An extended Freudenthal Magic Square in characteristic 3

Alberto Elduque

Universidad de Zaragoza

From Lie Algebras to Quantum Groups
Coimbra, 2006
Exceptional Lie algebras

$G_2, F_4, E_6, E_7, E_8$

$G_2 = \text{der} \mathcal{O}$  \hspace{1cm} (Cartan 1914)

$F_4 = \text{der} H_3(\mathcal{O})$  \hspace{1cm} (Chevalley-Schafer 1950)

$E_6 = \text{str}_0 H_3(\mathcal{O})$
Exceptional Lie algebras

\[ G_2, \ F_4, \ E_6, \ E_7, \ E_8 \]

\[
G_2 = \text{der} \mathcal{O} \quad \text{(Cartan 1914)}
\]

\[
F_4 = \text{der} \ H_3(\mathcal{O}) \quad \text{(Chevalley-Schafer 1950)}
\]

\[
E_6 = \text{str}_0 H_3(\mathcal{O})
\]
Tits construction (1966)

- $C$ a Hurwitz algebra (unital composition algebra),
- $J$ a central simple Jordan algebra of degree 3,

Then

$$T(C, J) = \text{der } C \oplus (C_0 \otimes J_0) \oplus \text{der } J$$

is a Lie algebra (char $\neq 3$) under a suitable Lie bracket.
Tits construction (1966)

- $C$ a Hurwitz algebra (unital composition algebra),
- $J$ a central simple Jordan algebra of degree 3,

Then

$$T(C, J) = \text{der } C \oplus (C_0 \otimes J_0) \oplus \text{der } J$$

is a Lie algebra (char $\neq 3$) under a suitable Lie bracket.
Tits construction (1966)

- $C$ a Hurwitz algebra (unital composition algebra),
- $J$ a central simple Jordan algebra of degree 3,

Then

$$\mathcal{T}(C, J) = \text{der } C \oplus (C_0 \otimes J_0) \oplus \text{der } J$$

is a Lie algebra (char $\neq 3$) under a suitable Lie bracket.
Tits construction (1966)

- $C$ a Hurwitz algebra (unital composition algebra),
- $J$ a central simple Jordan algebra of degree 3,

Then

$$T(C, J) = \text{der } C \oplus (C_0 \otimes J_0) \oplus \text{der } J$$

is a Lie algebra ($\text{char } \neq 3$) under a suitable Lie bracket.
### Freudenthal Magic Square

<table>
<thead>
<tr>
<th>$\mathcal{T}(C, J)$</th>
<th>$H_3(k)$</th>
<th>$H_3(k \times k)$</th>
<th>$H_3(\text{Mat}_2(k))$</th>
<th>$H_3(C(k))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$A_1$</td>
<td>$A_2$</td>
<td>$C_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$k \times k$</td>
<td>$A_2$</td>
<td>$A_2 \oplus A_2$</td>
<td>$A_5$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>$\text{Mat}_2(k)$</td>
<td>$C_3$</td>
<td>$A_5$</td>
<td>$D_6$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$C(k)$</td>
<td>$F_4$</td>
<td>$E_6$</td>
<td>$E_7$</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>
Tits construction rearranged

\[ J = H_3(C') \cong k^3 \oplus (\bigoplus_{i=0}^2 \nu_i(C')) , \]
\[ J_0 \cong k^2 \oplus (\bigoplus_{i=0}^2 \nu_i(C')) , \]
\[ \text{der } J \cong \text{tri}(C') \oplus (\bigoplus_{i=0}^2 \nu_i(C')) , \]

\[ \mathcal{T}(C, J) = \text{der } C \oplus (C_0 \otimes J_0) \oplus \text{der } J \]
\[ \cong \text{der } C \oplus (C_0 \otimes k^2) \oplus (\bigoplus_{i=0}^2 C_0 \otimes \nu_i(C')) \oplus (\text{tri}(C') \oplus (\bigoplus_{i=0}^2 \nu_i(C'))) \]
\[ \cong (\text{tri}(C) \oplus \text{tri}(C')) \oplus (\bigoplus_{i=0}^2 \nu_i(C \otimes C')) \]

(Barton-Sudbery, Landsberg-Manivel, Allison-Faulkner)
Tits construction rearranged

\[ J = H_3(C') \cong k^3 \oplus (\bigoplus_{i=0}^2 \nu_i(C')) , \]

\[ J_0 \cong k^2 \oplus (\bigoplus_{i=0}^2 \nu_i(C')) , \]

\[ \text{der} \, J \cong \text{tri}(C') \oplus (\bigoplus_{i=0}^2 \nu_i(C')) , \]

\[ \mathcal{T}(C, J) = \text{der} \, C \oplus (C_0 \otimes J_0) \oplus \text{der} \, J \]

\[ \cong \text{der} \, C \oplus (C_0 \otimes k^2) \oplus (\bigoplus_{i=0}^2 C_0 \otimes \nu_i(C')) \oplus (\text{tri}(C') \oplus (\bigoplus_{i=0}^2 \nu_i(C'))) \]

\[ \cong (\text{tri}(C) \oplus \text{tri}(C')) \oplus (\bigoplus_{i=0}^2 \nu_i(C \otimes C')) \]

(Barton-Sudbery, Landsberg-Manivel, Allison-Faulkner)
Tits construction rearranged

\[ J = H_3(C') \cong k^3 \oplus (\oplus_{i=0}^2 \nu_i(C')) , \]
\[ J_0 \cong k^2 \oplus (\oplus_{i=0}^2 \nu_i(C')) , \]
\[ \text{der } J \cong \text{tri}(C') \oplus (\oplus_{i=0}^2 \nu_i(C')) , \]

\[ \mathcal{T}(C,J) = \text{der } C \oplus (C_0 \otimes J_0) \oplus \text{der } J \]
\[ \cong \text{der } C \oplus (C_0 \otimes k^2) \oplus (\oplus_{i=0}^2 C_0 \otimes \nu_i(C')) \oplus (\text{tri}(C') \oplus (\oplus_{i=0}^2 \nu_i(C'))) \]
\[ \cong (\text{tri}(C) \oplus \text{tri}(C')) \oplus (\oplus_{i=0}^2 \nu_i(C \otimes C')) \]

(Barton-Sudbery, Landsberg-Manivel, Allison-Faulkner)
Symmetric composition algebras

Nicer formulas are obtained if symmetric composition algebras are used, instead of the more classical Hurwitz algebras.

\[(S, *, q)\]

\[
\begin{align*}
q(x * y) &= q(x)q(y), \\
q(x * y, z) &= q(x, y * z).
\end{align*}
\]
Nicer formulas are obtained if symmetric composition algebras are used, instead of the more classical Hurwitz algebras.

