Gradings on the octonions, the Albert algebra, and exceptional simple Lie algebras

Alberto Elduque
Universidad de Zaragoza

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Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading ($r$ being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.
In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than $D_4$, by arbitrary abelian groups were considered by Havlíček, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including $D_4$) over algebraically closed fields of characteristic zero has been obtained quite recently.
For any abelian group $G$, the classification of all $G$-gradings, up to isomorphism, on the classical simple Lie algebras other than $D_4$ over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.
For any abelian group $G$, the classification of all $G$-gradings, up to isomorphism, on the classical simple Lie algebras other than $D_4$ over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

The gradings on the octonions and on the Albert algebra are instrumental in obtaining the gradings on the exceptional simple Lie algebras.
Outline

Composition algebras

Freudenthal’s Magic Square

Gradings

Gradings on Octonions
Outline

The Albert algebra

$G_2$ and $F_4$

Jordan gradings on exceptional simple Lie algebras
Composition algebras

Freudenthal’s Magic Square

Gradings

Gradings on Octonions
Composition algebras

**Definition**

A *composition algebra* over a field $\mathbb{F}$ is a triple $(C, \cdot, n)$ where

- $C$ is a vector space over $\mathbb{F}$,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to \mathbb{F}$ is a multiplicative nondegenerate quadratic form:
  - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate,
  - $n(x \cdot y) = n(x)n(y) \ \forall x, y \in C$. 

The unital composition algebras will be called *Hurwitz algebras*. 
**Composition algebras**

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The unital composition algebras will be called *Hurwitz algebras*. 
**Hurwitz algebras**

*Hurwitz algebras* form a class of degree two algebras:

\[ x^2 - n(x, 1)x + n(x)1 = 0 \]

for any \( x \).

They are endowed with an antiautomorphism, the *standard conjugation*:

\[ \bar{x} = n(x, 1)1 - x, \]

satisfying

\[ \bar{x} = x, \quad x + \bar{x} = n(x, 1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1. \]
Cayley-Dickson doubling process

Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let $\lambda$ be a nonzero scalar in the ground field $\mathbb{F}$. Consider the direct sum of two copies of $B$:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on $B$:

$$(a + bu) \cdot (c + du) = (a \cdot c + \lambda \bar{d} \cdot b) + (d \cdot a + b \cdot \bar{c})u,$$

$$n(a + bu) = n(a) - \lambda n(b).$$

Then $(C, \cdot, n)$ is again a Hurwitz algebra, which is denoted by $CD(B, \lambda)$. 
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Notation: \(CD(A, \mu, \lambda) := CD(CD(A, \mu), \lambda)\).
Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following:

(i) The ground field $\mathbb{F}$ if its characteristic is $\neq 2$.

(ii) A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.

(iii) A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)

(iv) A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)
A composition algebra \((S, \ast, n)\) is said to be *symmetric* if the polar form of its norm is associative:

\[
n(x \ast y, z) = n(x, y \ast z),
\]
for any \(x, y, z \in S\).

This is equivalent to the condition:

\[
(x \ast y) \ast x = n(x)y = x \ast (y \ast x),
\]
for any \(x, y \in S\).
Examples

- **Para-Hurwitz algebras:** Given a Hurwitz algebra \((C, \cdot, n)\), its para-Hurwitz counterpart is the composition algebra \((\bar{C}, \bullet, n)\), where

  \[ x \bullet y = \bar{x} \cdot \bar{y}. \]

  This algebra will be denoted by \(\bar{C}\) for short.

- **Okubo algebras:** Assume \(\text{char } F \neq 3\) and \(\exists \omega \neq 1 = \omega^3\) in \(F\). Consider the algebra \(A_0\) of zero trace elements in a central simple degree 3 associative algebra with multiplication

  \[ x^* y = \omega xy - \omega^2 yx - \omega - \omega^2 3 \text{ tr}(xy)^1, \]

  and norm \(n(x) = -\frac{1}{2} \text{ tr}(x^2)\).

  (There is a more general definition valid over arbitrary fields.)
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x \ast y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy),
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**Classification**

**Theorem (E.-Myung 93, E. 97)**

*Any symmetric composition algebra is either:*

- a para-Hurwitz algebra,
- a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),
- an Okubo algebra.
Composition algebras

Freudenthal’s Magic Square

Gradings

Gradings on Octonions
Triality Lie algebra

Assume from now on that char $\mathbb{F} \neq 2$.

Let $(S, \ast, n)$ be any symmetric composition algebra and consider the corresponding orthogonal Lie algebra:

$$o(S, n) = \{ d \in \text{End}\mathbb{F}(S) : n(d(x), y) + n(x, d(y)) = 0 \ \forall x, y \in S \},$$

and the subalgebra of $o(S, n)^3$ (with componentwise multiplication):

$$\text{tti}(S, \ast, n) =$$

$$\{(d_1, d_2, d_3) \in o(S, n)^3 : d_3(x \ast y) = d_1(x) \ast y + x \ast d_2(y) \ \forall x, y \}$$
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and the subalgebra of $o(S, n)^3$ (with componentwise multiplication):

$$\text{tri}(S, *, n) = \{(d_1, d_2, d_3) \in o(S, n)^3 : d_3(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y \}$$

This is the triality Lie algebra.
Triality Lie algebra

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\mathfrak{tri}(S, \ast, n) = \{(d_1, d_2, d_3) \in \mathfrak{o}(S, n)^3 : d_3(x \ast y) = d_1(x) \ast y + x \ast d_2(y) \ \forall x, y \}\]

This is the triality Lie algebra.

The map: \( \theta : \mathfrak{tri}(S, \ast, n) \rightarrow \mathfrak{tri}(S, \ast, n), \ (d_1, d_2, d_3) \mapsto (d_3, d_1, d_2) \) is an automorphism of order 3, (triality automorphism).
Principle of Local Triality

Theorem (Principle of Local Triality)

Let \((S, \ast, n)\) be an eight dimensional symmetric composition algebra. Then the projection

\[ \pi_1 : \text{tri}(S, \ast, n) \longrightarrow \mathfrak{o}(S, n) \]

\[(d_1, d_2, d_3) \mapsto d_1,\]

is an isomorphism of Lie algebras.
Freudenthal’s Magic Square

Let \((S, *, n)\) and \((S', *, n')\) be two symmetric composition algebras. One can construct a Lie algebra as follows:

\[
g = g(S, S') = \left( \text{tri}(S) \oplus \text{tri}(S') \right) \oplus \left( \bigoplus_{i=1}^{3} \iota_i(S \otimes S') \right),
\]
Freudenthal’s Magic Square

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\[ g = g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus \left( \bigoplus_{i=1}^{3} \iota_i(S \otimes S') \right), \]

with bracket given by:

- the Lie bracket in \(\text{tri}(S) \oplus \text{tri}(S')\), which thus becomes a Lie subalgebra of \(g\),

- \[ [(d_1, d_2, d_3), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x'), \]

- \[ [(d'_1, d'_2, d'_3), \iota_i(x \otimes x')] = \iota_i(x \otimes d'_i(x')), \]

- \[ [\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}((x \ast y) \otimes (x' \ast y')) \text{ (indices modulo 3)}, \]

- \[ [\iota_i(x \otimes x'), \iota_i(y \otimes y')] = n'(x', y') \theta^i(t_{x,y}) + n(x, y) \theta'^i(t'_{x',y'}), \]
## Freudenthal’s Magic Square

<table>
<thead>
<tr>
<th>$g(S, S')$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>$A_1$</td>
<td>$A_2$</td>
<td>$C_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>2</td>
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<td>$A_2 \oplus A_2$</td>
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<td>$E_6$</td>
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<td>$A_5$</td>
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Composition algebras

Freudenthal’s Magic Square

Gradings

Gradings on Octonions
Definition

$G$ abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

$G$-grading on $\mathcal{A}$:

