradings up to isomorphism) are equivalent.
1 Gradings

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4 Gradings on $D_4$
Assume the ground field is algebraically closed of characteristic not two.

$B_n, C_n (n \geq 2), D_n (n \geq 5)$:

$\text{Aut}(L) \sim = \text{Aut}(M_{\text{r}}(F), \text{involution}).$

$A_n$: $\text{Aut}(L) \sim = \text{Aut}(M_{\text{r}}(F)^{+}),$

("Affine group scheme of automorphisms and antiautomorphisms of the matrix algebra")

Gradings on matrix algebras (with involution) have been dealt with by Bahturin et al.

The fine gradings are obtained by combining Pauli gradings and coarsenings of Cartan gradings.

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What about $D_4$?
There are, up to equivalence, two fine gradings on the octonions (E. 1998):
The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in Aut(O).
A \mathbb{Z}^2\mathbb{Z}^2-grading that appears naturally while constructing O from the ground field using the Cayley-Dickson doubling process.
The induced \mathbb{Z}^2\mathbb{Z}^2-grading on the simple Lie algebra of type G satisfies that L_0 = 0 and L_\alpha is a Cartan subalgebra of L for any 0 \neq \alpha \in \mathbb{Z}^2\mathbb{Z}^2.
There are, up to equivalence, two fine gradings on the octonions (E. 1998):

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\text{Aut}(O)$.
- A $\mathbb{Z}_3^2$-grading that appears naturally while constructing $O$ from the ground field using the Cayley-Dickson doubling process.

The induced $\mathbb{Z}_3^2$-grading on the simple Lie algebra of type $G_2$ satisfies that $L_0 = 0$ and $L_\alpha$ is a Cartan subalgebra of $L$ for any $0 \neq \alpha \in \mathbb{Z}_3^2$. 

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There are, up to equivalence, four fine gradings on the Albert algebra – Draper-Martín (char $F = 0$, 2009); E.-Kochetov (2012) –:

1. The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\text{Aut}(A)$.
2. A $\mathbb{Z} \times \mathbb{Z}_3^2$-grading related to the fine $\mathbb{Z}_3^2$-grading on the octonions.
3. A $\mathbb{Z}_3^3$-grading obtained by combining a natural $\mathbb{Z}_2^2$-grading on $3 \times 3$ hermitian matrices with the fine grading over $\mathbb{Z}_3^2$ of $\mathbb{O}$.
4. A $\mathbb{Z}_3^3$-grading with $\dim A_g = 1$ for all $g$ (char $F \neq 3$).

The induced $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type $F_4$ satisfies that $L_0 = 0$ and $L_{\alpha} \oplus L_{-\alpha}$ is a Cartan subalgebra of $L$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$. 

$F_4$
\[ \text{Aut}(\mathcal{L}) \cong \text{Aut}(\mathbb{A}), \text{ where } \mathbb{A} = H_3(\mathbb{O}) \text{ is the Albert algebra} \]

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- A \(\mathbb{Z}_3^3\)-grading with \(\dim \mathbb{A}_g = 1 \forall g \) (\(\text{char } \mathbb{F} \neq 3\)).
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Composition algebras

Definition

A composition algebra is a triple $(C, \ast, n)$, where $(C, \ast)$ is a (not necessarily associative) algebra, $n: C \to F$ is a nonsingular multiplicative quadratic form.

The unital composition algebras are called Hurwitz algebras.

For Hurwitz algebras, the map $x \mapsto \overline{x} = b_n(x, 1) - x$ is an involution such that $x \overline{x} = \overline{xx} = n(x)$ for any $x$. 


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Symmetric composition algebras

Definition

A composition algebra \((C, \ast, n)\) is said to be symmetric if its norm is associative:

\[ b_n(x \ast y, z) = b_n(x, y \ast z) \]

for any \(x, y, z \in C\).

Example

For any Hurwitz algebra \((C, \ast, n)\), its para-Hurwitz counterpart is \((C, \cdot, n)\), with

\[ x \cdot y = \bar{x} \ast \bar{y} \]

for any \(x, y \in C\). These are symmetric composition algebras.
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A composition algebra \((\mathbb{C}, *, n)\) is said to be symmetric if its norm is associative:

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b_n(x * y, z) = b_n(x, y * z)
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**Example**

For any Hurwitz algebra \((\mathbb{C}, *, n)\), its para-Hurwitz counterpart is \((\mathbb{C}, \bullet, n)\), with

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x \bullet y = \bar{x} \ast \bar{y}
\]

for any \(x, y \in \mathbb{C}\). These are symmetric composition algebras.
Okubo algebras

Example

Let $\omega \in \mathbb{F}$ be a primitive cube root of unity, then $\mathfrak{sl}_3(\mathbb{F})$, with
- multiplication: $x \ast y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy)$,
- norm: $n(x) = -\frac{1}{2} \text{tr}(x^2)$,

is a symmetric composition algebra.