\[(S, *, q)\]

\[
\begin{cases}
q(x * y) = q(x)q(y), \\
q(x * y, z) = q(x, y * z).
\end{cases}
\]
Symmetric composition algebras: examples

- **Para-Hurwitz algebras**: $C$ Hurwitz algebra with norm $q$ and standard involution $\overline{\cdot}$, but with new multiplication

$$x \ast y = \overline{x} \overline{y}.$$ 

- **Okubo algebras**: In characteristic $\neq 3$ these are the forms of $(\mathfrak{sl}_3, \ast, q)$ with

$$x \ast y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy)1,$$

$$q(x) = \frac{1}{2} \text{tr}(x^2), \quad q(x, y) = \text{tr}(xy).$$

$$(\omega \text{ a cubic root of } 1.)$$

A different definition is needed in characteristic 3.
Symmetric composition algebras: examples

- **Para-Hurwitz algebras**: A Hurwitz algebra with norm $q$ and standard involution $\bar{-}$, but with new multiplication

  $$ x \ast y = \bar{x}\bar{y}. $$

- **Okubo algebras**: In characteristic $\not= 3$ these are the forms of $(\mathfrak{sl}_3, \ast, q)$ with

  $$ x \ast y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy), $$

  $$ q(x) = \frac{1}{2} \text{tr}(x^2), \quad q(x, y) = \text{tr}(xy). $$

  $(\omega$ a cubic root of 1.)

A different definition is needed in characteristic 3.
Theorem (Okubo, Osborn, Myung, Pérez-Izquierdo, E.)

With some exceptions in dimension 2, any symmetric composition algebras is either

- a para-Hurwitz algebra (dimension 1, 2, 4 or 8), or
- an Okubo algebra (dimension 8).
Symmetric composition algebras: classification

Theorem (Okubo, Osborn, Myung, Pérez-Izquierdo, E.)

With some exceptions in dimension 2, any symmetric composition algebras is either

- a para-Hurwitz algebra (dimension 1, 2, 4 or 8), or
- an Okubo algebra (dimension 8).
Theorem (Okubo, Osborn, Myung, Pérez-Izquierdo, E.)

With some exceptions in dimension 2, any symmetric composition algebras is either

- a para-Hurwitz algebra (dimension 1, 2, 4 or 8), or
- an Okubo algebra (dimension 8).
Triality algebra

$(S, *, q)$ a symmetric composition algebra

\[ \text{tri}(S) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S, q)^3 : \\
\quad d_0(x * y) = d_1(x) * y + x * d_2(y) \quad \forall x, y \in S \} \]

is the triality Lie algebra of $S$.

\[
\text{tri}(S) = \begin{cases} 
0 & \text{if } \dim S = 1, \\
2\text{-dim’l abelian} & \text{if } \dim S = 2, \\
\mathfrak{so}(S_0, q)^3 & \text{if } \dim S = 4, \\
\mathfrak{so}(S, q) & \text{if } \dim S = 8. 
\end{cases}
\]
(S, *, q) a symmetric composition algebra

\[ \text{tri}(S) = \{ (d_0, d_1, d_2) \in so(S, q)^3 : \]
\[ d_0(x * y) = d_1(x) * y + x * d_2(y) \quad \forall x, y \in S \} \]

is the triality Lie algebra of S.

\[ \text{tri}(S) = \begin{cases} 
0 & \text{if dim } S = 1, \\
2\text{-dim'} abelian & \text{if dim } S = 2, \\
so(S_0, q)^3 & \text{if dim } S = 4, \\
so(S, q) & \text{if dim } S = 8.
\end{cases} \]
The Lie algebra \( g(S, S') \)

Let \( S \) and \( S' \) be two symmetric composition algebras. Consider

\[
g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus (\oplus_{i=0}^{2} \nu_i(S \otimes S')) ,
\]

where \( \nu_i(S \otimes S') \) is just a copy of \( S \otimes S' \), with bracket given by:

- \( \text{tri}(S) \oplus \text{tri}(S') \) is a Lie subalgebra of \( g(S, S') \),
- \( [(d_0, d_1, d_2), \nu_i(x \otimes x')] = \nu_i(d_i(x) \otimes x') \),
- \( [(d_0', d_1', d_2'), \nu_i(x \otimes x')] = \nu_i(x \otimes d_i'(x')) \),
- \( [\nu_i(x \otimes x'), \nu_i+1(y \otimes y')] = \nu_i+2((x' * y) \otimes (x' * y')) \) (indices modulo 3),
- \( [\nu_i(x \otimes x'), \nu_i(y \otimes y')] = q'(x', y') \theta^i(t_{x,y}) + q(x, y) \theta'^{i}(t'_{x', y'}) \),

where \( t_{x,y} = (q(x, .)y - q(y, .)x, \frac{1}{2} q(x, y)1 - r_x l_y, \frac{1}{2} q(x, y)1 - l_x r_y) \).
The Lie algebra $\mathfrak{g}(S, S')$

Let $S$ and $S'$ be two symmetric composition algebras. Consider

$$\mathfrak{g}(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus \left( \bigoplus_{i=0}^{2} \nu_i(S \otimes S') \right),$$

where $\nu_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\text{tri}(S) \oplus \text{tri}(S')$ is a Lie subalgebra of $\mathfrak{g}(S, S')$,
- $[(d_0, d_1, d_2), \nu_i(x \otimes x')] = \nu_i(d_i(x) \otimes x')$,
- $[(d'_0, d'_1, d'_2), \nu_i(x \otimes x')] = \nu_i(x \otimes d'_i(x'))$,
- $[\nu_i(x \otimes x'), \nu_{i+1}(y \otimes y')] = \nu_{i+2}((x \ast y) \otimes (x' \ast y'))$ (indices modulo 3),
- $[\nu_i(x \otimes x'), \nu_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta^{i'}(t_{x',y'})$,

where $t_{x,y} = (q(x, .)y - q(y, .)x, \frac{1}{2} q(x, y)1 - r_x l_y, \frac{1}{2} q(x, y)1 - l_x r_y)$.
The Lie algebra $g(S, S')$

Let $S$ and $S'$ be two symmetric composition algebras. Consider

$$g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus (\bigoplus_{i=0}^{2} \iota_i(S \otimes S')),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\text{tri}(S) \oplus \text{tri}(S')$ is a Lie subalgebra of $g(S, S')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x')$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i(x \otimes d'_i(x'))$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}((x * y) \otimes (x' * y'))$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

where $t_{x,y} = (q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - r_x l_y, \frac{1}{2}q(x, y)1 - l_x r_y)$.
Let $S$ and $S'$ be two symmetric composition algebras. Consider

$$g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus (\bigoplus_{i=0}^{2} \nu_i(S \otimes S')),$$

where $\nu_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\text{tri}(S) \oplus \text{tri}(S')$ is a Lie subalgebra of $g(S, S')$,
- $[(d_0, d_1, d_2), \nu_i(x \otimes x')] = \nu_i(d_i(x) \otimes x'),$
- $[(d'_0, d'_1, d'_2), \nu_i(x \otimes x')] = \nu_i(x \otimes d'_i(x')),$
- $[\nu_i(x \otimes x'), \nu_{i+1}(y \otimes y')] = \nu_{i+2}((x \ast y) \otimes (x' \ast y'))$ (indices modulo 3),
- $[\nu_i(x \otimes x'), \nu_i(y \otimes y')] = q'(x', y') \theta^i(t_{x, y}) + q(x, y) \theta'\theta^i(t'_{x', y'})$, where $t_{x, y} = (q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - r_x l_y, \frac{1}{2}q(x, y)1 - l_x r_y)$. 

