Gradings on the octonions and the Albert algebra

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Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading ($r$ being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.
In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than $D_4$, by arbitrary abelian groups were considered by Havlíček, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including $D_4$) over algebraically closed fields of characteristic zero has been obtained quite recently.
For any abelian group $G$, the classification of all $G$-gradings, up to isomorphism, on the classical simple Lie algebras other than $D_4$ over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.
For any abelian group $G$, the classification of all $G$-gradings, up to isomorphism, on the classical simple Lie algebras other than $D_4$ over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

The gradings on the octonions and on the Albert algebra are instrumental in obtaining the gradings on the exceptional simple Lie algebras.
1 Gradings

2 The algebra of Octonions

3 The Albert algebra

4 $G_2$ and $F_4$
Gradings

The algebra of Octonions

The Albert algebra

$G_2$ and $F_4$
Definition

$G$ abelian group, $A$ algebra over a field $\mathbb{F}$.

$G$-grading on $A$:

\[ A = \bigoplus_{g \in G} A_g, \]

\[ A_g A_h \subseteq A_{gh} \quad \forall g, h \in G. \]
Example: Pauli matrices

\[ A = \text{Mat}_n(\mathbb{F}) \]

\[
X = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \epsilon & 0 & \ldots & 0 \\
0 & 0 & \epsilon^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \epsilon^{n-1}
\end{pmatrix}
\]

\[
Y = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

(\( \epsilon \) a primitive \( n \)th root of 1)

\[ X^n = 1 = Y^n, \quad YX = \epsilon XY \]

\[ A = \bigoplus_{(\bar{i}, \bar{j}) \in \mathbb{Z}_n \times \mathbb{Z}_n} A(\bar{i}, \bar{j}), \quad A(\bar{i}, \bar{j}) = \mathbb{F} X^i Y^j. \]
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\end{pmatrix}, \quad Y = \begin{pmatrix}
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0 & 0 & 0 & \ldots & 1 \\
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\end{pmatrix} \]

(\( \epsilon \) a primitive \( n \)-th root of 1)

\[ X^n = 1 = Y^n, \quad YX = \epsilon XY \]

\[ \mathcal{A} = \bigoplus_{(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}(i,j), \quad \mathcal{A}(i,j) = \mathbb{F}X^iY^j. \]

\( \mathcal{A} \) becomes a graded division algebra.
Let $\Gamma : A = \bigoplus_{g \in G} A_g$ be a grading on $A$ (dim$_F A < \infty$, $F = \overline{F}$, char $F \neq 2$):
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- The *support* of $\Gamma$ is $\text{Supp} \, \Gamma = \{g \in G : \mathcal{A}_g \neq 0\}$. 

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Basic definitions (Patera-Zassenhaus)

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- The *support* of $\Gamma$ is $\text{Supp} \ \Gamma = \{ g \in G : \mathcal{A}_g \neq 0 \}$.

- The *universal grading group* of $\Gamma$ is the group $U(\Gamma)$ generated by $\text{Supp} \ \Gamma$ subject to the relations $g_1 g_2 = g_3$ if $0 \neq \mathcal{A}_{g_1} \mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_3}$.
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Gradings on $O$ and $A$
Exceptional Algebras&Groups
Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a grading on $\mathcal{A}$ \ ($\dim_{\mathbb{F}} \mathcal{A} < \infty$, $\mathbb{F} = \overline{\mathbb{F}}$, $\text{char} \mathbb{F} \neq 2$):

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The grading $\Gamma$ is then a grading too by $U(\Gamma)$. 
There appear several groups attached to $\Gamma$:

- The *automorphism group*

$$\text{Aut}(\Gamma) = \{ \varphi \in \text{Aut} \mathcal{A} : \exists \alpha \in \text{Sym}(\text{Supp} \, \Gamma) \text{ s.t. } \varphi(\mathcal{A}_g) \subseteq \mathcal{A}_{\alpha(g)} \, \forall g \}.$$
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- **The stabilizer group**

  $$\text{Stab}(\Gamma) = \{ \varphi \in \text{Aut}(\Gamma) : \varphi(A_g) \subseteq A_g \ \forall g \}.$$
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- The *diagonal group*
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  \text{Diag}(\Gamma) = \{ \varphi \in \text{Aut}(\Gamma) : \forall g \in \text{Supp} \ \Gamma \ \exists \lambda_g \in \mathbb{F}^\times \text{ s.t. } \varphi|_{\mathcal{A}_g} = \lambda_g \text{ id} \}. 
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- The quotient $W(\Gamma) = \text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ is the *Weyl group* of $\Gamma$. 
Each $\varphi \in \text{Aut}(\Gamma)$ determines a self-bijection $\alpha$ of $\text{Supp} \, \Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\text{Aut}(\Gamma) \to \text{Aut}(U(\Gamma))$$

with kernel $\text{Stab}(\Gamma)$.