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \quad \forall g, h \in G.$$
Example: Pauli matrices

\[ \mathcal{A} = \text{Mat}_n(\mathbb{F}) \]

\[ X = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \epsilon & 0 & \ldots & 0 \\ 0 & 0 & \epsilon^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \epsilon^{n-1} \end{pmatrix} \]

\[ Y = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix} \]

(\(\epsilon\) a primitive \(n\)th root of 1)

\[ X^n = 1 = Y^n, \quad YX = \epsilon XY \]

\[ \mathcal{A} = \bigoplus_{(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}(i,j), \quad \mathcal{A}(i,j) = \mathbb{F} X^i Y^j. \]
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\end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix}
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\[ \mathcal{A} = \bigoplus_{(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(i,j)}, \quad \mathcal{A}_{(i,j)} = \mathbb{F}X^iY^j. \]

\[ \mathcal{A} \text{ becomes a graded division algebra.} \]
Let $\Gamma : A = \bigoplus_{g \in G} A_g$ be a grading on $A$ (dim$_F A < \infty$, $F = \overline{F}$, char $F \neq 2$):
Basic definitions (Patera-Zassenhaus)

Let $\Gamma : A = \bigoplus_{g \in G} A_g$ be a grading on $A$ (dim$_F A < \infty$, $F = \overline{F}$, char $F \neq 2$):

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- The *support* of $\Gamma$ is $\text{Supp} \ \Gamma = \{g \in G : \mathcal{A}_g \neq 0\}$.
- The *universal grading group* of $\Gamma$ is the group $U(\Gamma)$ generated by $\text{Supp} \ \Gamma$ subject to the relations $g_1g_2 = g_3$ if $0 \neq \mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_3}$. 
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The grading $\Gamma$ is then a grading too by $U(\Gamma)$. 
Basic definitions (Patera-Zassenhaus)

There appear several groups attached to $\Gamma$:

- The automorphism group $\text{Aut}(\Gamma) = \{ \phi \in \text{Aut}(A) : \exists \alpha \in \text{Sym}(\text{Supp} \Gamma) \text{ s.t. } \phi(A_g) \subseteq A_{\alpha(g)} \ \forall \ g \}$.

- The stabilizer group $\text{Stab}(\Gamma) = \{ \phi \in \text{Aut}(\Gamma) : \phi(A_g) \subseteq A_g \ \forall \ g \}$.

- The diagonal group $\text{Diag}(\Gamma) = \{ \phi \in \text{Aut}(\Gamma) : \forall g \in \text{Supp} \Gamma \exists \lambda_g \in F \times \text{s.t. } \phi|_{A_g} = \lambda_g \text{id} \}$.

- The quotient $W(\Gamma) = \text{Aut}(\Gamma) / \text{Stab}(\Gamma)$ is the Weyl group of $\Gamma$. 
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- **The quotient** $W(\Gamma) = \text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ is the *Weyl group* of $\Gamma$. 

$W(\Gamma)$ acts by automorphisms on $U(\Gamma)$

Each $\varphi \in \text{Aut}(\Gamma)$ determines a self-bijection $\alpha$ of $\text{Supp} \Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\text{Aut}(\Gamma) \to \text{Aut}(U(\Gamma))$$

with kernel $\text{Stab}(\Gamma)$.

Thus, the Weyl group embeds naturally in $\text{Aut}(U(\Gamma))$, i.e., there is a natural action of the Weyl group on $U(\Gamma)$ by automorphisms.
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**Remark**

$\text{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$. 
Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}_{g'}$ be two gradings on $\mathcal{A}$:

- $\Gamma$ is a refinement of $\Gamma'$ if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
- Then $\Gamma'$ is a coarsening of $\Gamma$.
- For example, if $\alpha : G \to H$ is a group homomorphism, then $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$, with $\mathcal{A}_h = \bigoplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$, is a coarsening.
- If $G = U(\Gamma)$, any coarsening of $\Gamma$ is obtained in this way.
- $\Gamma$ is fine if it admits no proper refinement.
- Any grading is a coarsening of a fine grading.
Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}_{g'}$ be two gradings on $\mathcal{A}$:

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$\Gamma$ is fine if it admits no proper refinement. Any grading is a coarsening of a fine grading.
Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

- $\Gamma$ is a *refinement* of $\Gamma'$ if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}'_{g'}$. Then $\Gamma'$ is a *coarsening* of $\Gamma$.

For example, if $\alpha : G \to H$ is a group homomorphism, then $\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$, with $\mathcal{A}_h = \bigoplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$, is a coarsening.
Let $\Gamma : A = \bigoplus_{g \in G} A_g$ and $\Gamma' : A = \bigoplus_{g' \in G'} A_{g'}$ be two gradings on $A$:

- $\Gamma$ is a refinement of $\Gamma'$ if for any $g \in G$ there is a $g' \in G'$ such that $A_g \subseteq A_{g'}$.

Then $\Gamma'$ is a coarsening of $\Gamma$.

For example, if $\alpha : G \to H$ is a group homomorphism, then $A = \bigoplus_{h \in H} A_h$, with $A_h = \bigoplus_{g \in \alpha^{-1}(h)} A_g$, is a coarsening.

If $G = U(\Gamma)$, any coarsening of $\Gamma$ is obtained in this way.
Basic definitions (Patera-Zassenhaus)

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Any grading is a coarsening of a fine grading.
Basic definitions (Patera-Zassenhaus)

- Γ and Γ' are equivalent if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) such that for any \( g \in G \) there is a \( g' \in G' \) with \( \varphi(\mathcal{A}_g) = \mathcal{A}_{g'} \).

- Γ and Γ' are weakly isomorphic if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) and an isomorphism \( \alpha : G \rightarrow G' \) such that for any \( g \in G \) \( \varphi(\mathcal{A}_g) = \mathcal{A}_{\alpha(g)} \).

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Basic definitions (Patera-Zassenhaus)

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Composition algebras

Freudenthal's Magic Square

Gradings

Gradings on Octonions
The split octonions

Cayley-Dickson process:

\[ K = \mathbb{F} \oplus \mathbb{F} i, \quad i^2 = -1, \]
\[ H = K \oplus K j, \quad j^2 = -1, \]
\[ \mathbb{O} = H \oplus H l, \quad l^2 = -1, \]

\( \mathbb{O} \) is \( \mathbb{Z}_2^3 \)-graded with

\[ \deg(i) = (\bar{1}, 0, 0), \quad \deg(j) = (0, \bar{1}, 0), \quad \deg(l) = (0, 0, \bar{1}). \]
Cartan grading on the Octonions

\( \mathcal{O} \) contains canonical bases:

\[ \mathcal{B} = \{ e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3 \} \]

with

\[ n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{otherwise} \ 0. \]

\[ e_1^2 = e_1, \quad e_2^2 = e_2, \]

\[ e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3) \]

\[ u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3) \]

\[ u_i u_{i+1} = -u_{i+1} u_i = v_{i+2}, \quad v_i v_{i+1} = -v_{i+1} v_i = u_{i+2}, \quad (\text{indices modulo 3}) \]

otherwise 0.
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\( \mathbb{O} \) contains canonical bases:

\[ \mathcal{B} = \{ e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3 \} \]

with

\[ n(e_1, e_2) = n(u_i, \nu_i) = 1, \quad \text{otherwise 0}. \]

\[ e_1^2 = e_1, \quad e_2^2 = e_2, \]

\[ e_1 u_i = u_i e_2 = u_i, \quad e_2 \nu_i = \nu_i e_1 = \nu_i, \quad (i = 1, 2, 3) \]

\[ u_i \nu_i = -e_1, \quad \nu_i u_i = -e_2, \quad (i = 1, 2, 3) \]

\[ u_i u_{i+1} = -u_{i+1} u_i = \nu_{i+2}, \quad \nu_i \nu_{i+1} = -\nu_{i+1} \nu_i = u_{i+2}, \quad \text{(indices modulo 3)} \]

otherwise 0.