Alberto Elduque (Universidad de Zaragoza)
Freudenthal Magic Supersquare
June 30, 2006 10 / 29
The Lie algebra $g(S, S')$

Let $S$ and $S'$ be two symmetric composition algebras. Consider

$$g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus (\oplus_{i=0}^{2} \iota_i(S \otimes S')),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\text{tri}(S) \oplus \text{tri}(S')$ is a Lie subalgebra of $g(S, S')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x')$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i(x \otimes d'_i(x'))$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}((x * y) \otimes (x' * y'))$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y') \theta^i(t_{x,y}) + q(x, y) \theta'^i(t'_{x',y'})$,

where $t_{x,y} = (q(x, .) y - q(y, .) x, \frac{1}{2} q(x, y) 1 - r_x l_y, \frac{1}{2} q(x, y) 1 - l_x r_y)$. 
Let $S$ and $S'$ be two symmetric composition algebras. Consider

$$g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus \left( \bigoplus_{i=0}^{2} \nu_i(S \otimes S') \right),$$

where $\nu_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\text{tri}(S) \oplus \text{tri}(S')$ is a Lie subalgebra of $g(S, S')$,
- $[(d_0, d_1, d_2), \nu_i(x \otimes x')] = \nu_i(d_i(x) \otimes x')$,
- $[(d'_0, d'_1, d'_2), \nu_i(x \otimes x')] = \nu_i(x \otimes d'_i(x'))$,
- $[\nu_i(x \otimes x'), \nu_{i+1}(y \otimes y')] = \nu_{i+2}((x \ast y) \otimes (x' \ast y'))$ (indices modulo 3),
- $[\nu_i(x \otimes x'), \nu_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

where $t_{x,y} = (q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - r_x l_y, \frac{1}{2}q(x, y)1 - l_x r_y)$.
Freudenthal Magic Square again (2004)

<table>
<thead>
<tr>
<th>( g(S, S') )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( A_1 )</td>
<td>( A_2 )</td>
<td>( C_3 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( A_2 )</td>
<td>( A_2 \oplus A_2 )</td>
<td>( A_5 )</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>( C_3 )</td>
<td>( A_5 )</td>
<td>( D_6 )</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>( F_4 )</td>
<td>( E_6 )</td>
<td>( E_7 )</td>
</tr>
</tbody>
</table>

\( \text{dim } S \)

<table>
<thead>
<tr>
<th>( \text{dim } S' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>

(characteristic \( \neq 3 \))
Freudenthal Magic Square (char 3)

<table>
<thead>
<tr>
<th>$g(S, S')$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_1$</td>
<td>$\tilde{A}_2$</td>
<td>$C_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{A}_2$</td>
<td>$\tilde{A}_2 \oplus \tilde{A}_2$</td>
<td>$\tilde{A}_5$</td>
<td>$\tilde{E}_6$</td>
</tr>
<tr>
<td>4</td>
<td>$C_3$</td>
<td>$\tilde{A}_5$</td>
<td>$D_6$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>8</td>
<td>$F_4$</td>
<td>$\tilde{E}_6$</td>
<td>$E_7$</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>

- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.
Freudenthal Magic Square (char 3)

<table>
<thead>
<tr>
<th>dim $S$</th>
<th>dim $S'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(S,S')$</td>
<td>$A_1$ $\tilde{A}_2$ $C_3$ $F_4$</td>
</tr>
<tr>
<td>1</td>
<td>$\tilde{A}_2$ $\tilde{A}_2 \oplus \tilde{A}_2$ $\tilde{A}_5$ $\tilde{E}_6$</td>
</tr>
<tr>
<td>2</td>
<td>$C_3$ $\tilde{A}_5$ $D_6$ $E_7$</td>
</tr>
<tr>
<td>4</td>
<td>$F_4$ $\tilde{E}_6$ $E_7$ $E_8$</td>
</tr>
</tbody>
</table>

- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.
A superalgebra $C = C_0 \oplus C_1$, endowed with a regular quadratic superform $q = (q_0, b)$, called the norm, is said to be a composition superalgebra in case

\begin{align*}
q_0(x_0 y_0) &= q_0(x_0)q_0(y_0), \\
b(x_0 y_0, x_0 z) &= q_0(x_0)b(y, z) = b(yx_0, zx_0), \\
b(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|} b(zy, xt) &= (-1)^{|y||z|} b(x, z)b(y, t),
\end{align*}

The unital composition superalgebras are termed Hurwitz superalgebras.
Composition superalgebras: examples

\[ B(1, 2) = k1 \oplus V, \]

\[
\text{char } k = 3, \ V \text{ a two dim’l vector space with a nonzero alternating bilinear form } \langle .| . \rangle, \text{ with }
\]

\[ 1x = x1 = x, \quad uv = \langle u|v \rangle 1, \quad q_0(1) = 1, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

Fix a symplectic basis \( \{u, v\} \) of \( V \) and \( \lambda \in k \).
\( \varphi : 1 \mapsto 1, \ u \mapsto u + \lambda v, \ v \mapsto v, \) is an automorphism of \( B(1, 2), \) \( \varphi^3 = 1 \)
and

\[ S^\lambda_{1, 2} = B(1, 2) \text{ with same norm but } x \ast y = \varphi(\bar{x}) \varphi^2(\bar{y}) \]

is a symmetric composition superalgebra.
Composition superalgebras: examples

\[ B(1, 2) = k1 \oplus V, \]

\text{char } k = 3, \ V \text{ a two dim'l vector space with a nonzero alternating bilinear form } \langle .|.| \rangle, \text{ with}

\[ 1x = x1 = x, \quad uv = \langle u|v \rangle 1, \quad q_0(1) = 1, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