Thus, the Weyl group embeds naturally in $\text{Aut}(U(\Gamma))$, i.e., there is a natural action of the Weyl group on $U(\Gamma)$ by automorphisms.
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**Remark**

$\text{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$. 
Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}_{g'}$, be two gradings on $\mathcal{A}$:
Let $\Gamma : A = \bigoplus_{g \in G} A_g$ and $\Gamma' : A = \bigoplus_{g' \in G'} A'_{g'}$ be two gradings on $A$:

- $\Gamma$ is a \textit{refinement} of $\Gamma'$ if for any $g \in G$ there is a $g' \in G'$ such that $A_g \subseteq A_{g'}$.
- Then $\Gamma'$ is a \textit{coarsening} of $\Gamma$.
Let $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

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For example, if $\alpha : G \to H$ is a group homomorphism, then

$\mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$, with $\mathcal{A}_h = \bigoplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$, is a coarsening.
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If $G = U(\Gamma)$, any coarsening of $\Gamma$ is obtained in this way.
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- $\Gamma$ is **fine** if it admits no proper refinement.
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If $G = U(\Gamma)$, any coarsening of $\Gamma$ is obtained in this way.

- $\Gamma$ is fine if it admits no proper refinement.

  Any grading is a coarsening of a fine grading.
Basic definitions (Patera-Zassenhaus)

- Γ and Γ′ are *equivalent* if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) such that for any \( g \in G \) there is a \( g' \in G' \) with \( \varphi(\mathcal{A}_g) = \mathcal{A}_{g'} \).

- Γ and Γ′ are *weakly isomorphic* if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) and an isomorphism \( \alpha : G \rightarrow G' \) such that for any \( g \in G \) \( \varphi(\mathcal{A}_g) = \mathcal{A}_{\alpha(g)} \).

- For \( G = G' \), Γ and Γ′ are *isomorphic* if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) such that \( \varphi(\mathcal{A}_g) = \mathcal{A}_{g'} \) for any \( g \in G \).
Basic definitions (Patera-Zassenhaus)

- **Γ and Γ′ are equivalent** if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) such that for any \( g \in G \) there is a \( g' \in G' \) with \( \varphi(\mathcal{A}g) = \mathcal{A}'g' \).

- **Γ and Γ′ are weakly isomorphic** if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) and an isomorphism \( \alpha : G \rightarrow G' \) such that for any \( g \in G \) \( \varphi(\mathcal{A}g) = \mathcal{A}'_{\alpha(g)} \).
• Γ and Γ’ are equivalent if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) such that for any \( g \in G \) there is a \( g' \in G' \) with \( \varphi(A_g) = A'_{g'} \).

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• For \( G = G' \), Γ and Γ’ are isomorphic if there is an automorphism \( \varphi \in \text{Aut} \mathcal{A} \) such that \( \varphi(A_g) = A'_{g} \) for any \( g \in G \).
1 Gradings

2 The algebra of Octonions

3 The Albert algebra

4 $G_2$ and $F_4$
Cayley-Dickson process:

\[ K = F \oplus F i, \quad i^2 = -1, \]
\[ H = K \oplus K j, \quad j^2 = -1, \]
\[ O = H \oplus H l, \quad l^2 = -1, \]

\( O \) is \( \mathbb{Z}_2^3 \)-graded with

\[ \text{deg}(i) = (\bar{1}, \bar{0}, \bar{0}), \quad \text{deg}(j) = (\bar{0}, \bar{1}, \bar{0}), \quad \text{deg}(l) = (\bar{0}, \bar{0}, \bar{1}). \]
The Cartan grading is the $\mathbb{Z}_2$-grading determined by:

\[ \deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1). \]
Cartan grading on the Octonions