The Cartan grading is the \( \mathbb{Z}^2 \)-grading determined by:

\[ \deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1). \]
Up to equivalence, the fine gradings on $\mathbb{O}$ are

- the Cartan grading, and

- the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.
Fine gradings on the Octonions

Sketch of proof:
Fine gradings on the Octonions

Sketch of proof:

- The Cayley-Hamilton equation: $x^2 - n(x, 1)x + n(x)1 = 0$, implies that the norm has a well behavior relative to the grading:

  $$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$
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- If there is a \( g \in \text{Supp } \Gamma \) with either order \( > 2 \) or \( \dim \mathbb{O}_g \geq 2 \), there are elements \( x \in \mathbb{O}_g, \ y \in \mathbb{O}_{g-1} \) with \( n(x) = 0 = n(y), \ n(x, y) = 1 \). Then \( e_1 = x\bar{y} \) and \( e_2 = y\bar{x} \) are orthogonal primitive idempotents in \( \mathbb{O}_e \), and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.
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- The Cayley-Hamilton equation: \( x^2 - n(x, 1)x + n(x)1 = 0 \), implies that the norm has a well behavior relative to the grading:

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- Otherwise \( \dim \mathbb{O}_g = 1 \) and \( g^2 = e \) for any \( g \in \text{Supp } \Gamma \). We get the \( \mathbb{Z}_2^3 \)-grading.
$\mathbb{Z}_2^3$-grading: Octonions as a twisted group algebra
Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

\[ \mathcal{O} = \mathbb{F}_\sigma [\mathbb{Z}_2^3], \]

where

\[ e^\alpha e^\beta = \sigma(\alpha, \beta) e^{\alpha+\beta} \]

for \( \alpha, \beta \in \mathbb{Z}_2^3 \), with

\[ \sigma(\alpha, \beta) = (-1)^{\psi(\alpha, \beta)}, \]

\[ \psi(\alpha, \beta) = \beta_1 \alpha_2 \alpha_3 + \alpha_1 \beta_2 \alpha_3 + \alpha_1 \alpha_2 \beta_3 + \sum_{i \leq j} \alpha_i \beta_j. \]
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This allows to consider the algebra of octonions as an “associative algebra in a suitable category”. 
Cartan grading: Weyl group

Let $S$ be the vector subspace spanned by $(1, 1, 1)$ in $\mathbb{R}^3$ and consider the two-dimensional real vector space $E = \mathbb{R}^3/S$. Take the elements

$$
\epsilon_1 = (1, 0, 0) + S, \quad \epsilon_2 = (0, 1, 0) + S, \quad \epsilon_3 = (0, 0, 1) + S.
$$

The subgroup $G = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3$ is isomorphic to $\mathbb{Z}^2$, and we may think of the Cartan grading $\Gamma$ on the octonions $\mathbb{O}$ as the grading in which

$$
\deg(e_1) = 0 = \deg(e_2),
$$
$$
\deg(u_i) = \epsilon_i = -\deg(v_i), \quad i = 1, 2, 3.
$$
Then $\text{Supp} \; \Gamma = \{0\} \cup \{\pm \epsilon_i \mid i = 1, 2, 3\}$ and $G$ is the universal group.
The set

$$\Phi := \left(\text{Supp} \; \Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \text{Supp} \; \Gamma, \alpha \neq \pm \beta\}\right) \setminus \{0\}$$

is the root system of type $G_2$. 
Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\text{Aut}(G)$, and this with a subgroup of $\text{GL}(E)$, we have:

\[ W(\Gamma) \subset \{ \mu \in \text{Aut}(G) \mid \mu(\text{Supp} \ \Gamma) = \text{Supp} \ \Gamma \} \]
\[ \subset \{ \mu \in \text{GL}(E) \mid \mu(\Phi) = \Phi \} =: \text{Aut} \ \Phi. \]

The latter group is the automorphism group of the root system $\Phi$, which coincides with its Weyl group.
Identifying the Weyl group \( W(\Gamma) \) with a subgroup of \( \text{Aut}(G) \), and this with a subgroup of \( GL(E) \), we have:

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The latter group is the automorphism group of the root system \( \Phi \), which coincides with its Weyl group.

**Theorem**

*Let \( \Gamma \) be the Cartan grading on the octonions. Identify \( \text{Supp } \Gamma \setminus \{0\} \) with the short roots in the root system \( \Phi \) of type \( G_2 \). Then \( W(\Gamma) = \text{Aut } \Phi \).*
Theorem

Let $\Gamma$ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \text{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$. 
Theorem

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Remark

As any $\varphi \in Stab(\Gamma)$ multiplies each of the elements $i$, $j$, $l$ by either $1$ or $-1$, we see that $Stab(\Gamma) = Diag(\Gamma)$ is isomorphic to $\mathbb{Z}_2^3$. Therefore, the group $Aut(\Gamma)$ is a (non-split) extension of $\mathbb{Z}_2^3$ by $W(\Gamma) \cong GL_3(2)$. 
Gradings on para-Hurwitz algebras of dimension 4 or 8. Gradings on their Hurwitz counterparts. Therefore, any para-Cayley algebra is endowed with a $\mathbb{Z}_3^2$-grading.
Gradings on para-Hurwitz algebras

Theorem

Gradings on para-Hurwitz algebras of dimension 4 or 8

\[\uparrow\]

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Gradings on para-Hurwitz algebras

**Theorem**

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\[ \Leftrightarrow \]

Gradings on their Hurwitz counterparts.

Therefore, any para-Cayley algebra is endowed with a \( \mathbb{Z}_2^3 \)-grading.
Gradings on Okubo algebras

Assuming $F$ is a field of characteristic $\neq 3$ containing a primitive third root $\omega$ of 1, then the matrix algebra $\text{Mat}_3(F)$ is generated by the order 3 matrices:

$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$

$y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$

and the assignment $\deg(x) = (\overline{1}, \overline{0}), \deg(y) = (\overline{0}, \overline{1})$ gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $\text{Mat}_3(F)$, which is inherited by the Okubo algebra $(\mathfrak{sl}_3(F), \ast, n)$.

Over algebraically closed fields, any grading on an Okubo algebra is a coarsening of either the natural $\mathbb{Z}_2$-grading (Cartan grading) or this $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.
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and the assignment

$$\text{deg}(x) = (\bar{1}, \bar{0}), \quad \text{deg}(y) = (\bar{0}, \bar{1}),$$

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Over algebraically closed fields, any grading on an Okubo algebra is a coarsening of either the natural $\mathbb{Z}^2$-grading (Cartan grading) or this $\mathbb{Z}_3^2$-grading.
Consider the order three automorphism $\tau$ of $\mathbb{O}$:

$\tau(e_i) = e_i, \ i = 1, 2, \ \tau(u_j) = u_{j+1}, \ \tau(v_j) = v_{j+1}, \ j = 1, 2, 3,$

and define a new multiplication on $\mathbb{O}$:

$x \ast y = \tau(\bar{x})\tau^2(\bar{y}).$
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and define a new multiplication on $\mathcal{O}$:

$x \ast y = \tau(\bar{x}) \tau^2(\bar{y}).$

It turns out that this is too the (split) Okubo algebra, defined in a characteristic free way, and the $\mathbb{Z}_3^2$-grading is now given by setting

$\deg e_1 = (\bar{1}, \bar{0}) \ \text{and} \ \deg u_1 = (\bar{0}, \bar{1}).$
Multiplication table of the (split) Okubo algebra

<table>
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<tr>
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<th>$e_1$</th>
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</tr>
</tbody>
</table>
The Albert algebra

$G_2$ and $F_4$

Jordan gradings on exceptional simple Lie algebras
Albert algebra

\[ \mathbb{A} = H_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}, \ a_1, a_2, a_3 \in \mathbb{O} \right\} \]

= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}),

where

\[ E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ \iota_1(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & \bar{a} & 0 \end{pmatrix}, \quad \iota_2(a) = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2 \begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
Albert algebra

The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$ (char $\mathbb{F} \neq 2$, $\mathbb{F} = \overline{\mathbb{F}}$).