Its forms are called Okubo algebras.

(Okubo algebras need a different definition in characteristic three.)
Theorem

With a few exceptions in dimension 2, any symmetric composition algebra is either a para-Hurwitz algebra or an Okubo algebra.
Symmetric composition algebras

Theorem

With a few exceptions in dimension 2, any symmetric composition algebra is either a para-Hurwitz algebra or an Okubo algebra.

Two para-Hurwitz algebras are isomorphic if and only if so are their Hurwitz counterparts.

In characteristic not three, the classification of Okubo algebras, up to isomorphism, is given in terms of central simple associative algebras of degree three.

In characteristic three it follows a different path.
Let \((\mathcal{C}, \ast, n)\) be an eight-dimensional symmetric composition algebra. The linear map
\[
\mathcal{C} \longrightarrow \operatorname{End}_F(\mathcal{C} \oplus \mathcal{C})
\]
\[
\chi \mapsto \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix}
\]
induces an algebra isomorphism
\[
\alpha : \mathcal{C}_{\bar{0}}(\mathcal{C}, n) \rightarrow \operatorname{End}_F(\mathcal{C}) \times \operatorname{End}_F(\mathcal{C}).
\]
Triality

For any $u \in \text{Spin}(\mathbb{C}, n)$, if $\alpha(u) = (\rho_u^+, \rho_u^-)$, then

$$\chi_u(x \ast y) = \rho_u^-(x) \ast \rho_u^+(y)$$

for any $x, y \in \mathbb{C}$. (Here $\chi_u(x) = u \cdot x \cdot u^{-1}$ is the natural representation of $\text{Spin}(\mathbb{C}, n)$ on $\mathbb{C}$.)
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This provides a group isomorphism:

\[
\text{Spin}(\mathbb{C}, n) \rightarrow \text{Tri}(\mathbb{C}, *, n)
\]

\[ u \rightarrow (\chi_u, \rho_u^-, \rho_u^+) \]

where the triality group is defined by

\[
\text{Tri}(\mathbb{C}, *, n) := \{(f_1, f_2, f_3) \in O(\mathbb{C}, n)^3 : f_1(x \ast y) = f_2(x) \ast f_3(y) \ \forall x, y \in \mathbb{C} \}.
\]
Triality

For any $u \in \text{Spin}(\mathbb{C}, n)$, if $\alpha(u) = (\rho_u^+, \rho_u^-)$, then

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$$\text{Tri}(\mathbb{C}, *, n) := \{(f_1, f_2, f_3) \in O(\mathbb{C}, n)^3 : \}$$

$$f_1(x \ast y) = f_2(x) \ast f_3(y) \forall x, y \in \mathbb{C}\}.$$ 

(This isomorphism can be defined at the level of the corresponding affine group schemes.)
Definition
A cyclic composition is a 5-tuple $(V, L, \rho, \ast, Q)$ consisting of a cubic étale $F$-algebra $L$ with an $F$-automorphism $\rho$ of order 3, a free $L$-module $V$ of rank 8, a quadratic form $Q: V \to L$ with nondegenerate polar form $b_Q$, an $F$-bilinear multiplication $\ast: V \times V \to V$ such that, for any $x, y, z \in V$ and $\ell \in L$:

$$(\ell x) \ast y = \rho(\ell)(x \ast y),$$
$$x \ast (\ell y) = \rho^2(\ell)(x \ast y),$$
$$Q(x \ast y) = \rho(Q(x)) \rho^2(Q(y)),$$
$$b_Q(x \ast y, z) = \rho(b_Q(y \ast z, x)) = \rho^2(b_Q(z \ast x, y)).$$
Cyclic compositions (Springer)

**Definition**

A cyclic composition is a 5-tuple \((V, \mathbb{L}, \rho, \ast, Q)\) consisting of