\( \mathcal{O} \) contains canonical bases:

\[ \mathcal{B} = \{ e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3 \} \]

with

\[ n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{otherwise 0.} \]

\[ e_1^2 = e_1, \quad e_2^2 = e_2, \]

\[ e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3) \]

\[ u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3) \]

\[ u_i u_{i+1} = -u_{i+1} u_i = v_{i+2}, \quad v_i v_{i+1} = -v_{i+1} v_i = u_{i+2}, \quad \text{(indices modulo 3)} \]

The Cartan grading is the \( \mathbb{Z}^2 \)-grading determined by:

\[ \deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1). \]
Theorem (E. 1998)

Up to equivalence, the fine gradings on $\mathbb{O}$ are

- the Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$), and

- the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.
Fine gradings on the Octonions

Sketch of proof:

The Cayley-Hamilton equation:

\[ x^2 - n(x,1)x + n(x)1 = 0, \]
implies that the norm has a well behavior relative to the grading:

\[ n(O_g) = 0 \text{ unless } g^2 = e, \]
\[ n(O_g, O_h) = 0 \text{ unless } gh = e. \]

If there is a \( g \in \text{Supp } \Gamma \) with either order > 2 or \( \dim O_g \geq 2 \), there are elements \( x \in O_g, y \in O_g^{-1} \) with

\[ n(x) = 0 = n(y), \]
\[ n(x, y) = 1. \]

Then \( e_1 = \overline{x} \overline{y} \) and \( e_2 = \overline{y} \overline{x} \) are orthogonal primitive idempotents in \( O_e \), and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.

Otherwise \( \dim O_g = 1 \) and \( g^2 = e \) for any \( g \in \text{Supp } \Gamma \). We get the \( Z_3 \)-grading.

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Gradings on \( \mathbb{O} \) and \( \mathbb{A} \) Exceptional Algebras & Groups
Sketch of proof:

- The Cayley-Hamilton equation: $x^2 - n(x, 1)x + n(x)1 = 0$, implies that the norm has a well behavior relative to the grading:

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- If there is a \( g \in \text{Supp} \Gamma \) with either order \( > 2 \) or \( \dim \mathbb{O}_g \geq 2 \), there are elements \( x \in \mathbb{O}_g, y \in \mathbb{O}_{g^{-1}} \) with \( n(x) = 0 = n(y), n(x, y) = 1. \) Then \( e_1 = x\bar{y} \) and \( e_2 = y\bar{x} \) are orthogonal primitive idempotents in \( \mathbb{O}_e \), and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.
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- Otherwise $\dim \mathbb{O}_g = 1$ and $g^2 = e$ for any $g \in \text{Supp} \ \Gamma$. We get the $\mathbb{Z}_2^3$-grading.
Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

$$O = \mathbb{F}\sigma[Z_2]$$

where

$$e_\alpha e_\beta = \sigma(\alpha,\beta) e_\alpha + \beta$$

for \(\alpha, \beta \in Z_2^3\), with \(\sigma(\alpha,\beta) = (-1)^{\psi(\alpha,\beta)}\),

$$\psi(\alpha,\beta) = \beta_1 \alpha_2 \alpha_3 + \alpha_1 \beta_2 \alpha_3 + \alpha_1 \alpha_2 \beta_3 + \sum_{i \leq j} \alpha_i \beta_j.$$
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This allows to consider the algebra of octonions as an “associative algebra in a suitable category”.
Let $S$ be the vector subspace spanned by $(1, 1, 1)$ in $\mathbb{R}^3$ and consider the two-dimensional real vector space $E = \mathbb{R}^3 / S$. Take the elements

$$
\epsilon_1 = (1, 0, 0) + S, \quad \epsilon_2 = (0, 1, 0) + S, \quad \epsilon_3 = (0, 0, 1) + S.
$$

The subgroup $G = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3$ is isomorphic to $\mathbb{Z}^2$, and we may think of the Cartan grading $\Gamma$ on the octonions $\mathbb{O}$ as the grading in which

$$
deg(e_1) = 0 = deg(e_2),
$$
$$
deg(u_i) = \epsilon_i = -deg(v_i), \quad i = 1, 2, 3.
$$

Then $\text{Supp} \, \Gamma = \{0\} \cup \{\pm \epsilon_i \mid i = 1, 2, 3\}$ and $G$ is the universal group. The set

$$
\Phi := \left( \text{Supp} \, \Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \text{Supp} \, \Gamma, \alpha \neq \pm \beta\} \right) \setminus \{0\}
$$

is the root system of type $G_2$. 