Then $E_i$ are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2} \iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \quad \iota_i(a) \circ \iota_i(b) = 2n(a, b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3, where $a \bullet b = \overline{a \overline{b}}$ is the para-Hurwitz multiplication.
Cartan grading

Consider the following elements in $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$a_1 = (1, 0, 0, 0), \quad a_2 = (0, 1, 0, 0), \quad a_3 = (-1, -1, 0, 0),$$
$$g_1 = (0, 0, 1, 0), \quad g_2 = (0, 0, 0, 1), \quad g_3 = (0, 0, -1, -1).$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$. 
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\[
\begin{align*}
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\end{align*}
\]

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

Take a canonical basis of the octonions. The assignment

\[
\text{deg } e_1 = \text{deg } e_2 = 0, \quad \text{deg } u_i = g_i = -\text{deg } v_i
\]

gives the Cartan grading on $\mathbb{O}$. 
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Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

Take a canonical basis of the octonions. The assignment

\[\deg e_1 = \deg e_2 = 0, \quad \deg u_i = g_i = -\deg v_i\]

gives the Cartan grading on $\mathbb{O}$.

Now, the Cartan grading on $\mathbb{A}$ is given by:

\[\deg E_i = 0, \quad \deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2),\]
\[\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i),\]
\[\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}),\]
\[\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}).\]
The universal group of the Cartan grading is $\mathbb{Z}^4$, which is contained in $E = \mathbb{R}^4$. Consider the following elements of $\mathbb{Z}^4$:

$$
\begin{align*}
\epsilon_0 &= \deg \iota_1(e_1) = a_1 = (1, 0, 0, 0), \\
\epsilon_1 &= \deg \iota_1(u_1) = g_1 = (0, 0, 1, 0), \\
\epsilon_2 &= \deg \iota_1(u_2) = a_3 + g_2 = (-1, -1, 0, 1), \\
\epsilon_3 &= \deg \iota_1(u_3) = -a_2 + g_3 = (0, -1, -1, -1).
\end{align*}
$$

Note that the $\epsilon_i$'s, $0 \leq i \leq 3$, are linearly independent, but do not form a basis of $\mathbb{Z}^4$. For instance,

$$
\begin{align*}
\deg \iota_2(e_1) &= a_2 = \frac{1}{2}(-\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3), \\
\deg \iota_3(e_1) &= a_3 = \frac{1}{2}(-\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3).
\end{align*}
$$
The supports of the Cartan grading \( \Gamma \) on each of the subspaces \( \iota_i(\mathbb{O}) \) are:

\[
\text{Supp } \iota_1(\mathbb{O}) = \{ \pm \epsilon_i \mid 0 \leq i \leq 3 \},
\]

\[
\text{Supp } \iota_2(\mathbb{O}) = \text{Supp } \iota_1(\mathbb{O}) (\iota_3(e_1) + \iota_3(e_2))
\]
\[= \{ \frac{1}{2}(\pm \epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{even number of } + \text{ signs} \},
\]

\[
\text{Supp } \iota_3(\mathbb{O}) = \text{Supp } \iota_1(\mathbb{O}) (\iota_2(e_1) + \iota_2(e_2))
\]
\[= \{ \frac{1}{2}(\pm \epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{odd number of } + \text{ signs} \}.
\]
Cartan grading: Weyl group

\[ \Phi := \left( \text{Supp } \Gamma \cup \{ \alpha + \beta \mid \alpha, \beta \in \text{Supp } \iota_1(\mathcal{O}), \ \alpha \neq \pm \beta \} \right) \setminus \{0\} \]

\[ = \text{Supp } \iota_1(\mathcal{O}) \cup \text{Supp } \iota_2(\mathcal{O}) \cup \text{Supp } \iota_3(\mathcal{O}) \]

\[ \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3 \}, \]

is the root system of type $F_4$. (Note that the $\epsilon_i$’s, $i = 0, 1, 2, 3$, form an orthogonal basis of $E$ relative to the unique (up to scalar) inner product that is invariant under the Weyl group of $\Phi$.)
Cartan grading: Weyl group

\[
\Phi := \left( \text{Supp } \Gamma \cup \{ \alpha + \beta \mid \alpha, \beta \in \text{Supp } \iota_1(O), \alpha \neq \pm \beta \} \right) \setminus \{0\}
\]

\[
= \text{Supp } \iota_1(O) \cup \text{Supp } \iota_2(O) \cup \text{Supp } \iota_3(O)
\]

\[
\cup \{ \pm \epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3 \},
\]

is the root system of type $F_4$. (Note that the $\epsilon_i$'s, $i = 0, 1, 2, 3$, form an orthogonal basis of $E$ relative to the unique (up to scalar) inner product that is invariant under the Weyl group of $\Phi$.)

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\text{Aut}(\mathbb{Z}^4)$, and this with a subgroup of $GL(E)$, we have:

**Theorem**

*Let $\Gamma$ be the Cartan grading on the Albert algebra. Identify $\text{Supp } \Gamma \setminus \{0\}$ with the short roots in the root system $\Phi$ of type $F_4$. Then $W(\Gamma) = \text{Aut} \Phi$.***
$\mathbb{Z}_2^5$-grading

$\mathbb{A}$ is naturally $\mathbb{Z}_2^2$-graded with

\[ A_{(\bar{0},\bar{0})} = FE_1 + FE_2 + FE_3, \]
\[ A_{(\bar{1},\bar{0})} = \iota_1(\emptyset), \quad A_{(\bar{0},\bar{1})} = \iota_2(\emptyset), \quad A_{(\bar{1},\bar{1})} = \iota_3(\emptyset). \]
$\mathbb{Z}_2^5$-grading

$A$ is naturally $\mathbb{Z}_2^2$-graded with

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$$A_{(\bar{1},\bar{0})} = \iota_1(\emptyset),$$

$$A_{(\bar{0},\bar{1})} = \iota_2(\emptyset),$$

$$A_{(\bar{1},\bar{1})} = \iota_3(\emptyset).$$

This $\mathbb{Z}_2^2$-grading may be combined with the fine $\mathbb{Z}_2^3$-grading on $\emptyset$ to obtain a fine $\mathbb{Z}_2^5$-grading:

$$\deg E_i = (\bar{0},\bar{0},\bar{0},\bar{0},\bar{0}), \ i = 1, 2, 3,$$

$$\deg \iota_1(x) = (\bar{1},\bar{0},\deg x),$$

$$\deg \iota_2(x) = (\bar{0},\bar{1},\deg x),$$

$$\deg \iota_3(x) = (\bar{1},\bar{1},\deg x).$$
$\mathbb{Z}_2^5$-grading: Weyl group

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading $\Gamma$ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$$
$$\deg \iota_3(i) = a + b + c_1, \quad \deg \iota_3(j) = a + b + c_2, \quad \deg \iota_3(l) = a + b + c_3.$$ 

**Theorem**

Let $\Gamma$ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \oplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{ \mu \in \text{Aut}(\mathbb{Z}_2^5) : \mu(T) = T \}.$$
Theorem

Let $\Gamma$ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \bigoplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{ \mu \in \text{Aut}(\mathbb{Z}_2^5) : \mu(T) = T \}.$$ 

Remark

Any $\psi \in \text{Stab}(\Gamma)$ fixes $E_i$ and multiplies $\iota_1(1), \iota_2(1), \iota_3(i), \iota_3(j), \iota_3(l)$, by either 1 or $-1$. Hence $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^5$. 

