Jordan gradings on exceptional simple Lie algebras

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1. Jordan subgroups

2. Composition algebras

3. Freudenthal Magic Square

4. Exceptional Jordan gradings
1. Jordan subgroups

2. Composition algebras

3. Freudenthal Magic Square

4. Exceptional Jordan gradings
Definition (Alekseevski˘ı 1974)

Given a simple Lie algebra $g$ and a complex Lie group $G$ with $\text{Int}(g) \leq G \leq \text{Aut}(g)$, an abelian subgroup $A$ of $G$ is a Jordan subgroup if:

(i) its normalizer $N_G(A)$ is finite,
(ii) $A$ is a minimal normal subgroup of its normalizer, and
(iii) its normalizer is maximal among the normalizers of those abelian subgroups satisfying (i) and (ii).
Jordan subgroups

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The Jordan subgroups are elementary ($\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime number $p$), and they induce gradings, called Jordan gradings, in the Lie algebra $g$.

The classification of Jordan subgroups by Alekseevski splits in two types: classical and exceptional.
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Jordan subgroups: classical cases

The dimension of all nonzero homogeneous spaces is always 1 in these classical cases, which are well-known.
Jordan subgroups: classical cases

<table>
<thead>
<tr>
<th>$g$</th>
<th>$A$</th>
</tr>
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<tbody>
<tr>
<td>$A_{p^n-1}$</td>
<td>$\mathbb{Z}_p^{2n}$</td>
</tr>
<tr>
<td>$B_n \ (n \geq 3)$</td>
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</tr>
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<td>$C_{2^{n-1}} \ (n \geq 2)$</td>
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<td>$D_{n+1} \ (n \geq 3)$</td>
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**Models of these gradings?**
1. Jordan subgroups

2. Composition algebras

3. Freudenthal Magic Square

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Composition algebras

Definition

A composition algebra over a field $F$ is a triple $(C, \cdot, n)$ where $C$ is a vector space over $F$, $\cdot : C \times C \to C$ is a bilinear multiplication, and $n : C \to F$ is a multiplicative nondegenerate quadratic form: its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate, $n(x \cdot y) = n(x)n(y)$ for all $x, y \in C$.

The unital composition algebras will be called Hurwitz algebras.

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A *composition algebra* over a field $\mathbb{F}$ is a triple $(C, \cdot, n)$ where

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Hurwitz algebras

form a class of degree two algebras:

\[ x \cdot 2 - n(x,1)x + n(x)1 = 0 \]

for any \( x \).

They are endowed with an antiautomorphism, the standard conjugation:

\[ \bar{x} = n(x,1)1 - x, \]

satisfying

\[ \bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1. \]
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Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let $\lambda$ be a nonzero scalar in the ground field $F$. Consider the direct sum of two copies of $B$: 

$C = B \oplus Bu,$

with the following multiplication and nondegenerate quadratic form that extend those on $B$:

$\left( a + bu \right) \cdot \left( c + du \right) = \left( a \cdot c + \lambda \overline{d} \cdot b \right) + \left( d \cdot a + b \cdot \overline{c} \right)u,$

$n \left( a + bu \right) = n \left( a \right) - \lambda n \left( b \right).$

Then $(C, \cdot, n)$ is again a Hurwitz algebra, which is denoted by $CD(B, \lambda)$.

Notation: $CD(A, \mu, \lambda) := CD(CD(A, \mu), \lambda)$.
Cayley-Dickson doubling process

Let \((B, \cdot, n)\) be an associative Hurwitz algebra, and let \(\lambda\) be a nonzero scalar in the ground field \(\mathbb{F}\). Consider the direct sum of two copies of \(B\):

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Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field $F$ is isomorphic to one of the following:

(i) The ground field $F$ if its characteristic is $\neq 2$.

(ii) A quadratic commutative and associative separable algebra $K(\mu) = F_1 + Fv$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.

(iii) A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four-dimensional algebras are associative but not commutative.)