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Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\text{Aut}(G)$, and this with a subgroup of $\text{GL}(E)$, we have:

$$W(\Gamma) \subset \{ \mu \in \text{Aut}(G) \mid \mu(\text{Supp } \Gamma) = \text{Supp } \Gamma \}$$
$$\subset \{ \mu \in \text{GL}(E) \mid \mu(\Phi) = \Phi \} =: \text{Aut } \Phi.$$

The latter group is the automorphism group of the root system $\Phi$, which coincides with its Weyl group.

**Theorem**

Let $\Gamma$ be the Cartan grading on the octonions. Identify $\text{Supp } \Gamma \setminus \{0\}$ with the short roots in the root system $\Phi$ of type $G_2$. Then $W(\Gamma) = \text{Aut } \Phi$. 
Theorem

Let $\Gamma$ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \text{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$. 

Remark

As any $\phi \in \text{Stab}(\Gamma)$ multiplies each of the elements $i$, $j$, $l$ by either $1$ or $-1$, we see that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^3$. Therefore, the group $\text{Aut}(\Gamma)$ is a (non-split) extension of $\mathbb{Z}_2^3$ by $W(\Gamma) \cong GL_3(2)$. 

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Theorem

Let $\Gamma$ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \text{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

Remark

As any $\varphi \in \text{Stab}(\Gamma)$ multiplies each of the elements $i, j, l$ by either 1 or $-1$, we see that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^3$. Therefore, the group $\text{Aut}(\Gamma)$ is a (non-split) extension of $\mathbb{Z}_2^3$ by $W(\Gamma) \cong GL_3(2)$. 
1. Gradings

2. The algebra of Octonions

3. The Albert algebra

4. $G_2$ and $F_4$
Albert algebra

\[ A = H_3(\mathbb{O}, \ast) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}, \ a_1, a_2, a_3 \in \mathbb{O} \right\} \]

\[ = \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}), \]

where

\[ E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ \iota_1(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2 \begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$.

Then $E_i$ are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \quad \iota_i(a) \circ \iota_i(b) = 2n(a, b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3, where $a \bullet b = \bar{a}\bar{b}$. 
Cartan grading

Consider the following elements in $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$a_1 = (1, 0, 0, 0), \quad a_2 = (0, 1, 0, 0), \quad a_3 = (-1, -1, 0, 0),$$
$$g_1 = (0, 0, 1, 0), \quad g_2 = (0, 0, 0, 1), \quad g_3 = (0, 0, -1, -1).$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$. Take a “good basis” of the octonions. The assignment $\deg e_1 = \deg e_2 = 0$, $\deg u_i = g_i = -\deg v_i$ gives the Cartan grading on $\mathbb{O}$.

Now, the Cartan grading on $\mathbb{A}$ is given by the assignment

$$\deg E_i = 0,$$
$$\deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2),$$
$$\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i),$$
$$\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}),$$
$$\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}).$$
The universal group of the Cartan grading is $\mathbb{Z}^4$, which is contained in $E = \mathbb{R}^4$. Consider the following elements of $\mathbb{Z}^4$:

$$
\begin{align*}
\epsilon_0 &= \deg \nu_1(e_1) = a_1 = (1, 0, 0, 0), \\
\epsilon_1 &= \deg \nu_1(u_1) = g_1 = (0, 0, 1, 0), \\
\epsilon_2 &= \deg \nu_1(u_2) = a_3 + g_2 = (-1, -1, 0, 1), \\
\epsilon_3 &= \deg \nu_1(u_3) = -a_2 + g_3 = (0, -1, -1, -1).
\end{align*}
$$

Note that the $\epsilon_i$'s, $0 \leq i \leq 3$, are linearly independent, but do not form a basis of $\mathbb{Z}^4$. We have for instance $\deg \nu_2(e_1) = a_2 = \frac{1}{2}(-\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3)$ and $\deg \nu_3(e_1) = \frac{1}{2}(-\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3)$.
The supports of the Cartan grading $\Gamma$ on each of the subspaces $\nu_i(\mathbb{O})$ are:

$\text{Supp } \nu_1(\mathbb{O}) = \{ \pm \epsilon_i \mid 0 \leq i \leq 3 \}$,

$\text{Supp } \nu_2(\mathbb{O}) = \text{Supp } \nu_1(\mathbb{O})(\nu_3(e_1) + \nu_3(e_2))$

$= \left\{ \frac{1}{2} (\pm \epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{even number of + signs} \right\}$,

$\text{Supp } \nu_3(\mathbb{O}) = \text{Supp } \nu_1(\mathbb{O})(\nu_2(e_1) + \nu_2(e_2))$

$= \left\{ \frac{1}{2} (\pm \epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{odd number of + signs} \right\}$. 
Cartan grading: Weyl group

\[ \Phi := \left( \text{Supp } \Gamma \cup \{ \alpha + \beta \mid \alpha, \beta \in \text{Supp } \nu_1(O), \alpha \neq \pm \beta \} \right) \setminus \{0\} \]

\[ = \text{Supp } \nu_1(O) \cup \text{Supp } \nu_2(O) \cup \text{Supp } \nu_3(O) \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3 \}, \]

is the root system of type \( F_4 \). (Note that the \( \epsilon_i \)'s, \( i = 0, 1, 2, 3 \), form an orthogonal basis of \( E \) relative to the unique (up to scalar) inner product that is invariant under the Weyl group of \( \Phi \).)
Cartan grading: Weyl group

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\[ = \text{Supp } \nu_1(\mathbb{O}) \cup \text{Supp } \nu_2(\mathbb{O}) \cup \text{Supp } \nu_3(\mathbb{O}) \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3 \}, \]

is the root system of type $F_4$. (Note that the $\epsilon_i$'s, $i = 0, 1, 2, 3$, form an orthogonal basis of $E$ relative to the unique (up to scalar) inner product that is invariant under the Weyl group of $\Phi$.)

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\text{Aut}(\mathbb{Z}^4)$, and this with a subgroup of $\text{GL}(E)$, we have:

**Theorem**

*Let $\Gamma$ be the Cartan grading on the Albert algebra. Identify $\text{Supp } \Gamma \setminus \{0\}$ with the short roots in the root system $\Phi$ of type $F_4$. Then $W(\Gamma) = \text{Aut } \Phi$.***
\( \mathbb{Z}_2^5 \)-grading

\( A \) is naturally \( \mathbb{Z}_2^2 \)-graded with

\[
A(\bar{0},\bar{0}) = FE_1 + FE_2 + FE_3,
\]
\[
A(\bar{1},\bar{0}) = \nu_1(O), \quad A(\bar{0},\bar{1}) = \nu_2(O), \quad A(\bar{1},\bar{1}) = \nu_3(O).
\]
\( \mathbb{Z}_2^5 \)-grading

\( \mathbb{A} \) is naturally \( \mathbb{Z}_2^2 \)-graded with

\[
\mathbb{A}(\bar{0},\bar{0}) = F E_1 + F E_2 + F E_3, \\
\mathbb{A}(\bar{1},\bar{0}) = \iota_1(\mathbb{O}), \\
\mathbb{A}(\bar{0},\bar{1}) = \iota_2(\mathbb{O}), \\
\mathbb{A}(\bar{1},\bar{1}) = \iota_3(\mathbb{O}).
\]

This \( \mathbb{Z}_2^2 \)-grading may be combined with the fine \( \mathbb{Z}_2^3 \)-grading on \( \mathbb{O} \) to obtain a fine \( \mathbb{Z}_2^5 \)-grading:

\[
\deg E_i = (\bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{0}), \quad i = 1, 2, 3, \\
\deg \iota_1(x) = (\bar{1}, \bar{0}, \deg x), \quad \deg \iota_2(x) = (\bar{0}, \bar{1}, \deg x), \quad \deg \iota_3(x) = (\bar{1}, \bar{1}, \deg x).
\]
Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading $\Gamma$ is defined by setting

$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$
$\deg \iota_3(i) = a + b + c_1, \quad \deg \iota_3(j) = a + b + c_2, \quad \deg \iota_3(l) = a + b + c_3.$

**Theorem**

Let $\Gamma$ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \bigoplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$W(\Gamma) = \{ \mu \in \text{Aut}(\mathbb{Z}_2^5) : \mu(T) = T \}.$
Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading $\Gamma$ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$$
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**Theorem**