$\mathbb{Z}_2^5$-grading: Weyl group

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading $\Gamma$ is defined by setting

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\( \mathbb{Z} \times \mathbb{Z}_2^3 \)-grading

Take an element \( i \in \mathbb{F} \) with \( i^2 = -1 \) and consider the following elements in \( \mathbb{A} \):

\[
E = E_1, \quad \tilde{E} = 1 - E = E_2 + E_3,
\]

\[
\nu(a) = i \nu_1(a) \quad \text{for all} \quad a \in \mathbb{O}_0,
\]

\[
\nu_\pm(x) = \nu_2(x) \pm i \nu_3(x) \quad \text{for all} \quad x \in \mathbb{O},
\]

\[
S_\pm = E_3 - E_2 \pm \frac{i}{2} \nu_1(1).
\]

\( \mathbb{A} \) is then 5-graded:

\[
\mathbb{A} = \mathbb{A}_{-2} \oplus \mathbb{A}_{-1} \oplus \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2,
\]

with \( \mathbb{A}_{\pm 2} = \mathbb{F} S_\pm \), \( \mathbb{A}_{\pm 1} = \nu_\pm(\mathbb{O}) \), and \( \mathbb{A}_0 = \mathbb{F} E \oplus (\mathbb{F} \tilde{E} \oplus \nu(\mathbb{O}_0)) \).
**$\mathbb{Z} \times \mathbb{Z}_2^3$-grading**

The $\mathbb{Z}_2^3$-grading on $\mathcal{O}$ combines with this $\mathbb{Z}$-grading

$$A = FS^- \oplus \nu^-(\mathcal{O}) \oplus A_0 \oplus \nu^+(\mathcal{O}) \oplus FS^+$$

to give a fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading as follows:

$$\deg S^\pm = (\pm 2, \bar{0}, \bar{0}, \bar{0}),$$
$$\deg \nu^\pm(x) = (\pm 1, \deg x),$$
$$\deg E = 0 = \deg \tilde{E},$$
$$\deg \nu(a) = (0, \deg a),$$

for homogeneous elements $x \in \mathcal{O}$ and $a \in \mathcal{O}_0$. 
\( \mathbb{Z} \times \mathbb{Z}_2^3 \)-grading: Weyl group

**Theorem**

Let \( \Gamma \) be the \( \mathbb{Z} \times \mathbb{Z}_2^3 \)-grading on the Albert algebra. Then

\[
W(\Gamma) = \text{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).
\]
$\mathbb{Z} \times \mathbb{Z}_2^3$-grading: Weyl group

**Theorem**

Let $\Gamma$ be the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the Albert algebra. Then

$$W(\Gamma) = \text{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).$$

**Remark**

One can show that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$, which is isomorphic to $\mathbb{F}^\times \times \mathbb{Z}_2^3$. 
Recall that the Okubo algebra can be defined on the octonions, with new multiplication:

\[ x * y = \tau(\bar{x})\tau^2(\bar{y}). \]

where \( \tau \) is the order three automorphism of \( \mathbb{O} \) given by:

\[ \tau(e_i) = e_i, \quad i = 1, 2, \quad \tau(u_j) = u_{j+1}, \quad \tau(v_j) = v_{j+1}, \quad j = 1, 2, 3. \]
Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in O$. Then the multiplication in the Albert algebra

$$\mathbb{A} = \bigoplus_{i=1}^{3}(\mathbb{F}E_i \oplus \tilde{\iota}_i(O))$$

becomes:

$$E_i \circ 2 = E_i, \quad E_i \circ E_{i+1} = 0,$$

$$E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2} \tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),$$

$$\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x \ast y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),$$

for $i = 1, 2, 3$ and $x, y \in O$. 
Assume now char $\mathbb{F} \neq 3$. Then the $\mathbb{Z}_3^2$-grading on the Okubo algebra is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \text{Aut}(\mathbb{O}, *)$:

$$
\begin{align*}
\varphi_1(e_1) &= \omega e_1, \\
\varphi_1(u_1) &= u_1, \\
\varphi_2(e_1) &= e_1, \\
\varphi_2(u_1) &= \omega u_1,
\end{align*}
$$

where $\omega$ is a primitive cubic root of unity in $\mathbb{F}$. 

**$\mathbb{Z}_3^3$-grading**
The commuting order 3 automorphisms $\varphi_1$, $\varphi_2$ of $(\mathcal{O}, \ast)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j(\tilde{i}_i(x)) = \tilde{i}_i(\varphi_j(x)).$$

On the other hand, the linear map $\varphi_3 \in \text{End}(\mathbb{A})$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3(\tilde{i}_i(x)) = \tilde{i}_{i+1}(x),$$

is another order 3 automorphism, which commutes with $\varphi_1$ and $\varphi_2$. 
The commuting order 3 automorphisms $\varphi_1$, $\varphi_2$ of $(\mathcal{O}, \ast)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

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The subgroup of $\text{Aut}(\mathcal{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathcal{A}$. 

$\mathbb{Z}_3^3$-grading
The commuting order 3 automorphisms $\varphi_1$, $\varphi_2$ of $(\emptyset, \ast)$ extend to commuting order 3 automorphisms of $A$: 

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The subgroup of $\text{Aut}(A)$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $A$.

All the homogeneous components have dimension 1.
The $\mathbb{Z}_3^3$-grading is determined by

\[
\begin{align*}
\text{deg} \left( \sum_{i=1}^{3} \tilde{i}_i(e_1) \right) &= (1, 0, 0), \\
\text{deg} \left( \sum_{i=1}^{3} \tilde{i}_i(u_1) \right) &= (0, 1, 0), \\
\text{deg} \left( \sum_{i=1}^{3} \omega^{-i} E_i \right) &= (0, 0, 1),
\end{align*}
\]
The $\mathbb{Z}_3^3$-grading is determined by

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\deg\left(\sum_{i=1}^3 \mathfrak{i}_i(e_1)\right) = (1, 0, 0),
\]
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\]
\[
\deg\left(\sum_{i=1}^3 \omega^{-i}E_i\right) = (0, 0, 1),
\]

**Theorem**

Let $\Gamma$ be the $\mathbb{Z}_3^3$-grading on the Albert algebra. Then $W(\Gamma)$ is the commutator subgroup of $\text{Aut}(\mathbb{Z}_3^3)$, i.e.,

\[
W(\Gamma) \cong SL_3(3).
\]
$\mathbb{Z}_3^3$-grading: Weyl group

Why $SL_3(3)$ and not $GL_3(3)$?
Why $SL_3(3)$ and not $GL_3(3)$?

Consider the $\mathbb{Z}_3^3$-grading $\Gamma^-$ determined by

\[
\begin{align*}
\deg(\sum_{i=1}^3 \tilde{\iota}_i(e_1)) &= (\bar{0}, \bar{1}, \bar{0}), \\
\deg(\sum_{i=1}^3 \tilde{\iota}_i(u_1)) &= (\bar{1}, \bar{0}, \bar{0}), \\
\deg(\sum_{i=1}^3 \omega^{-i}E_i) &= (\bar{0}, \bar{0}, \bar{1}),
\end{align*}
\]
Consider the $\mathbb{Z}_3^3$-grading $\Gamma$ determined by

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\deg\left(\sum_{i=1}^3 \omega^{-i}E_i\right) &= (\bar{0}, \bar{0}, \bar{1}),
\end{align*}
\]

Then, for $X_1 \in A(\bar{1}, \bar{0}, \bar{0})$, $X_2 \in A(\bar{0}, \bar{1}, \bar{0})$, $X_3 \in A(\bar{0}, \bar{0}, \bar{1})$, we have:

\[
(X_1 \circ X_2) \circ X_3 = \begin{cases} 
\omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma, \\
\omega^{-1}X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^{-}.
\end{cases}
\]
Why $SL_3(3)$ and not $GL_3(3)$?