(iv) A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight-dimensional algebras are alternative, but not associative.)
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The Cayley-Dickson doubling process induces a \( \mathbb{Z}_2 \)-grading on the resulting algebra. Hence if the characteristic of the ground field \( F \) is \( \neq 2 \), any Cayley algebra appears as

\[
\mathbb{C} = \mathbb{C}D(F, \alpha, \beta, \gamma) = \mathbb{Q} \oplus \mathbb{Q}z = (K \oplus K_y) \oplus (K \oplus K_y)z = (F \oplus F_x) \oplus (F \oplus F_x)y \oplus ((F \oplus F_x)y \oplus (F \oplus F_x)y)z,
\]

and it is naturally graded over \( \mathbb{Z}_3^2 \), with

- \( \mathbb{C}(\bar{1}, \bar{0}, \bar{0}) = F_x \)
- \( \mathbb{C}(\bar{0}, \bar{1}, \bar{0}) = F_y \)
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$$\oplus \left( (\mathbb{F} \oplus \mathbb{F}x) \oplus (\mathbb{F} \oplus \mathbb{F}x)y \right)z,$$
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$$C = CD(\mathbb{F}, \alpha, \beta, \gamma) = Q \oplus Qz$$
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Symmetric composition algebras

Definition

A composition algebra \((S, \ast, n)\) is said to be symmetric if the polar form of its norm is associative:

\[ n((x \ast y), z) = n(x, y \ast z), \]

for any \(x, y, z \in S\).

This is equivalent to the condition:

\[ (x \ast y) \ast x = n(x) y = x \ast (y \ast x), \]

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Examples

Para-Hurwitz algebras

Given a Hurwitz algebra \((\mathbb{C}, \cdot, n)\), its para-Hurwitz counterpart is the composition algebra \((\overline{\mathbb{C}}, \cdot, n)\), where \(x \cdot y = \overline{x} \cdot \overline{y}\).

This algebra will be denoted by \(\overline{\mathbb{C}}\) for short.

Okubo algebras

Assume \(\text{char } F \neq 3\) and \(\exists \omega \neq 1, -1, \omega^3 \in F\).

Consider the algebra \(A_0\) of zero trace elements in a central simple degree 3 associative algebra with multiplication \(x^*y = \omega xy - \omega^2 yx - \omega^2 3 \text{tr}(xy)^{1/3}\), and norm \(n(x) = -\frac{1}{2} \text{tr}(x^2)\).

(There is a more general definition valid over arbitrary fields.)
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- **Okubo algebras:** Assume \(\text{char } \mathbb{F} \neq 3\) and \(\exists \omega \neq 1 = \omega^3\) in \(\mathbb{F}\). Consider the algebra \(A_0\) of zero trace elements in a central simple degree 3 associative algebra with multiplication

  \[ x \ast y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy)1, \]

  and norm \(n(x) = -\frac{1}{2} \text{tr}(x^2)\).

  (There is a more general definition valid over arbitrary fields.)
Classification

Theorem (E.-Myung 93, E. 97)

Any symmetric composition algebra is either:

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- a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),
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Gradings on para-Hurwitz algebras

Therefore, any para-Cayley algebra over a field of characteristic $\neq 2$ is endowed with a $\mathbb{Z}_3$-grading.
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Theorem

Gradings on para-Hurwitz algebras of dimension 4 or 8

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Gradings on Okubo algebras

Assuming $F$ is a field of characteristic $\neq 3$ containing a primitive third root $\omega$ of 1, then the matrix algebra $\text{Mat}_3(F)$ is generated by the order 3 matrices:

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix},$$
$$y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and the assignment $\deg(x) = (\bar{1}, \bar{0})$, $\deg(y) = (\bar{0}, \bar{1})$, gives a $\mathbb{Z}_2^3$-grading of $\text{Mat}_3(F)$, which is inherited by the Okubo algebra $(\text{sl}_3(F), \ast, n)$.