Fix a symplectic basis \( \{u, v\} \) of \( V \) and \( \lambda \in k \).
\( \varphi : 1 \mapsto 1, \ u \mapsto u + \lambda v, \ v \mapsto v, \) is an automorphism of \( B(1, 2), \varphi^3 = 1 \) and

\[ S^\lambda_{1,2} = B(1, 2) \] with same norm but \( x \ast y = \varphi(\bar{x})\varphi^2(\bar{y}) \)

is a symmetric composition superalgebra.
Composition superalgebras: examples

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

\( k \) and \( V \) as before, and where \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto \bar{f}, \) \( (\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle) \), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
- \( \nu \cdot f = f(\nu) = \bar{f} \cdot \nu \) for any \( f \in \text{End}_k(V) \) and \( \nu \in V \),
- \( u \cdot v = \langle .|u \rangle v \ (w \mapsto \langle w|u \rangle v) \in \text{End}_k(V) \) for any \( u, v \in V \),

and with quadratic superform

\[ q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

\( S_{4,2} \) will denote the associated para-Hurwitz superalgebra.
Composition superalgebras: examples

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

\( k \) and \( V \) as before, and where \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto \bar{f} \), \( (\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle) \), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
- \( v \cdot f = f(v) = \bar{f} \cdot v \) for any \( f \in \text{End}_k(V) \) and \( v \in V \),
- \( u \cdot v = \langle .|u \rangle v \) \( (w \mapsto \langle w|u \rangle v) \in \text{End}_k(V) \) for any \( u, v \in V \),

and with quadratic superform

\[ q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

\( S_{4,2} \) will denote the associated para-Hurwitz superalgebra.
Composition superalgebras: examples

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

\( k \) and \( V \) as before, and where \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto \bar{f}, (\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle) \), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
- \( v \cdot f = f(v) = \bar{f} \cdot v \) for any \( f \in \text{End}_k(V) \) and \( v \in V \),
- \( u \cdot v = \langle .|u \rangle v (w \mapsto \langle w|u \rangle v) \in \text{End}_k(V) \) for any \( u, v \in V \),

and with quadratic superform

\[ q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

\( S_{4,2} \) will denote the associated para-Hurwitz superalgebra.
Composition superalgebras: examples

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

\( k \) and \( V \) as before, and where \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto \bar{f}, (\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle) \), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
- \( v \cdot f = f(v) = \bar{f} \cdot v \) for any \( f \in \text{End}_k(V) \) and \( v \in V \),
- \( u \cdot v = \langle .|u \rangle v \left( w \mapsto \langle w|u \rangle v \right) \in \text{End}_k(V) \) for any \( u, v \in V \),

and with quadratic superform

\[ q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

\( S_{4,2} \) will denote the associated para-Hurwitz superalgebra.
Composition superalgebras: examples

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

\( k \) and \( V \) as before, and where \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto \bar{f}, (\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle) \), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
- \( v \cdot f = f(v) = \bar{f} \cdot v \) for any \( f \in \text{End}_k(V) \) and \( v \in V \),
- \( u \cdot v = \langle .|u \rangle v \) \((w \mapsto \langle w|u \rangle v) \in \text{End}_k(V)\) for any \( u, v \in V \),

and with quadratic superform

\[ q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

\( S_{4,2} \) will denote the associated para-Hurwitz superalgebra.
$B(4, 2) = \text{End}_k(V) \oplus V$,

$k$ and $V$ as before, and where $\text{End}_k(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, $(\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle)$, with multiplication given by:

- the usual multiplication (composition of maps) in $\text{End}_k(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \text{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle \cdot |u \rangle v \ (w \mapsto \langle w|u \rangle v) \in \text{End}_k(V)$ for any $u, v \in V$,

and with quadratic superform

$$q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle,$$

is a Hurwitz superalgebra.

$S_{4,2}$ will denote the associated para-Hurwitz superalgebra.
Composition superalgebras: examples

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

\( k \) and \( V \) as before, and where \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto \bar{f} \), \((\langle f(u)|v \rangle = \langle u|\bar{f}(v) \rangle)\), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
- \( v \cdot f = f(v) = \bar{f} \cdot v \) for any \( f \in \text{End}_k(V) \) and \( v \in V \),
- \( u \cdot v = \langle .|u \rangle v \left( w \mapsto \langle w|u \rangle v \right) \in \text{End}_k(V) \) for any \( u, v \in V \),

and with quadratic superform

\[ q_{0}(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]

is a Hurwitz superalgebra.

\( S_{4,2} \) will denote the associated para-Hurwitz superalgebra.
Theorem (E.-Okubo 02)

- Any unital composition superalgebra is either:
  - a Hurwitz algebra,
  - a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,
  - isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.

- Any symmetric composition superalgebra is either:
  - a symmetric composition algebra,
  - a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic 2,
  - isomorphic to either $S_{1,2}$ or $S_{4,2}$ in characteristic 3.
Theorem (E.-Okubo 02)

- Any unital composition superalgebra is either:
  - a Hurwitz algebra,
  - a \( \mathbb{Z}_2 \)-graded Hurwitz algebra in characteristic 2,
  - isomorphic to either \( B(1,2) \) or \( B(4,2) \) in characteristic 3.

- Any symmetric composition superalgebra is either:
  - a symmetric composition algebra,
  - a \( \mathbb{Z}_2 \)-graded symmetric composition algebra in characteristic 2,
  - isomorphic to either \( S_{1,2}^\lambda \) or \( S_{4,2} \) in characteristic 3.
Theorem (E.-Okubo 02)

- Any unital composition superalgebra is either:
  - a Hurwitz algebra,
  - a \(\mathbb{Z}_2\)-graded Hurwitz algebra in characteristic 2,
  - isomorphic to either \(B(1,2)\) or \(B(4,2)\) in characteristic 3.

- Any symmetric composition superalgebra is either:
  - a symmetric composition algebra,
  - a \(\mathbb{Z}_2\)-graded symmetric composition algebra in characteristic 2,
  - isomorphic to either \(S_{1,2}\) or \(S_{4,2}\) in characteristic 3.
Any unital composition superalgebra is either:

- a Hurwitz algebra,
- a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,
- isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.

Any symmetric composition superalgebra is either:

- a symmetric composition algebra,
- a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic 2,
- isomorphic to either $S_{1,2}$ or $S_{4,2}$ in characteristic 3.
Any unital composition superalgebra is either:
- a Hurwitz algebra,
- a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,
- isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.

Any symmetric composition superalgebra is either:
- a symmetric composition algebra,
- a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic 2,
- isomorphic to either $S_{1,2}^\lambda$ or $S_{4,2}$ in characteristic 3.
Theorem (E.-Okubo 02)

- Any unital composition superalgebra is either:
  - a Hurwitz algebra,
  - a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,
  - isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.