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$$

Then, for $X_1 \in A(\bar{1}, \bar{0}, \bar{0})$, $X_2 \in A(\bar{0}, \bar{1}, \bar{0})$, $X_3 \in A(\bar{0}, \bar{0}, \bar{1})$, we have:

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Hence $\Gamma$ and $\Gamma^-$ are equivalent, but NOT isomorphic, gradings.
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$$

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\omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-.
\end{cases}
$$

Hence $\Gamma$ and $\Gamma^-$ are equivalent, but NOT isomorphic, gradings. Besides, any fine $\mathbb{Z}_3^3$-grading on $A$ is isomorphic to either $\Gamma$ or $\Gamma^-$, so $W(\Gamma)$ has index two in $\text{Aut}(U(\Gamma)) \cong GL_3(3)$. 

Why $SL_3(3)$ and not $GL_3(3)$?
\(\mathbb{Z}_3^3\)-grading and the Tits construction

Let \(\mathcal{R} = \text{Mat}_3(\mathbb{F})\). Then

\[ \mathbb{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2, \]

with \(\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2\) copies of \(\mathcal{R}\).
\[ \mathbb{Z}_3^3 \text{-grading and the Tits construction} \]

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\]

with \( \mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2 \) copies of \( \mathcal{R} \).

The product in \( \mathbb{A} \) satisfies \( \mathcal{R}_i \circ \mathcal{R}_j \subseteq \mathcal{R}_{i+j} \pmod{3} \) and:

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>( a'_0 )</th>
<th>( b'_1 )</th>
<th>( c'_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>((a \circ a')_0)</td>
<td>((\bar{a}b')_1)</td>
<td>((c'\bar{a})_2)</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>((\bar{a}'b)_1)</td>
<td>((b \times b')_2)</td>
<td>((bc')_2)</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>((c\bar{a}')_2)</td>
<td>((\overline{b'}c)_0)</td>
<td>((c \times c')_1)</td>
</tr>
</tbody>
</table>

where

\[ a \circ a' = \frac{1}{2}(aa' + a'a), \]
\[ a \times b = a \circ b - \frac{1}{2}(\text{tr}(a)b + \text{tr}(b)a) + \frac{1}{2}(\text{tr}(a)\text{tr}(b) - \text{tr}(ab))1, \]
\[ \bar{a} = a \times 1 = \frac{1}{2}(\text{tr}(a)1 - a). \]
$\mathbb{Z}_3^3$-grading and the Tits construction

Assume $\text{char } \mathbb{F} \neq 3$. Take Pauli matrices in $\mathcal{R}$:

\[
x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

where $\omega, \omega^2$ are the primitive cubic roots of 1, which satisfy

\[
x^3 = 1 = y^3, \quad yx = \omega xy.
\]
\[ \mathbb{Z}_3^3 \text{-grading and the Tits construction} \]

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\]

These Pauli matrices give a grading by \( \mathbb{Z}_3^2 \) on \( \mathcal{R} \), with

\[
\mathcal{R}_{(\alpha_1,\alpha_2)} = \mathbb{F} x^{\alpha_1} y^{\alpha_2}.
\]
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\[
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\]

This grading combines with the \( \mathbb{Z}_3^3 \)-grading on \( \mathbb{A} \) induced by Tits construction, to give the unique, up to equivalence, fine grading by \( \mathbb{Z}_3^3 \) of the Albert algebra.
For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^{\alpha} := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq \mathbb{A}.$$
For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq A.$$ 

Then, for any $\alpha, \beta \in \mathbb{Z}_3^3$:

$$Z^\alpha \circ Z^\beta = \begin{cases} 
\omega \tilde{\psi}(\alpha, \beta) Z^{\alpha + \beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3^3 \alpha + \mathbb{Z}_3^3 \beta) \leq 1, \\
-\frac{1}{2} \omega \tilde{\psi}(\alpha, \beta) Z^{\alpha + \beta} & \text{otherwise},
\end{cases}$$

where

$$\tilde{\psi}(\alpha, \beta) = (\alpha_2 \beta_1 - \alpha_1 \beta_2)(\alpha_3 - \beta_3) - (\alpha_1 \beta_2 + \alpha_2 \beta_1).$$
\(\mathbb{Z}_3^3\)-grading and the Tits construction

Consider now the elements (Racine 1990, unpublished)

\[
W^\alpha := \omega^{-\alpha_1 \alpha_2} Z^\alpha.
\]
$\mathbb{Z}_3^3$-grading and the Tits construction

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$$W^\alpha \circ W^\beta = \omega^{-\alpha_1 \alpha_2 - \beta_1 \beta_2} Z^\alpha \circ Z^\beta$$

$$= \begin{cases} 
\omega \tilde{\psi}(\alpha, \beta) - (\alpha_1 \alpha_2 + \beta_1 \beta_2) Z^\alpha + Z^\beta & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\
-\frac{1}{2} \omega \tilde{\psi}(\alpha, \beta) - (\alpha_1 \alpha_2 + \beta_1 \beta_2) Z^\alpha + Z^\beta & \text{otherwise,}
\end{cases}$$

$$= \begin{cases} 
\omega \tilde{\psi}(\alpha, \beta) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) W^\alpha + W^\beta & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\
-\frac{1}{2} \omega \tilde{\psi}(\alpha, \beta) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) W^\alpha + W^\beta & \text{otherwise.}
\end{cases}$$
The Albert algebra as a twisted group algebra

**Theorem (Griess 1990)**

The Albert algebra is, up to isomorphism, the twisted group algebra

\[ A = \mathbb{F}_\sigma [\mathbb{Z}_3^3], \]

with

\[ \sigma (\alpha, \beta) = \begin{cases} \omega^\psi (\alpha, \beta) & \text{if } \dim_{\mathbb{Z}_3} (\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\ -\frac{1}{2} \omega^\psi (\alpha, \beta) & \text{otherwise}, \end{cases} \]

where

\[ \psi (\alpha, \beta) = (\alpha_2 \beta_1 - \alpha_1 \beta_2) (\alpha_3 - \beta_3). \]
Fine gradings on the Albert algebra

Theorem (Draper–Martín-González 2009 (char = 0), E.–Kochetov 2012)

Up to equivalence, the fine gradings of the Albert algebra are:

1. **The Cartan grading** (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \text{Der}(\mathbb{A})$).
2. **The $\mathbb{Z}_2^5$-grading** obtained by combining the natural $\mathbb{Z}_2^3$-grading on $3 \times 3$ hermitian matrices with the fine grading by $\mathbb{Z}_2^3$ of $\mathbb{O}$.
3. **The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading** obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.
4. **The $\mathbb{Z}_3^3$-grading** with $\dim \mathbb{A}_g = 1 \ \forall g$ (char $\mathbb{F} \neq 3$).
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4. The $\mathbb{Z}_3^3$-grading with $\text{dim } A_g = 1 \ \forall g$ (char $\mathbb{F} \neq 3$).