Over algebraically closed fields, any grading on an Okubo algebra is a coarsening of either the natural $\mathbb{Z}_2$-grading (Cartan grading) or this $\mathbb{Z}_2^3$-grading.
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Assume from now on that $\text{char } F \neq 2, 3$ and $\omega \in F$.

Let $(S, \ast, n)$ be any symmetric composition algebra and consider the corresponding orthogonal Lie algebra:

$$o(S, n) = \{ d \in \text{End}_F(S) : n(d(x), y) + n(x, d(y)) = 0 \forall x, y \in S \},$$

and the subalgebra of $o(S, n)$ (with componentwise multiplication):

$$\text{tri}(S, \ast, n) = \{ (d_0, d_1, d_2) \in o(S, n)^3 : d_0(x \ast y) = d_1(x) \ast y + x \ast d_2(y) \forall x, y \in S \}.$$

This is the triality Lie algebra.

The map:

$$\theta : \text{tri}(S, \ast, n) \to \text{tri}(S, \ast, n), \quad (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$$

is an automorphism of order 3.
Assume from now on that char $\mathbb{F} \neq 2, 3$ and $\omega \in \mathbb{F}$.

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$$\mathfrak{tri}(S, *, n) = \{ (d_0, d_1, d_2) \in \mathfrak{o}(S, n)^3 : d_0(x* y) = d_1(x)* y + x* d_2(y) \ \forall x, y \}$$

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Principle of Local Triality

Theorem (Principle of Local Triality)

Let \((S, \ast, n)\) be an eight dimensional symmetric composition algebra. Then the projection \(\pi_0: \text{tri}(S, \ast, n) \rightarrow o(S, n) = (d_0, d_1, d_2) \mapsto d_0\) is an isomorphism of Lie algebras.
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Gradings on $D_4$

By taking together gradings on a symmetric composition algebra and the order 3 automorphism given by triality, one obtains the following gradings on $D_4$:

Theorem

A $\mathbb{Z}_3 \times \mathbb{Z}_2$-grading of a para-Cayley algebra $(\overline{C}, \cdot, n)$ induces a $\mathbb{Z}_3 \times \mathbb{Z}_2$-grading of the orthogonal Lie algebra $o(C, n)$ of type $(14, 7)$.

The standard $\mathbb{Z}_2$-grading on an Okubo algebra $(O, *, n)$ induces a $\mathbb{Z}_3$-grading on the orthogonal Lie algebra $o(O, n)$ of type $(24, 2)$.

Remark

A $\mathbb{Z}_3 \times \mathbb{Z}_2$-grading of a para-Cayley algebra $(\overline{C}, \cdot, n)$ also induces a $\mathbb{Z}_3$-grading of its Lie algebra of derivations (which is an exceptional simple Lie algebra of type $G_2$). The type of this grading is $(0, 7)$. 
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- The standard $\mathbb{Z}_3^2$-grading on an Okubo algebra $(\mathcal{O}, \ast, n)$ induces a $\mathbb{Z}_3^3$-grading on the orthogonal Lie algebra $\mathfrak{o}(\mathcal{O}, n)$ of type (24, 2).
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Let $(S, \ast, n)$ and $(S', \star, n')$ be two symmetric composition algebras. One can construct a Lie algebra as follows:

$$g = g(S, S') = \text{tri}(S) \oplus \text{tri}(S') \oplus \bigoplus_{i=0}^{2} \iota_i(x \otimes x')$$

with bracket given by:

- the Lie bracket in $\text{tri}(S) \oplus \text{tri}(S')$, which thus becomes a Lie subalgebra of $g$,

$$\left[ (d_0, d_1, d_2), \iota_i(x \otimes x') \right] = \iota_i(d_i(x) \otimes x'),$$

$$\left[ (d_0', d_1', d_2'), \iota_i(x \otimes x') \right] = \iota_i(x \otimes d_i'(x')),$$

$$\left[ \iota_i(x \otimes x'), \iota_i+1(y \otimes y') \right] = \iota_{i+2}((x \ast y) \otimes (x' \star y')) \pmod{3},$$

$$\left[ \iota_i(x \otimes x'), \iota_i(y \otimes y') \right] = n'(x', y') \theta_i(t x, y) + n(x, y) \theta'_i(t' x', y'),$$
Freudenthal Magic Square