- Any symmetric composition superalgebra is either:
  - a symmetric composition algebra,
  - a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic 2,
  - isomorphic to either $S^{\lambda}_{1,2}$ or $S_{4,2}$ in characteristic 3.
Theorem (E.-Okubo 02)

Any unital composition superalgebra is either:

- a Hurwitz algebra,
- a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,
- isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.

Any symmetric composition superalgebra is either:

- a symmetric composition algebra,
- a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic 2,
- isomorphic to either $S^\lambda_{1,2}$ or $S_{4,2}$ in characteristic 3.
Theorem (E.-Okubo 02)

Any unital composition superalgebra is either:

- a Hurwitz algebra,
- a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,
- isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.

Any symmetric composition superalgebra is either:

- a symmetric composition algebra,
- a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic 2,
- isomorphic to either $S_{1,2}^\lambda$ or $S_{4,2}$ in characteristic 3.
<table>
<thead>
<tr>
<th>$\mathfrak{g}(S, S')$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_4$</th>
<th>$S_8$</th>
<th>$S_{1,2}$</th>
<th>$S_{4,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$A_1$</td>
<td>$\tilde{A}_2$</td>
<td>$C_3$</td>
<td>$F_4$</td>
<td>(6,8)</td>
<td>(21,14)</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$\tilde{A}_2 \oplus \tilde{A}_2$</td>
<td>$\tilde{A}_5$</td>
<td>$\tilde{E}_6$</td>
<td></td>
<td>(11,14)</td>
<td>(35,20)</td>
</tr>
<tr>
<td>$S_4$</td>
<td></td>
<td>$D_6$</td>
<td>$E_7$</td>
<td></td>
<td>(24,26)</td>
<td>(66,32)</td>
</tr>
<tr>
<td>$S_8$</td>
<td></td>
<td></td>
<td>$E_8$</td>
<td></td>
<td>(55,50)</td>
<td>(133,56)</td>
</tr>
<tr>
<td>$S_{1,2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(21,16)</td>
<td>(36,40)</td>
</tr>
<tr>
<td>$S_{4,2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(78,64)</td>
</tr>
</tbody>
</table>
### Lie superalgebras in Freudenthal Magic Supersquare

<table>
<thead>
<tr>
<th></th>
<th>$S_{1,2}$</th>
<th>$S_{4,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$\mathfrak{psl}_{2,2}$</td>
<td>$\mathfrak{sp}_6 \oplus (14)$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus (2 \otimes \mathfrak{psl}_3)$</td>
<td>$\mathfrak{pgl}_6 \oplus (20)$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus (2 \otimes (13))$</td>
<td>$\mathfrak{so}<em>{12} \oplus \mathfrak{spin}</em>{12}$</td>
</tr>
<tr>
<td>$S_8$</td>
<td>$(\mathfrak{sl}_2 \oplus f_4) \oplus (2 \otimes (25))$</td>
<td>$e_7 \oplus (56)$</td>
</tr>
<tr>
<td>$S_{1,2}$</td>
<td>$\mathfrak{so}_7 \oplus 2\mathfrak{spin}_7$</td>
<td>$\mathfrak{sp}_8 \oplus (40)$</td>
</tr>
<tr>
<td>$S_{4,2}$</td>
<td>$\mathfrak{sp}_8 \oplus (40)$</td>
<td>$\mathfrak{so}<em>{13} \oplus \mathfrak{spin}</em>{13}$</td>
</tr>
</tbody>
</table>

All these Lie superalgebras are simple, with the exception of $g(S_2, S_{1,2})$ and $g(S_2, S_{4,2})$, both of which contain a codimension one simple ideal.
All these Lie superalgebras are simple, with the exception of $g(S_2, S_{1,2})$ and $g(S_2, S_{4,2})$, both of which contain a codimension one simple ideal.
\[ g(S_{1,2}, S) \]

\[ \text{tri}(S_{1,2}) = \{(d, d, d) : d \in \mathfrak{osp}(S_{1,2})\} \simeq \mathfrak{osp}(S_{1,2}) \simeq \mathfrak{sp}(V) \oplus V. \]

\[ g(S_{1,2}, S) = (\text{tri}(S_{1,2}) \oplus \text{tri}(S)) \oplus (\bigoplus_{i=0}^{2} \iota_i(S_{1,2} \otimes S)) \]
\[ = ((\mathfrak{sp}(V) \oplus V) \oplus \text{tri}(S)) \oplus (\bigoplus_{i=0}^{2} \iota_i(1 \otimes S)) \oplus (\bigoplus_{i=0}^{2} \iota_i(V \otimes S)) \]

\[ g(S_{1,2}, S)_0 = \mathfrak{sp}(V) \oplus \text{tri}(S) \oplus (\bigoplus_{i=0}^{2} \iota_i(S)) \]
\[ \simeq \mathfrak{sp}(V) \oplus \text{der} J \quad (J = H_3(\tilde{S})) \]

\[ g(S_{1,2}, S)_{\overline{1}} = V \oplus (\bigoplus_{i=0}^{2} \iota_i(V \otimes S)) \]
\[ \simeq V \otimes (k \oplus (\bigoplus_{i=0}^{2} \iota_i(S))) \]
\[ \mathfrak{g}(S_{1,2}, S) = \{ (d, d, d) : d \in \mathfrak{osp}(S_{1,2}) \} \cong \mathfrak{osp}(S_{1,2}) \cong \mathfrak{sp}(V) \oplus V. \]

\[ \mathfrak{g}(S_{1,2}, S) = (\text{tri}(S_{1,2}) \oplus \text{tri}(S)) \oplus (\bigoplus_{i=0}^{2} \iota_i(S_{1,2} \otimes S)) \]
\[ = ((\mathfrak{sp}(V) \oplus V) \oplus \text{tri}(S)) \oplus (\bigoplus_{i=0}^{2} \iota_i(1 \otimes S)) \oplus (\bigoplus_{i=0}^{2} \iota_i(V \otimes S)) \]

\[ \mathfrak{g}(S_{1,2}, S)_0 = \mathfrak{sp}(V) \oplus \text{tri}(S) \oplus (\bigoplus_{i=0}^{2} \iota_i(S)) \]
\[ \cong \mathfrak{sp}(V) \oplus \text{der } J \quad (J = H_3(\bar{S})) \]

\[ \mathfrak{g}(S_{1,2}, S)_{1} = V \oplus (\bigoplus_{i=0}^{2} \iota_i(V \otimes S)) \]
\[ \cong V \otimes (k \oplus (\bigoplus_{i=0}^{2} \iota_i(S)))) \]
\[ \tri(S_{1,2}) = \{(d, d, d) : d \in \osp(S_{1,2})\} \simeq \osp(S_{1,2}) \simeq \sp(V) \oplus V. \]