All the gradings up to isomorphism on $A$ have been classified too (E.–Kochetov).
The Albert algebra

$G_2$ and $F_4$

Jordan gradings on exceptional simple Lie algebras
Gradings and comodule algebras

\[ G\text{-grading} \iff \text{comodule algebra over the group algebra } F_G \]
Gradings and comodule algebras

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \Rightarrow \quad \rho_\Gamma : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathbb{F}G \]

\[ x_g \mapsto x_g \otimes g \]

(algebra morphism and comodule str.)
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\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \Rightarrow \quad \rho_{\Gamma} : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathbb{F}G \]

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\[ \Gamma_{\rho} : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \Leftrightarrow \quad \rho : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathbb{F}G \]

\[ (\mathcal{A}_g = \{ x \in \mathcal{A} : \rho(x) = x \otimes g \}) \]
A comodule algebra map

\[ \rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \]

induces a \textit{generic automorphism} of \( \mathbb{F}G \)-algebras

\[ \mathcal{A} \otimes \mathbb{F}G \longrightarrow \mathcal{A} \otimes \mathbb{F}G \]

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Gradings and comodule algebras

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$$\mathcal{A} \otimes \mathbb{F}G \longrightarrow \mathcal{A} \otimes \mathbb{F}G$$

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All the information on the grading $\Gamma$ attached to $\rho$ is contained in this single automorphism!
Gradings and affine group schemes

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \iff \rho_\Gamma : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathbb{F} G \]

(comodule algebra structure)
Gradings and affine group schemes

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(comodule algebra structure)

Now,

\[ \rho_\Gamma : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \quad \iff \quad \eta_\Gamma : G^D \to \textbf{Aut} \, \mathcal{A} \]

(comodule algebra) \quad (morphism of affine group schemes)
Gradings and affine group schemes

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(comodule algebra) (morphism of affine group schemes)

For any \( \varphi \in G^D(\mathcal{R}) \), \( \eta_\Gamma(\varphi) \in \text{Aut}_\mathcal{R}(\mathcal{A} \otimes \mathcal{R}) \) is given by:

\[ \eta_\Gamma(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g)r. \]
Gradings and affine group schemes

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and \( \rho_{\Gamma} \) is recovered as

\[ \rho_{\Gamma}(x) = \eta_{\Gamma}(id_{\mathbb{F}G})(x \otimes 1) \quad \left( \eta_{\Gamma}(id_{\mathbb{F}G}) \in \text{Aut}_{\mathbb{F}G}(\mathcal{A} \otimes \mathbb{F}G) \right) \]
Consider a homomorphism $\Phi : \text{Aut} \mathcal{A} \longrightarrow \text{Aut} \mathcal{A}'$ of affine group schemes.
Gradings and affine group schemes

Consider a homomorphism $\Phi : \text{Aut} \mathcal{A} \rightarrow \text{Aut} \mathcal{A}'$ of affine group schemes.

Then any grading $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ induces a grading $\Gamma' : \mathcal{A}' = \bigoplus_{g \in G} \mathcal{A}'_g$ by means of:

$$\eta_{\Gamma'} : G^D \xrightarrow{\eta_{\Gamma}} \text{Aut} \mathcal{A} \xrightarrow{\Phi} \text{Aut} \mathcal{A}'$$
Gradings and affine group schemes

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$$\eta_{\Gamma'} : G^D \xrightarrow{\eta_{\Gamma}} \textbf{Aut} \mathcal{A} \overset{\Phi}{\longrightarrow} \textbf{Aut} \mathcal{A}' .$$

If $\Gamma_1 : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma_2 : \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}'_h$ are weakly isomorphic through the automorphisms $\psi \in \text{Aut} \mathcal{A}$ and $\varphi : G \rightarrow H$, then the induced gradings $\Gamma'_1$ and $\Gamma'_2$ on $\mathcal{A}'$ are weakly isomorphic too through the automorphisms $\Phi_{\mathcal{F}}(\psi) \in \text{Aut} \mathcal{A}'$ and $\varphi : G \rightarrow H$. 
For $G = \textbf{Aut} \ A$, $\text{Lie}(G) = \text{Der}(A)$, so

$$\text{Ad} : \textbf{Aut} \ A \rightarrow \textbf{Aut}(\text{Der}(A))$$

is a homomorphism, and any grading $\Gamma : A = \bigoplus_{g \in G} A_g$ induces a grading

$$\Gamma' : \text{Der}(A) = \bigoplus_{g \in G} \text{Der}(A)_g,$$

$$\text{Der}(A)_g = \{ d \in \text{Der}(A) : d(A_h) \subseteq A_{gh} \ \forall h \in G \}.$$
Gradings on $G_2$ and $F_4$

If $\text{Aut} \mathcal{A} \cong \text{Aut} \mathcal{B}$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $\mathcal{A}$ and $\mathcal{B}$ are equivalent.
Gradings on $G_2$ and $F_4$

If $\text{Aut} \mathcal{A} \cong \text{Aut} \mathcal{B}$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

If the characteristic of the ground field $\mathbb{F}$ is $\neq 2, 3$, then

$$\text{Ad} : \text{Aut} \mathbb{O} \rightarrow \text{Aut} g_2$$

is an isomorphism, and (assuming just char $\mathbb{F} \neq 2$),

$$\text{Ad} : \text{Aut} \mathcal{A} \rightarrow \text{Aut} f_4$$

is an isomorphism too.
Theorem

Up to equivalence, the fine gradings on $\mathfrak{g}_2$ are

- the Cartan grading, and

- a $\mathbb{Z}_2^3$-grading with $(\mathfrak{g}_2)_0 = 0$ and where $(\mathfrak{g}_2)_g$ is a Cartan subalgebra of $\mathfrak{g}_2$ for any $0 \neq g \in \mathbb{Z}_2^3$. 

Gradings on $G_2$
Up to equivalence, the fine gradings on \( \mathfrak{f}_4 \) are

- the Cartan grading,
- a grading by \( \mathbb{Z}_2^5 \), obtained by combining the \( \mathbb{Z}_2^2 \)-grading given by the decomposition \( \mathfrak{f}_4 = \mathfrak{o}_4 \oplus \text{natural} \oplus \text{spin} \oplus \text{spin} \), with the \( \mathbb{Z}_2^3 \)-grading on the octonions (which is the vector space behind the natural and spin representations of \( \mathfrak{o}_4 \)).
- a grading by \( \mathbb{Z} \times \mathbb{Z}_2^3 \), obtained by looking at \( \mathfrak{f}_4 \) as the Kantor Lie algebra of a structurable algebra: \( \mathfrak{f}_4 = \mathcal{K}(\mathbb{O}, -) \), and combining the natural 5-grading on \( \mathcal{K}(\mathbb{O}, -) \) and the \( \mathbb{Z}_2^3 \)-grading on \( \mathbb{O} \).
- a \( \mathbb{Z}_3^3 \)-grading (only if \( \text{char } F \neq 3 \)), with \( (\mathfrak{f}_4)_0 = 0 \) and where \( (\mathfrak{f}_4)_g \oplus (\mathfrak{f}_4)_{-g} \) is a Cartan subalgebra of \( \mathfrak{f}_4 \) for any \( 0 \neq g \in \mathbb{Z}_3^3 \).
The Albert algebra

$G_2$ and $F_4$

Jordan gradings on exceptional simple Lie algebras
Jordan subgroups

**Definition (Alekseevskii 1974)**

Given a simple Lie algebra $\mathfrak{g}$ and a complex Lie group $G$ with $\text{Int}(\mathfrak{g}) \leq G \leq \text{Aut}(\mathfrak{g})$, an abelian subgroup $A$ of $G$ is a *Jordan subgroup* if:

1. its normalizer $N_G(A)$ is finite,
2. $A$ is a minimal normal subgroup of its normalizer, and
3. its normalizer is maximal among the normalizers of those abelian subgroups satisfying (i) and (ii).
The Jordan subgroups are elementary ($\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime number $p$), and they induce gradings, called *Jordan gradings*, in the Lie algebra $\mathfrak{g}$. 
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The classification of Jordan subgroups by Alekseevskii splits in two types: classical and exceptional.
Jordan subgroups: classical cases

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{p^{n-1}}$</td>
<td>$\mathbb{Z}_p^{2n}$</td>
</tr>
<tr>
<td>$B_n \ (n \geq 3)$</td>
<td>$\mathbb{Z}_2^{2n}$</td>
</tr>
<tr>
<td>$C_{2^{n-1}} \ (n \geq 2)$</td>
<td>$\mathbb{Z}_2^{2n}$</td>
</tr>
<tr>
<td>$D_{n+1} \ (n \geq 3)$</td>
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<tr>
<td>$D_{2^{n-1}} \ (n \geq 3)$</td>
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</tbody>
</table>
Jordan subgroups: classical cases

The dimension of all nonzero homogeneous spaces is always 1 in these classical cases, which are well-known.