Let \((S, *, n)\) and \((S', *, n')\) be two symmetric composition algebras. One can construct a Lie algebra as follows:

\[
g = g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus \left( \bigoplus_{i=0}^{2} \iota_i(S \otimes S') \right),
\]

with bracket given by:

- the Lie bracket in \(\text{tri}(S) \oplus \text{tri}(S')\), which thus becomes a Lie subalgebra of \(g\),

- \([\left( d_0, d_1, d_2 \right), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x')\),

- \([\left( d'_0, d'_1, d'_2 \right), \iota_i(x \otimes x')] = \iota_i(x \otimes d'_i(x'))\),

- \([\iota_i(x \otimes x') , \iota_{i+1}(y \otimes y')] = \iota_{i+2}((x * y) \otimes (x' * y')) \) (indices modulo 3),

- \([\iota_i(x \otimes x') , \iota_i(y \otimes y')] = n'(x', y')\theta^i(t_{x,y}) + n(x, y)\theta'^i(t'_{x',y'}),\)
<table>
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<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
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<td>C3</td>
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<td>D6</td>
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<tr>
<td>8</td>
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<td>E6</td>
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</tr>
</tbody>
</table>

Alberto Elduque (Universidad de Zaragoza) Exceptional Jordan gradings July 2009
### Freudenthal Magic Square

<table>
<thead>
<tr>
<th>( g(S, S') )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( dim S )</td>
<td>( A_1 )</td>
<td>( A_2 )</td>
<td>( C_3 )</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>2</td>
<td>( A_2 \oplus A_2 )</td>
<td>( A_5 )</td>
<td>( E_6 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( C_3 )</td>
<td>( A_5 )</td>
<td>( D_6 )</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>8</td>
<td>( F_4 )</td>
<td>( E_6 )</td>
<td>( E_7 )</td>
<td>( E_8 )</td>
</tr>
</tbody>
</table>
The Lie algebra $g(S, S')$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with $g(\bar{0}, \bar{0}) = \text{tri}(S) \oplus \text{tri}(S')$, $g(\bar{1}, \bar{0}) = \iota_0(S \otimes S')$, $g(\bar{0}, \bar{1}) = \iota_1(S \otimes S')$, and $g(\bar{1}, \bar{1}) = \iota_2(S \otimes S')$.

Also, the order 3 automorphisms $\theta$ and $\theta'$ extend to an order 3 automorphism $\Theta$ of $g(S, S')$. The eigenspaces of $\Theta$ constitute a $\mathbb{Z}_3$-grading of $g(S, S')$. 
The Lie algebra \( g(S, S') \) is naturally \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded with

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g(\bar{0}, \bar{0}) &= \text{tri}(S) \oplus \text{tri}(S'), \\
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The Lie algebra $\mathfrak{g}(S, S')$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with

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Also, the order 3 automorphisms $\theta$ and $\theta'$ extend to an order 3 automorphism $\Theta$ of $\mathfrak{g}(S, S')$. The eigenspaces of $\Theta$ constitute a $\mathbb{Z}_3$-grading of $\mathfrak{g}(S, S')$. 
1. Jordan subgroups

2. Composition algebras

3. Freudenthal Magic Square

4. Exceptional Jordan gradings
Induced gradings

From now on, assume that our ground field $F$ is algebraically closed of characteristic 0.

The previous $\mathbb{Z}_2^2$ and $\mathbb{Z}_3$-gradings on the Lie algebras $g(S, S')$ can be complemented with gradings on the symmetric composition algebras $S$ and $S'$ in several ways. The $\mathbb{Z}_3^3$-grading on the Okubo algebra $O$ induces a $\mathbb{Z}_3^3$-grading on both the simple Lie algebra $g(F, O)$ of type $F_4$ and the simple Lie algebra $g(S, O)$ (for the two-dimensional para-Hurwitz algebra $S$) of type $E_6$. In both cases $g_0 = 0$ and $g_\alpha \oplus g_{-\alpha}$ is a Cartan subalgebra of $g$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.