\[ g(S_{1,2}, S) = (\tri(S_{1,2}) \oplus \tri(S)) \oplus \left( \bigoplus_{i=0}^{2} \nu_i(S_{1,2} \otimes S) \right) \]
\[ = ((\sp(V) \oplus V) \oplus \tri(S)) \oplus \left( \bigoplus_{i=0}^{2} \nu_i(1 \otimes S) \right) \oplus \left( \bigoplus_{i=0}^{2} \nu_i(V \otimes S) \right) \]

\[ g(S_{1,2}, S)_{\bar{0}} = \sp(V) \oplus \tri(S) \oplus \left( \bigoplus_{i=0}^{2} \nu_i(S) \right) \]
\[ \simeq \sp(V) \oplus \der J \quad (J = H_3(\bar{S})) \]

\[ g(S_{1,2}, S)_{\bar{1}} = V \oplus \left( \bigoplus_{i=0}^{2} \nu_i(V \otimes S) \right) \]
\[ \simeq V \otimes \left( k \oplus \left( \bigoplus_{i=0}^{2} \nu_i(S) \right) \right) \]
Thus,\[\begin{cases} g(S_{1,2}, S) \tilde{0} \cong \text{sp}(V) \oplus \text{der } J, & \text{(direct sum of ideals)} \\ g(S_{1,2}, S) \tilde{1} \cong V \otimes \hat{J}. \end{cases}\]

\(\hat{J}\) is then an orthogonal triple system with
\[
[\hat{x}\hat{y}\hat{z}] = (x \circ (y \circ z) - y \circ (x \circ z))^\wedge
\]
\((\hat{x} = x + k1)\)
\( g(S_{1,2}, S) \) (cont.)

\[
k \oplus \left( \bigoplus_{i=0}^{2} \nu_i(S) \right) \cong J_0/k1 =: \hat{J}
\]

Thus,

\[
\begin{cases}
  g(S_{1,2}, S)_{\overline{0}} \cong \text{sp}(V) \oplus \text{der} J, & \text{(direct sum of ideals)} \\
  g(S_{1,2}, S)_{\overline{1}} \cong V \otimes \hat{J}.
\end{cases}
\]

\( \hat{J} \) is then an orthogonal triple system with

\[
[\hat{x}\hat{y}\hat{z}] = (x \circ (y \circ z) - y \circ (x \circ z))^\wedge
\]

\((\hat{x} = x + k1)\)
\[ k \oplus (\bigoplus_{i=0}^{2} \nu_i(S)) \cong J_0/k1 =: \hat{J} \]

Thus,

\[
\begin{cases}
g(S_{1,2}, S)_{\bar{0}} \cong \text{sp}(V) \oplus \text{der } J, \quad \text{(direct sum of ideals)} \\
g(S_{1,2}, S)_{\bar{1}} \cong V \otimes \hat{J}.
\end{cases}
\]

\(\hat{J}\) is then an orthogonal triple system with

\[
[\hat{x}\hat{y}\hat{z}] = (x \circ (y \circ z) - y \circ (x \circ z))^{\hat{\flat}}
\]

\((\hat{x} = x + k1)\)
$S$ a para-Hurwitz algebra,

$J = H_3(\tilde{S})$ the associated central simple degree 3 Jordan algebra.

**Theorem (Cunha-E.)**

The Lie superalgebra $g(S_{1,2}, S)$ is the Lie superalgebra associated to the orthogonal triple system $\hat{J} = J_0/k1$, for $J = H_3(\tilde{S})$. 
$g(S_{1,2}, S)$ (cont.)

$S$ a para-Hurwitz algebra,

$J = H_3(\bar{S})$ the associated central simple degree 3 Jordan algebra.

**Theorem (Cunha-E.)**

The Lie superalgebra $g(S_{1,2}, S)$ is the Lie superalgebra associated to the orthogonal triple system $\hat{J} = J_0/k1$, for $J = H_3(\bar{S})$. 
\( g(S_{4,2}, S) \)

- \( \dim V = 2 \), \( \langle .|. \rangle \) a nonzero alternating bilinear form.
- The split para-quaternion algebra is
  \[ S_4 = \text{Mat}_2(k) \simeq \text{End}_k(V) \simeq V_1 \otimes V_2 \ (\langle x.|. \rangle y \leftrightarrow x \otimes y). \]
- The split para-Cayley algebra is
  \[ S_8 = \text{End}_k(V) \oplus \text{End}_k(V) \simeq (V_1 \otimes V_2) \oplus (V_3 \otimes V_4). \]
- The para-Hurwitz superalgebra attached to \( B(4, 2) \) is
  \[ S_{4,2} = \text{End}_k(V) \oplus V \simeq (V_1 \otimes V_2) \oplus V_3. \]
\[ g(S_{4,2}, S) \]

- \( \dim V = 2, \langle .| . \rangle \) a nonzero alternating bilinear form.
- The split para-quaternion algebra is
  \[ S_4 = \overline{\text{Mat}_2(k)} \cong \text{End}_k(V) \cong V_1 \otimes V_2 \ (\langle x|.\rangle y \leftrightarrow x \otimes y). \]
- The split para-Cayley algebra is
  \[ S_8 = \text{End}_k(V) \oplus \text{End}_k(V) \cong (V_1 \otimes V_2) \oplus (V_3 \otimes V_4). \]
- The para-Hurwitz superalgebra attached to \( B(4, 2) \) is
  \[ S_{4,2} = \text{End}_k(V) \oplus V \cong (V_1 \otimes V_2) \oplus V_3. \]
\( g(S_{4,2}, S) \)

- \( \dim V = 2, \langle .| . \rangle \) a nonzero alternating bilinear form.

- The split para-quaternion algebra is

\[ S_4 = \overline{\text{Mat}_2(k)} \cong \text{End}_k(V) \cong V_1 \otimes V_2 \left( \langle x.|. \rangle y \leftrightarrow x \otimes y \right) . \]

- The split para-Cayley algebra is

\[ S_8 = \text{End}_k(V) \oplus \text{End}_k(V) \cong (V_1 \otimes V_2) \oplus (V_3 \otimes V_4) . \]

- The para-Hurwitz superalgebra attached to \( B(4, 2) \) is

\[ S_{4,2} = \text{End}_k(V) \oplus V \cong (V_1 \otimes V_2) \oplus V_3 . \]
\( s(\mathbb{S}_{4,2}, S) \)

- \( \dim V = 2, \langle .|.\rangle \) a nonzero alternating bilinear form.