- a cubic étale \(\mathbb{F}\)-algebra \(\mathbb{L}\) with an \(\mathbb{F}\)-automorphism \(\rho\) of order 3,
- a free \(\mathbb{L}\)-module \(V\) of rank 8,
- a quadratic form \(Q : V \to \mathbb{L}\) with nondegenerate polar form \(b_Q\),
- an \(\mathbb{F}\)-bilinear multiplication \(\ast : V \times V \to V\) such that, for any \(x, y, z \in V\) and \(\ell \in L\):

\[
(\ell x) \ast y = \rho(\ell)(x \ast y), \quad x \ast (\ell y) = \rho^2(\ell)(x \ast y),
\]

\[
Q(x \ast y) = \rho(Q(x))\rho^2(Q(y)),
\]

\[
b_Q(x \ast y, z) = \rho(Q(y \ast z, x)) = \rho^2(b_Q(z \ast x, y)).
\]
Cyclic compositions

Example

Let \((\mathcal{C}, \star, n)\) be a symmetric composition algebra (over \(\mathbb{F}\)) and let \(L = \mathbb{F} \times \mathbb{F} \times \mathbb{F}\) and \(\rho : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_2, \alpha_3, \alpha_1)\). Then \((\mathcal{C} \otimes_\mathbb{F} L, L, \rho, \star, Q)\), with \(Q = (n, n, n)\) and

\[(x_1, x_2, x_3) \star (y_1, y_2, y_3) = (x_2 \star y_3, x_3 \star y_1, x_1 \star y_2)\]

for any \(x_1, \ldots, y_3 \in \mathcal{C}\), is a cyclic composition.
Example

Let \((C, \star, n)\) be a symmetric composition algebra (over \(F\)) and let \(L = F \times F \times F\) and \(\rho : (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_2, \alpha_3, \alpha_1)\).

Then \((C \otimes_F L, L, \rho, \ast, Q)\), with \(Q = (n, n, n)\) and

\[(x_1, x_2, x_3) \ast (y_1, y_2, y_3) = (x_2 \ast y_3, x_3 \ast y_1, x_1 \ast y_2)\]

for any \(x_1, \ldots, y_3 \in C\), is a cyclic composition.

In this example, the automorphism group scheme is given by:

\[\text{Aut}_F(V, L, \rho, \ast, Q) = \text{Tri}(C, \ast, n) \rtimes A_3 \cong \text{Spin}(C, n) \rtimes A_3.\]
Let \((V, L, \rho, \ast, Q)\) be a cyclic composition. The associative algebra \(E = \text{End}_L(V)\) is endowed with the involution \(\sigma\) determined by \(Q\) and an isomorphism \(\alpha: \text{Cl}(E, \sigma) \to \rho E \times \rho E\), where the superscripts denote the twist of scalar multiplication (i.e., \(\rho E\) is \(E\) as an \(F\)-algebra with involution, but with the new \(L\)-module structure defined by \(\ell \cdot a = \rho(\ell) a\)).

(In the example above, this isomorphism is induced by the isomorphism \(\text{Cl}^\sim_0(C, n) \cong \text{End}_F(C) \times \text{End}_F(C)\).)

The quadruple \((E, L, \sigma, \alpha)\) is an example of a trialitarian algebra.
Let \((V, L, \rho, *, Q)\) be a cyclic composition.

The associative algebra \(E = \text{End}_{L}(V)\) is endowed with the involution \(\sigma\) determined by \(Q\) and an isomorphism

\[
\alpha : \mathcal{Cl}(E, \sigma) \xrightarrow{\sim} \rho E \times \rho^2 E,
\]

where the superscripts denote the twist of scalar multiplication (i.e., \(\rho E\) is \(E\) as an \(F\)-algebra with involution, but with the new \(L\)-module structure defined by \(\ell \cdot a = \rho(\ell) a\)).

(In the example above, this isomorphism is induced by the isomorphism \(\mathcal{Cl}_0(\mathbb{C}, n) \simeq \text{End}_F(\mathbb{C}) \times \text{End}_F(\mathbb{C})\).)
Let \((V, \mathbb{L}, \rho, *, Q)\) be a cyclic composition.

The associative algebra \(E = \text{End}_{\mathbb{L}}(V)\) is endowed with the involution \(\sigma\) determined by \(Q\) and an isomorphism

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where the superscripts denote the twist of scalar multiplication (i.e., \(\rho E\) is \(E\) as an \(\mathbb{F}\)-algebra with involution, but with the new \(\mathbb{L}\)-module structure defined by \(\ell \cdot a = \rho(\ell)a\)).