The $\mathbb{Z}_3^2$-grading on a para-Cayley algebra $\bar{C}$ induces a $\mathbb{Z}_5^2$-grading on the simple Lie algebra $g(\bar{C}, \bar{C})$ of type $E_8$. Moreover, $g_0 = 0$ and $g_\alpha$ is a Cartan subalgebra of $g$ for any $0 \neq \alpha \in \mathbb{Z}_5^2$. 

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Induced gradings

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The previous $\mathbb{Z}_2^2$ and $\mathbb{Z}_3$-gradings on the Lie algebras $\mathfrak{g}(S, S')$ can be complemented with gradings on the symmetric composition algebras $S$ and $S'$ in several ways.
Induced gradings

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The previous $\mathbb{Z}_2^2$ and $\mathbb{Z}_3$-gradings on the Lie algebras $\mathfrak{g}(S, S')$ can be complemented with gradings on the symmetric composition algebras $S$ and $S'$ in several ways.

- The $\mathbb{Z}_2^2$-grading on the Okubo algebra $\mathcal{O}$ induces a $\mathbb{Z}_3^3$-grading on both the simple Lie algebra $\mathfrak{g}(\mathbb{F}, \mathcal{O})$ of type $F_4$ and the simple Lie algebra $\mathfrak{g}(S, \mathcal{O})$ (for the two-dimensional para-Hurwitz algebra $S$) of type $E_6$.
  In both cases $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$. 

Induced gradings

(From now on, assume that our ground field $\mathbb{F}$ is algebraically closed of characteristic 0.)

The previous $\mathbb{Z}_2^2$ and $\mathbb{Z}_3$-gradings on the Lie algebras $\mathfrak{g}(S, S')$ can be complemented with gradings on the symmetric composition algebras $S$ and $S'$ in several ways.

- The $\mathbb{Z}_3^2$-grading on the Okubo algebra $\mathcal{O}$ induces a $\mathbb{Z}_3^3$-grading on both the simple Lie algebra $\mathfrak{g}(\mathbb{F}, \mathcal{O})$ of type $F_4$ and the simple Lie algebra $\mathfrak{g}(S, \mathcal{O})$ (for the two-dimensional para-Hurwitz algebra $S$) of type $E_6$.
  In both cases $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.

- The $\mathbb{Z}_2^3$-grading on a para-Cayley algebra $\tilde{\mathcal{C}}$ induces a $\mathbb{Z}_2^5$-grading on the simple Lie algebra $\mathfrak{g}(\tilde{\mathcal{C}}, \tilde{\mathcal{C}})$ of type $E_8$.
  Moreover, $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_2^5$. 
Theorem

The previous gradings:

1. A grading on the simple Lie algebra of type $G_2$ induced by the $\mathbb{Z}_3 \oplus \mathbb{Z}_2$-grading of the Cayley algebra,

2. A grading on the simple Lie algebra of type $D_4$ induced by the $\mathbb{Z}_3 \oplus \mathbb{Z}_2$-grading of the Cayley algebra,

3. A grading on the simple Lie algebra of type $F_4$ induced by the $\mathbb{Z}_2 \oplus \mathbb{Z}_3$-grading of the Okubo algebra,

4. A grading on the simple Lie algebra of type $E_6$ induced by the $\mathbb{Z}_2 \oplus \mathbb{Z}_3$-grading of the Okubo algebra,