- The split para-quaternion algebra is
  \[
  S_4 = \overline{\text{Mat}_2(k)} \cong \text{End}_k(V) \cong V_1 \otimes V_2 (\langle x|.\rangle y \leftrightarrow x \otimes y).
  \]

- The split para-Cayley algebra is
  \[
  S_8 = \text{End}_k(V) \oplus \text{End}_k(V) \cong (V_1 \otimes V_2) \oplus (V_3 \otimes V_4).
  \]

- The para-Hurwitz superalgebra attached to \( B(4, 2) \) is
  \[
  S_{4,2} = \text{End}_k(V) \oplus V \cong (V_1 \otimes V_2) \oplus V_3.
  \]
Then

\[ \text{tri}(S_8) \cong so(8) \]
\[ \cong (\text{sp}(V_1) \oplus \text{sp}(V_2) \oplus \text{sp}(V_3) \oplus \text{sp}(V_4)) \oplus (V_1 \otimes V_2 \otimes V_3 \otimes V_4) \]

\[ \text{tri}(S_4) \cong \text{sp}(V_1) \oplus \text{sp}(V_2) \oplus \text{sp}(V_3) \]

\[ \text{tri}(S_{4,2}) \cong (\text{sp}(V_1) \oplus \text{sp}(V_2) \oplus \text{sp}(V_3)) \oplus (V_1 \otimes V_2 \otimes V_3) \]
\[ g(S_8, S) \simeq (\bigoplus_{i=1}^{4} \mathfrak{sp}(V_i)) \oplus (\bigotimes_{i=1}^{4} V_i) \oplus \text{tri}(S) \]
\[ \oplus (\bigoplus (V_1 \otimes V_2) \oplus (V_3 \otimes V_4)) \otimes \iota_0(S) \]
\[ \oplus (\bigoplus (V_2 \otimes V_3) \oplus (V_1 \otimes V_4)) \otimes \iota_1(S) \]
\[ \oplus (\bigoplus (V_1 \otimes V_3) \oplus (V_2 \otimes V_4)) \otimes \iota_2(S) \]

\[ g(S_4, S) \simeq (\bigoplus_{i=1}^{3} \mathfrak{sp}(V_i)) \oplus \text{tri}(S) \]
\[ \oplus (V_1 \otimes V_2) \otimes \iota_0(S) \oplus (V_2 \otimes V_3) \otimes \iota_1(S) \oplus (V_1 \otimes V_3) \otimes \iota_2(S) \]

\[ g(S_{4,2}, S) \simeq (\bigoplus_{i=1}^{3} \mathfrak{sp}(V_i)) \oplus (\bigotimes_{i=1}^{3} V_i) \oplus \text{tri}(S) \]
\[ \oplus (\bigoplus ((V_1 \otimes V_2) \oplus V_3) \otimes \iota_0(S) \]
\[ \oplus (\bigoplus ((V_2 \otimes V_3) \oplus V_1) \otimes \iota_1(S) \]
\[ \oplus (\bigoplus ((V_1 \otimes V_3) \oplus V_2) \otimes \iota_2(S) \]
\( \mathfrak{g}(S_{8}, S) \simeq (\mathfrak{sp}(V_{4}) \oplus \mathfrak{g}(S_{4}, S)) \)
\[
\oplus V_{4} \otimes ((V_{1} \otimes V_{2} \otimes V_{3})
\oplus (V_{3} \otimes \iota_{0}(S)) \oplus (V_{1} \otimes \iota_{1}(S)) \oplus (V_{2} \otimes \iota_{2}(S)))
\]

\( \mathfrak{g}(S_{4,2}, S) \simeq \mathfrak{g}(S_{4}, S) \)
\[
\oplus ((V_{1} \otimes V_{2} \otimes V_{3})
\oplus (V_{3} \otimes \iota_{0}(S)) \oplus (V_{1} \otimes \iota_{1}(S)) \oplus (V_{2} \otimes \iota_{2}(S)))
\]
Given any $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$ with

\[
\begin{cases}
g_0 = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals),}
g_1 = V \otimes T & \text{(as a module for } g_0),
\end{cases}
\]

then

\[
[u \otimes x, v \otimes y] = (x|y)(\langle u| \cdot \rangle v + \langle v| \cdot \rangle u) + \langle u|v \rangle d_{x,y}
\]

for some alternating bilinear form $(\cdot|\cdot) : T \times T \to k$ and skewsymmetric bilinear map $d_{\cdot, \cdot} : T \times T \to \mathfrak{s}$.

$T$ becomes a symplectic triple system under $[xyz] = d_{x,y}(z)$.

**Theorem (E. 05)**

In characteristic 3, $\mathfrak{s} \oplus T$ is a Lie superalgebra with the natural bracket.
Given any $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$ with

\[
\begin{align*}
g_0 &= \mathfrak{sp}(V) \oplus \mathfrak{s} \quad \text{(direct sum of ideals),} \\
g_1 &= V \otimes T \quad \text{(as a module for $g_0$),}
\end{align*}
\]

then

\[
[u \otimes x, v \otimes y] = (x|y)(\langle u|\cdot\rangle v + \langle v|\cdot\rangle u) + \langle u|v\rangle d_{x,y}
\]

for some alternating bilinear form $(\cdot|\cdot) : T \times T \to k$ and skewsymmetric bilinear map $d_{\cdot,\cdot} : T \times T \to \mathfrak{s}$.

$T$ becomes a symplectic triple system under $[xyz] = d_{x,y}(z)$.

**Theorem (E. 05)**

*In characteristic 3, $\mathfrak{s} \oplus T$ is a Lie superalgebra with the natural bracket.*
Given any $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$ with

\[
\begin{align*}
g_0 &= \mathfrak{sp}(V) \oplus \mathfrak{s} \quad \text{(direct sum of ideals)}, \\
g_1 &= V \otimes T \quad \text{(as a module for } g_0),
\end{align*}
\]

then

\[
[u \otimes x, v \otimes y] = (x|y)(\langle u|.|v + \langle v|.|u \rangle + \langle u|v \rangle d_{x,y}
\]

for some alternating bilinear form $(.|.): T \times T \rightarrow k$ and skewsymmetric bilinear map $d_{.,.}: T \times T \rightarrow \mathfrak{s}$.

$T$ becomes a symplectic triple system under $[xyz] = d_{x,y}(z)$.

**Theorem (E. 05)**

*In characteristic 3, $\mathfrak{s} \oplus T$ is a Lie superalgebra with the natural bracket.*
Given any $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$ with
\[
\begin{cases}
g_0 = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals)}, \\
g_1 = V \otimes T & \text{(as a module for $g_0$)},
\end{cases}
\]
then
\[
[u \otimes x, v \otimes y] = (x|y)(\langle u|\cdot\rangle v + \langle v|\cdot\rangle u) + \langle u|v\rangle d_{x,y}
\]
for some alternating bilinear form $(\cdot|\cdot) : T \times T \to k$ and skewsymmetric bilinear map $d_{\cdot,\cdot} : T \times T \to \mathfrak{s}$.