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The quadruple \((E, \mathbb{L}, \sigma, \alpha)\) is an example of a trialitarian algebra.
The subspace

$$\mathcal{L}(E) := \{ x \in \text{Skew}(E, \sigma) : \alpha(“x”) = (x, x) \}$$

is a central simple Lie algebra of type $D_4$. 
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**Theorem**

$$\text{Aut}(\mathcal{L}(E)) \simeq \text{Aut}(E, \mathbb{L}, \sigma, \alpha).$$
The subspace

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**Theorem**

$$\text{Aut}(\mathcal{L}(E)) \cong \text{Aut}(E, \mathbb{L}, \sigma, \alpha).$$

**Remark**

Conjugation gives a natural morphism

$$\text{Int} : \text{Aut}(V, \mathbb{L}, \rho, *, Q) \rightarrow \text{Aut}(E, \mathbb{L}, \sigma, \alpha).$$
1 Gradings

2 Gradings on simple classical Lie algebras

3 (Cyclic) composition algebras

4 Gradings on $D_4$
Type I, II, III gradings

From now on the ground field $\mathbb{F}$ will be assumed to be algebraically closed of characteristic not two.
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Let $\mathcal{L}$ be the simple Lie algebra of type $D_4$.

$$1 \rightarrow PGO^+_8 \rightarrow \text{Aut}_{\mathbb{F}}(\mathcal{L}) \xrightarrow{\pi} S_3 \rightarrow 1$$
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Let $\mathcal{L}$ be the simple Lie algebra of type $D_4$.

\[ 1 \rightarrow PGO_8^+ \rightarrow \text{Aut}_{\mathbb{F}}(\mathcal{L}) \rightarrow \pi \rightarrow S_3 \rightarrow 1 \]

If $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ is a grading and $\eta : G^D \rightarrow \text{Aut}(\mathcal{L})$ the corresponding morphism of group schemes, then the image of $\pi \eta$ is a diagonalizable subgroupscheme of the constant scheme $S_3$, so it corresponds to an abelian subgroup of the symmetric group $S_3$, and hence its order is 1, 2 or 3. The grading $\Gamma$ will be said to have Type I, II, or III respectively.
Type III gradings

- The classification of type I or II gradings follow the same lines as the classification of gradings for $D_n, n \geq 5$.
- Type III gradings do not appear in characteristic 3.
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- Type III gradings do not appear in characteristic 3.

From now on we will deal with type III gradings $\Gamma$ on $\mathcal{L}$. If $(E, \mathbb{L}, \sigma, \alpha)$ is the trialitarian algebra over $\mathbb{F}$, the isomorphism $\text{Aut}(\mathcal{L}(E)) \cong \text{Aut}(E, \mathbb{L}, \sigma, \alpha)$ allows us to transfer $\Gamma$ to a grading on $(E, \mathbb{L}, \sigma, \alpha)$.
Lifting to $\text{Aut}(V, \mathbb{L}, \rho, *, Q)$
Lifting to $\text{Aut}(V, \mathbb{L}, \rho, *, Q)$

**Theorem**

Any type III grading, identified with a morphism $\eta : G^D \to \text{Aut}(E, \mathbb{L}, \sigma, \alpha)$, can be lifted to a grading on the cyclic composition $(V, \mathbb{L}, \rho, *, Q)$:
Gradings on \((V, \mathbb{L}, \rho, *, Q)\)

Theorem

Let \(\Gamma\) be a Type III grading by an abelian group \(G\) on the cyclic composition \((V, \mathbb{L}, \rho, *, Q)\) over an algebraically closed field \(F\), where \(\text{char } F \neq 2, 3\), and let \(\Gamma_L\) be the induced grading on \(L\).

1. If \(V_e = 0\), then \((V, \mathbb{L}, \rho, *, Q)\) is isomorphic to \((O, \star, n) \otimes (L, \rho)\) as a graded cyclic composition algebra, where \((O, \star, n)\) is the Okubo algebra, endowed with a \(G\)-grading \(\Gamma_O\) with \(O_e = 0\), and the grading on \((O, \star, n) \otimes (L, \rho)\) is \(\Gamma_O \otimes \Gamma_L\).

2. Otherwise, \((V, \mathbb{L}, \rho, *, Q)\) is isomorphic to \((C, \cdot, n) \otimes (L, \rho)\) as a graded cyclic composition algebra, where \((C, \cdot, n)\) is the para-Cayley algebra, endowed with a \(G\)-grading \(\Gamma_C\), and the grading on \((C, \cdot, n) \otimes (L, \rho)\) is \(\Gamma_C \otimes \Gamma_L\).