5. A grading on the simple Lie algebra of type $E_8$ induced by the $\mathbb{Z}_3 \oplus \mathbb{Z}_2$-grading of the Cayley algebra,

are exceptional Jordan gradings.
The previous gradings:
Theorem

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4. a \( \mathbb{Z}_3^3 \)-grading on the simple Lie algebra of type \( E_6 \) induced by the \( \mathbb{Z}_3^2 \)-grading of the Okubo algebra,
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5. a $\mathbb{Z}_2^5$-grading on the simple Lie algebra of type $E_8$ induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,
Exceptional Jordan gradings

Theorem

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3. A \( \mathbb{Z}_3^3 \)-grading on the simple Lie algebra of type \( F_4 \) induced by the \( \mathbb{Z}_3^2 \)-grading of the Okubo algebra,

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5. A \( \mathbb{Z}_2^5 \)-grading on the simple Lie algebra of type \( E_8 \) induced by the \( \mathbb{Z}_2^3 \)-grading of the Cayley algebra,

are exceptional Jordan gradings.
Only one exceptional Jordan grading does not fit in the Theorem above: the $\mathbb{Z}_5$-grading on $E_8$.

Let $V_1$ and $V_2$ be two vector spaces over $F$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space $g = \bigoplus_{i=0}^4 g_{\bar{i}}$, where $g_{\bar{0}} = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$, $g_{\bar{1}} = V_1 \otimes \bigwedge^2 V_2$, $g_{\bar{2}} = \bigwedge^2 V_1 \otimes \bigwedge^4 V_2$, $g_{\bar{3}} = \bigwedge^3 V_1 \otimes V_2$, $g_{\bar{4}} = \bigwedge^4 V_1 \otimes \bigwedge^3 V_2$.

This is a $\mathbb{Z}_5$-graded Lie algebra in a unique way: the exceptional simple Lie algebra of type $E_8$. 

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The missing exceptional Jordan grading

Only one exceptional Jordan grading does not fit in the Theorem above: the $\mathbb{Z}_5^3$-grading on $E_8$. 

Alberto Elduque (Universidad de Zaragoza)
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Let $V_1$ and $V_2$ be two vector spaces over $\mathbb{F}$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space

$$G = \bigoplus_{i=0}^{4} G_i,$$

where

$$G_0 = sl(V_1) \oplus sl(V_2),$$

$$G_1 = V_1 \otimes \wedge^2 V_2,$$

$$G_2 = \wedge^2 V_1 \otimes \wedge^4 V_2,$$

$$G_3 = \wedge^3 V_1 \otimes V_2,$$

$$G_4 = \wedge^4 V_1 \otimes \wedge^3 V_2.$$

This is a $\mathbb{Z}_5$-graded Lie algebra in a unique way: the exceptional simple Lie algebra of type $E_8$. 
The missing exceptional Jordan grading

Up to conjugation in $\text{Aut} g$, there is a unique order 5 automorphism of the simple Lie algebra $g$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.

This uniqueness shows us that, up to conjugation, this is the automorphism of $g$ such that its restriction to $\bar{g}$ is $\xi$ times the identity, where $\xi$ is a fixed primitive fifth root of unity.
Up to conjugation in $\text{Aut}\, \mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.

The uniqueness shows us that, up to conjugation, this is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar{i}}$ is $\xi_i$ times the identity, where $\xi$ is a fixed primitive fifth root of unity.
Up to conjugation in $\text{Aut} \, \mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.

![Diagram showing a grading of the Lie algebra $E_8$ with indices 1, 2, 3, 4, 5, 6, 4, 2, and 3, where 5 is the fixed element under the automorphism.]
Up to conjugation in $\text{Aut} \, \mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.