$T$ becomes a symplectic triple system under $[xyz] = d_{x,y}(z)$.

**Theorem (E. 05)**

*In characteristic 3, $\mathfrak{s} \oplus T$ is a Lie superalgebra with the natural bracket.*
Corollary (Cunha-E.)

Let $S$ be a para-Hurwitz algebra, then:

- $T = (V_1 \otimes V_2 \otimes V_3) \oplus (V_3 \otimes \iota_0(S)) \oplus (V_1 \otimes \iota_1(S)) \oplus (V_2 \otimes \iota_2(S))$ is a symplectic triple system.

- $\mathfrak{g}(S_{4,2}, S)$ is the Lie superalgebra attached to this triple system.

Remark

$T \simeq \begin{pmatrix} k & J \\ J & k \end{pmatrix}$, \quad $J = H_3(\bar{S}) \simeq k^3 \oplus (\oplus_{i=0}^2 \iota_i(S))$. 
Corollary (Cunha-E.)

Let $S$ be a para-Hurwitz algebra, then:

- $T = (V_1 \otimes V_2 \otimes V_3) \oplus (V_3 \otimes \l_0(S)) \oplus (V_1 \otimes \l_1(S)) \oplus (V_2 \otimes \l_2(S))$ is a symplectic triple system.

- $\mathfrak{g}(S_{4,2}, S)$ is the Lie superalgebra attached to this triple system.

Remark

$$T \simeq \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(\tilde{S}) \simeq k^3 \oplus (\oplus_{i=0}^2 \l_i(S)).$$
Corollary (Cunha-E.)

Let $S$ be a para-Hurwitz algebra, then:

- $T = (V_1 \otimes V_2 \otimes V_3) \oplus (V_3 \otimes \iota_0(S)) \oplus (V_1 \otimes \iota_1(S)) \oplus (V_2 \otimes \iota_2(S))$ is a symplectic triple system.

- $\mathfrak{g}(S_{4,2}, S)$ is the Lie superalgebra attached to this triple system.

Remark

$T \simeq \begin{pmatrix} k & J \\ J & k \end{pmatrix}$, \quad $J = H_3(\tilde{S}) \simeq k^3 \oplus (\oplus_{i=0}^{2} \iota_i(S))$. 
Corollary (Cunha-E.)

Let $S$ be a para-Hurwitz algebra, then:

- $T = (V_1 \otimes V_2 \otimes V_3) \oplus (V_3 \otimes \iota_0(S)) \oplus (V_1 \otimes \iota_1(S)) \oplus (V_2 \otimes \iota_2(S))$ is a symplectic triple system.
- $\mathfrak{g}(S_{4,2}, S)$ is the Lie superalgebra attached to this triple system.

Remark

$$T \simeq \begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix}, \quad J = H_3(\bar{S}) \simeq k^3 \oplus (\oplus_{i=0}^{2} \iota_i(S)).$$
Conclusion on $g(S_{1,2}, S)$ and $g(S_{4,2}, S)$

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_4$</th>
<th>$S_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{1,2}$</td>
<td>Lie superalgebras attached to orthogonal triple systems $\hat{J} = J_0/k1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_{4,2}$</td>
<td>Lie superalgebras attached to symplectic triple systems $\begin{pmatrix} k &amp; J \ J &amp; k \end{pmatrix}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

($J$ a degree 3 central simple Jordan algebra)
Some final comments

- Only $g(S_{1,2}, S_1) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac’s classification in characteristic 0. The other Lie superalgebras in Freudenthal Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

- $g(S_{1,2}, S_{1,2})$ and $g(S_{1,2}, S_{4,2})$ are related to some orthosymplectic triple systems (Cunha-E.).

- $g(S_{1,2}, S_{1,2})$ is related to a “null orthogonal triple system”.

- The simple Lie superalgebra $[g(S_{1,2}, S_2), g(S_{1,2}, S_2)]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.
Some final comments

- Only $\mathfrak{g}(S_{1,2}, S_1) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac’s classification in characteristic 0. The other Lie superalgebras in Freudenthal Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

- $\mathfrak{g}(S_{1,2}, S_{1,2})$ and $\mathfrak{g}(S_{1,2}, S_{4,2})$ are related to some orthosymplectic triple systems (Cunha-E.).

- $\mathfrak{g}(S_{1,2}, S_{1,2})$ is related to a “null orthogonal triple system”.

- The simple Lie superalgebra $[\mathfrak{g}(S_{1,2}, S_2), \mathfrak{g}(S_{1,2}, S_2)]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.
Some final comments

- Only $g(S_{1,2}, S_{1}) \cong \mathfrak{psl}_{2,2}$ has a counterpart in Kac’s classification in characteristic 0. The other Lie superalgebras in Freudenthal Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

- $g(S_{1,2}, S_{1,2})$ and $g(S_{1,2}, S_{4,2})$ are related to some orthosymplectic triple systems (Cunha-E.).

- $g(S_{1,2}, S_{1,2})$ is related to a “null orthogonal triple system”.

- The simple Lie superalgebra $[g(S_{1,2}, S_{2}), g(S_{1,2}, S_{2})]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.
Some final comments

- Only $\mathfrak{g}(S_{1,2}, S_1) \cong \mathfrak{psl}_{2,2}$ has a counterpart in Kac’s classification in characteristic 0. The other Lie superalgebras in Freudenthal Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

- $\mathfrak{g}(S_{1,2}, S_{1,2})$ and $\mathfrak{g}(S_{1,2}, S_{4,2})$ are related to some orthosymplectic triple systems (Cunha-E.).

- $\mathfrak{g}(S_{1,2}, S_{1,2})$ is related to a “null orthogonal triple system”.

- The simple Lie superalgebra $[\mathfrak{g}(S_{1,2}, S_2), \mathfrak{g}(S_{1,2}, S_2)]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.
Some final comments

- Only $\mathfrak{g}(S_{1,2}, S_1) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac’s classification in characteristic 0. The other Lie superalgebras in Freudenthal Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

- $\mathfrak{g}(S_{1,2}, S_{1,2})$ and $\mathfrak{g}(S_{1,2}, S_{4,2})$ are related to some orthosymplectic triple systems (Cunha-E.).

- $\mathfrak{g}(S_{1,2}, S_{1,2})$ is related to a “null orthogonal triple system”.

- The simple Lie superalgebra $[\mathfrak{g}(S_{1,2}, S_2), \mathfrak{g}(S_{1,2}, S_2)]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.

That’s all. Thanks