<table>
<thead>
<tr>
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<tr>
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Jordan subgroups: exceptional cases

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</tr>
<tr>
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<td>2</td>
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</tr>
<tr>
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<td>$\mathbb{Z}_2^3$</td>
<td>8</td>
</tr>
<tr>
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Models of these gradings?
Gradings on Freudenthal’s Magic Square

Given two symmetric composition algebras, the Lie algebra $\mathfrak{g}(S, S')$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with

\[
\mathfrak{g}(\bar{0}, \bar{0}) = \text{tri}(S) \oplus \text{tri}(S'), \\
\mathfrak{g}(\bar{1}, \bar{0}) = \iota_1(S \otimes S'), \\
\mathfrak{g}(\bar{0}, \bar{1}) = \iota_2(S \otimes S'), \\
\mathfrak{g}(\bar{1}, \bar{1}) = \iota_3(S \otimes S').
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Gradings on Freudenthal’s Magic Square

Given two symmetric composition algebras, the Lie algebra \( g(S, S') \) is naturally \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded with

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g(\bar{1}, \bar{1}) = \iota_3(S \otimes S').
\]

Also, the triality automorphisms \( \theta \) and \( \theta' \) extend to an order 3 automorphism \( \Theta \) of \( g(S, S') \). The eigenspaces of \( \Theta \) constitute a \( \mathbb{Z}_3 \)-grading of \( g(S, S') \).
Induced gradings

(From now on, assume that our ground field \( \mathbb{F} \) is algebraically closed of characteristic 0.)
**Induced gradings**

(From now on, assume that our ground field $\mathbb{F}$ is algebraically closed of characteristic 0.)

The previous $\mathbb{Z}_2^2$ and $\mathbb{Z}_3$-gradings on the Lie algebras $\mathfrak{g}(S, S')$ can be complemented with gradings on the symmetric composition algebras $S$ and $S'$ in several ways.
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- The \( \mathbb{Z}_2^2 \) -grading on the Okubo algebra \( \mathcal{O} \) induces a \( \mathbb{Z}_3^3 \) -grading on both the simple Lie algebra \( g(\mathbb{F}, \mathcal{O}) \) of type \( F_4 \) (our fine \( \mathbb{Z}_3^3 \) -grading!!) and the simple Lie algebra \( g(S, \mathcal{O}) \) (for the two-dimensional para-Hurwitz algebra \( S \)) of type \( E_6 \).

In both cases \( g_0 = 0 \) and \( g_\alpha \oplus g_{-\alpha} \) is a Cartan subalgebra of \( g \) for any \( 0 \neq \alpha \in \mathbb{Z}_3^3 \).
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- The \( \mathbb{Z}_2^3 \)-grading on a para-Cayley algebra \( \bar{\mathcal{C}} \) induces a \( \mathbb{Z}_2^5 \)-grading on the simple Lie algebra \( \mathfrak{g}(\bar{\mathcal{C}}, \bar{\mathcal{C}}) \) of type \( E_8 \). Moreover, \( \mathfrak{g}_0 = 0 \) and \( \mathfrak{g}_\alpha \) is a Cartan subalgebra of \( \mathfrak{g} \) for any \( 0 \neq \alpha \in \mathbb{Z}_2^5 \).
Exceptional Jordan gradings

Theorem

The gradings:

1. a $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type $G_2$ induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,

2. a $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type $D_4$ induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,

3. a $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type $F_4$ induced by the $\mathbb{Z}_3^2$-grading of the Okubo algebra,

4. a $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type $E_6$ induced by the $\mathbb{Z}_3^2$-grading of the Okubo algebra,

5. a $\mathbb{Z}_2^5$-grading on the simple Lie algebra of type $E_8$ induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,

are exceptional Jordan gradings.
The missing exceptional Jordan grading

Only one exceptional Jordan grading does not fit in the Theorem above: the \( \mathbb{Z}_5^3 \)-grading on \( E_8 \).
The missing exceptional Jordan grading

Only one exceptional Jordan grading does not fit in the Theorem above: the $\mathbb{Z}_5^3$-grading on $E_8$.

Let $V_1$ and $V_2$ be two vector spaces over $\mathbb{F}$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space

$$g = \bigoplus_{i=0}^4 g_i,$$

where

$$g_0 = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2),$$
$$g_1 = V_1 \otimes \wedge^2 V_2,$$
$$g_2 = \wedge^2 V_1 \otimes \wedge^4 V_2,$$
$$g_3 = \wedge^3 V_1 \otimes V_2,$$
$$g_4 = \wedge^4 V_1 \otimes \wedge^3 V_2.$$

This is a $\mathbb{Z}_5$-graded Lie algebra in a unique way: the exceptional simple Lie algebra of type $E_8$. 

The missing exceptional Jordan grading

Up to conjugation in $\operatorname{Aut} \mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.
The missing exceptional Jordan grading

Up to conjugation in Aut $\mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \\
& & & & \bullet & & & \\
& & & & \downarrow & & & \\
& & & & & & 3 \\
\end{array}
\]
The missing exceptional Jordan grading

Up to conjugation in $\text{Aut}\, \mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type $E_8$ such that the dimension of the subalgebra of fixed elements is 48.

The uniqueness shows us that, up to conjugation, this is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\xi}$ is $\xi^i$ times the identity, where $\xi$ is a fixed primitive fifth root of unity.
The missing exceptional Jordan grading
The missing exceptional Jordan grading

Consider the following automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $\mathfrak{g}$:

$\sigma_1(x) = \xi^i x$ for any $x \in \mathfrak{g}_i$ and $0 \leq i \leq 4$,

$\sigma_2|_{\mathfrak{g}_i} = b_1 \otimes \wedge^2 b_2,$

$\sigma_3|_{\mathfrak{g}_i} = c_1 \otimes \wedge^2 c_2,$
The missing exceptional Jordan grading

Consider the following automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $g$:

$$\sigma_1(x) = \xi^i x \quad \text{for any } x \in g_{\bar{i}} \text{ and } 0 \leq i \leq 4,$$

$$\sigma_2|_{g_{\bar{1}}} = b_1 \otimes \wedge^2 b_2,$$

$$\sigma_3|_{g_{\bar{1}}} = c_1 \otimes \wedge^2 c_2,$$

where on fixed bases of $V_1$ and $V_2$, the coordinate matrices of $b_1, c_1, b_2, c_2$ are:

$$b_1 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}, \quad c_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$b_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 \\ 0 & 0 & \xi^4 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & \xi^3 \end{pmatrix}, \quad c_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$
The missing exceptional Jordan grading

The grading of $E_8$ induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading by $Z_{3^5}$.

$\forall 0 \neq \alpha \in Z_{3^5}, \bigoplus_{i=1}^{4} g_i^\alpha$ is a Cartan subalgebra of $g$.

There are models of the Jordan gradings of $F_4$ and $E_6$ by $Z_{3^3}$ constructed along the same lines.
The missing exceptional Jordan grading

Theorem

The grading of $E_8$ induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading by $\mathbb{Z}_5^3$. 

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The missing exceptional Jordan grading

**Theorem**

*The grading of $E_8$ induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading by $\mathbb{Z}_5^3$.***

\[ \forall 0 \neq \alpha \in \mathbb{Z}_5^3, \bigoplus_{i=1}^4 g_{i\alpha} \text{ is a Cartan subalgebra of } g. \]
The missing exceptional Jordan grading

Theorem

The grading of $E_8$ induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading by $\mathbb{Z}_5^3$.

∀ $0 \neq \alpha \in \mathbb{Z}_5^3$, $\bigoplus_{i=1}^4 g_{i\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$.

There are models of the Jordan gradings of $F_4$ and $E_6$ by $\mathbb{Z}_3^3$ constructed along the same lines.
That's all.
Thanks