The proof uses the fact that \(J(L, V) = L \oplus V\) is the Albert algebra, and there is a classification of the gradings on this algebra.
Gradings on \((V, L, \rho, *, Q)\)

**Theorem**

Let \(\Gamma\) be a Type III grading by an abelian group \(G\) on the cyclic composition \((V, L, \rho, *, Q)\) over an algebraically closed field \(F\), \(\text{char } F \neq 2, 3\), and let \(\Gamma_L\) be the induced grading on \(L\).

1. If \(V_e = 0\), then \((V, L, \rho, *, Q)\) is isomorphic to \((\emptyset, *, n) \otimes (L, \rho)\) as a graded cyclic composition algebra, where \((\emptyset, *, n)\) is the Okubo algebra, endowed with a \(G\)-grading \(\Gamma_{\emptyset}\) with \(\emptyset_e = 0\), and the grading on \((\emptyset, *, n) \otimes (L, \rho)\) is \(\Gamma_{\emptyset} \otimes \Gamma_L\).

2. Otherwise, \((V, L, \rho, *, Q)\) is isomorphic to \((\mathbb{C}, \bullet, n) \otimes (L, \rho)\) as a graded cyclic composition algebra, where \((\mathbb{C}, \bullet, n)\) is the para-Cayley algebra, endowed with a \(G\)-grading \(\Gamma_{\mathbb{C}}\), and the grading on \((\mathbb{C}, \bullet, n) \otimes (L, \rho)\) is \(\Gamma_{\mathbb{C}} \otimes \Gamma_L\).
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1. If \(V_e = 0\), then \((V, L, \rho, *, Q)\) is isomorphic to \((O, *, n) \otimes (L, \rho)\) as a graded cyclic composition algebra, where \((O, *, n)\) is the Okubo algebra, endowed with a \(G\)-grading \(\Gamma_O\) with \(O_e = 0\), and the grading on \((O, *, n) \otimes (L, \rho)\) is \(\Gamma_O \otimes \Gamma_L\).

2. Otherwise, \((V, L, \rho, *, Q)\) is isomorphic to \((C, \bullet, n) \otimes (L, \rho)\) as a graded cyclic composition algebra, where \((C, \bullet, n)\) is the para-Cayley algebra, endowed with a \(G\)-grading \(\Gamma_C\), and the grading on \((C, \bullet, n) \otimes (L, \rho)\) is \(\Gamma_C \otimes \Gamma_L\).

The proof uses the fact that \(\mathcal{J}(L, V) = L \oplus V\) is the Albert algebra, and there is a classification of the gradings on this algebra.
Gradings on $D_4$

Theorem
Up to equivalence, there are three fine gradings of Type III on the simple Lie algebra of type $D_4$ over an algebraically closed field $F$, $\text{char } F \neq 2, 3$. Their universal groups are $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3^2 \times \mathbb{Z}_3$ and $\mathbb{Z}_3^3$.

Theorem
Let $F$ be an algebraically closed field and let $L$ be the simple Lie algebra of type $D_4$ over $F$.

1. If $\text{char } F \neq 2, 3$ then there are, up to equivalence, 17 fine gradings on $L$.
2. If $\text{char } F = 3$ then there are, up to equivalence, 14 fine gradings on $L$. 
Gradings on $D_4$

**Theorem**

*Up to equivalence, there are three fine gradings of Type III on the simple Lie algebra of type $D_4$ over an algebraically closed field $\mathbb{F}$, char $\mathbb{F} \neq 2, 3$. Their universal groups are $\mathbb{Z}^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3^3$.***
Gradings on $D_4$

**Theorem**

Up to equivalence, there are three fine gradings of Type III on the simple Lie algebra of type $D_4$ over an algebraically closed field $\mathbb{F}$, $\text{char} \mathbb{F} \neq 2, 3$. Their universal groups are $\mathbb{Z}^2 \times \mathbb{Z}_3$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3^3$.

**Theorem**

Let $\mathbb{F}$ be an algebraically closed field and let $\mathcal{L}$ be the simple Lie algebra of type $D_4$ over $\mathbb{F}$.

1. If $\text{char} \mathbb{F} \neq 2, 3$ then there are, up to equivalence, 17 fine gradings on $\mathcal{L}$.

2. If $\text{char} \mathbb{F} = 3$ then there are, up to equivalence, 14 fine gradings on $\mathcal{L}$.
A. Elduque and M. Kochetov.

*Gradings on simple Lie algebras.*

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*Gradings on the Lie algebra $D_4$ revisited.*
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That’s all.
Thanks