The uniqueness shows us that, up to conjugation, this is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_i$ is $\xi^i$ times the identity, where $\xi$ is a fixed primitive fifth root of unity.
The missing exceptional Jordan grading

Consider the following automorphisms $\sigma_1$, $\sigma_2$, $\sigma_3$ of $g$:

$\sigma_1(x) = \xi_ix$ for any $x \in g_0$ and $0 \leq i \leq 4$,

$\sigma_2|_{g_1} = b_1 \otimes \wedge^2 b_2$,

$\sigma_3|_{g_1} = c_1 \otimes \wedge^2 c_2$,

where on fixed bases of $V_1$ and $V_2$, the coordinate matrices of $b_1$, $c_1$, $b_2$, $c_2$ are:

$b_1 \leftrightarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi_0 & 0 & 0 \\
0 & 0 & 0 & \xi_2 & 0 \\
0 & 0 & 0 & 0 & \xi_3 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$,

$c_1 \leftrightarrow \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}$,

$b_2 \leftrightarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi_2 & 0 & 0 \\
0 & 0 & 0 & \xi_4 & 0 \\
0 & 0 & 0 & 0 & \xi_0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$,

$c_2 \leftrightarrow \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 1 & 0 & 0 \\
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\end{pmatrix}$. 
Consider the following automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $g$:

\[
\begin{align*}
\sigma_1(x) &= \xi^i x \quad \text{for any } x \in g_{\overline{i}} \text{ and } 0 \leq i \leq 4, \\
\sigma_2|_{g_{\overline{1}}} &= b_1 \otimes \wedge^2 b_2, \\
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\end{align*}
\]
Consider the following automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $\mathfrak{g}$:

\[
\sigma_1(x) = \xi^i x \quad \text{for any } x \in \mathfrak{g}_i \text{ and } 0 \leq i \leq 4,
\]

\[
\sigma_2|_{\mathfrak{g}_1} = b_1 \otimes \wedge^2 b_2,
\]

\[
\sigma_3|_{\mathfrak{g}_1} = c_1 \otimes \wedge^2 c_2,
\]

where on fixed bases of $V_1$ and $V_2$, the coordinate matrices of $b_1, c_1, b_2, c_2$ are:

\[
\begin{align*}
    b_1 & \leftrightarrow \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & \xi & 0 & 0 & 0 \\
        0 & 0 & \xi^2 & 0 & 0 \\
        0 & 0 & 0 & \xi^3 & 0 \\
        0 & 0 & 0 & 0 & \xi^4 
    \end{pmatrix}, &
    c_1 & \leftrightarrow \begin{pmatrix}
        0 & 0 & 0 & 0 & 1 \\
        1 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 \\
        0 & 0 & 1 & 0 & 0 \\
        0 & 0 & 0 & 1 & 0 
    \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
    b_2 & \leftrightarrow \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & \xi^2 & 0 & 0 & 0 \\
        0 & 0 & \xi^4 & 0 & 0 \\
        0 & 0 & 0 & \xi^3 & 0 \\
        0 & 0 & 0 & 0 & \xi^0 
    \end{pmatrix}, &
    c_2 & \leftrightarrow \begin{pmatrix}
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        1 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 \\
        0 & 0 & 1 & 0 & 0 \\
        0 & 0 & 0 & 1 & 0 
    \end{pmatrix}.
\end{align*}
\]
The missing exceptional Jordan grading

The grading of $E_8$ induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading over $\mathbb{Z}_{3^5}$.

$\forall 0 \neq \alpha \in \mathbb{Z}_{3^5}, \bigoplus_{i=1}^{4} g_{\alpha}$ is a Cartan subalgebra of $g$.

There are models of the Jordan gradings of $F_4$ and $E_6$ over $\mathbb{Z}_{3^3}$ constructed along the same lines.

That's all. Thanks.
The missing exceptional Jordan grading

Theorem

The grading of $E_8$ induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading over $\mathbb{Z}_5^3$. 

∀ $0 \neq \alpha \in \mathbb{Z}_5^3$, $\bigoplus_{i=1}^{4} g_i \alpha$ is a Cartan subalgebra of $g$. 

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\[ \forall 0 \neq \alpha \in \mathbb{Z}_5^3, \bigoplus_{i=1}^{4} g_{i\alpha} \text{ is a Cartan subalgebra of } g. \]
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There are models of the Jordan gradings of $F_4$ and $E_6$ over $\mathbb{Z}_3^3$ constructed along the same lines.

That’s all. Thanks