Let $\Gamma$ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \bigoplus_{i=1}^{3} \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{ \mu \in \text{Aut}(\mathbb{Z}_2^5) : \mu(T) = T \}.$$ 

**Remark**

Any $\psi \in \text{Stab}(\Gamma)$ fixes $E_i$ and multiplies $\iota_i(i), \iota_i(j), \iota_i(l), i = 1, 2, 3$, by either 1 or $-1$. Hence $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^5$. 
Take an element $i \in \mathbb{F}$ with $i^2 = -1$ and consider the following elements in $\mathbb{A}$:

\[
E = E_1, \quad \tilde{E} = 1 - E = E_2 + E_3,
\]

\[
\nu(a) = i\nu_1(a) \quad \text{for all} \quad a \in O_0,
\]

\[
\nu_\pm(x) = \nu_2(x) \pm i\nu_3(\bar{x}) \quad \text{for all} \quad x \in O,
\]

\[
S_\pm = E_3 - E_2 \pm \frac{i}{2}\nu_1(1).
\]

$\mathbb{A}$ is then 5-graded:

\[
\mathbb{A} = \mathbb{A}_{-2} \oplus \mathbb{A}_{-1} \oplus \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2,
\]

with $\mathbb{A}_{\pm 2} = \mathbb{F}S_\pm$, $\mathbb{A}_{\pm 1} = \nu_\pm(O)$, and $\mathbb{A}_0 = \mathbb{F}E \oplus \left(\mathbb{F}\tilde{E} \oplus \nu(O_0)\right)$. 
The \( \mathbb{Z}_2^3 \)-grading on \( \mathfrak{O} \) combines with this \( \mathbb{Z} \)-grading

\[
A = \mathbb{F} S^- \oplus \nu^- (\mathfrak{O}) \oplus A_0 \oplus \nu^+ (\mathfrak{O}) \oplus \mathbb{F} S^+
\]

to give a fine \( \mathbb{Z} \times \mathbb{Z}_2^3 \)-grading as follows:

\[
\deg S^\pm = (\pm 2, 0, 0, 0),
\]

\[
\deg \nu_\pm (x) = (\pm 1, \deg x),
\]

\[
\deg E = 0 = \deg \tilde{E},
\]

\[
\deg \nu (a) = (0, \deg a),
\]

for homogeneous elements \( x \in \mathfrak{O} \) and \( a \in \mathfrak{O}_0 \).
Let $\Gamma$ be the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the Albert algebra. Then

$$W(\Gamma) = \text{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).$$
Let \( \Gamma \) be the \( \mathbb{Z} \times \mathbb{Z}^3_2 \)-grading on the Albert algebra. Then

\[
W(\Gamma) = \text{Aut}(\mathbb{Z} \times \mathbb{Z}^3_2).
\]

Remark

One can show that \( \text{Stab}(\Gamma) = \text{Diag}(\Gamma) \), which is isomorphic to \( \mathbb{F}^\times \times \mathbb{Z}^3_2 \).
Consider the order three automorphism $\tau$ of $\mathcal{O}$:

$$\tau(e_i) = e_i, \quad i = 1, 2, \quad \tau(u_j) = u_{j+1}, \quad \tau(v_j) = v_{j+1}, \quad j = 1, 2, 3,$$

and define a new multiplication on $\mathcal{O}$:

$$x \ast y = \tau(\bar{x})\tau^2(\bar{y}).$$
Consider the order three automorphism $\tau$ of $\mathbb{O}$:

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and define a new multiplication on $\mathbb{O}$:

$$x \ast y = \tau(\bar{x})\tau^2(\bar{y}).$$

This is the Okubo algebra, which is $\mathbb{Z}_3^2$-graded by setting

$$\text{deg } e_1 = (\bar{1}, \bar{0}) \quad \text{and} \quad \text{deg } u_1 = (\bar{0}, \bar{1}).$$
\[
\begin{array}{cccccccc}
\text{e}_1 & \text{e}_2 & \text{u}_1 & \text{v}_1 & \text{u}_2 & \text{v}_2 & \text{u}_3 & \text{v}_3 \\
\hline
\text{e}_1 & \text{e}_2 & 0 & -\text{v}_3 & 0 & -\text{v}_1 & 0 & -\text{v}_2 \\
\text{e}_2 & 0 & \text{e}_1 & -\text{u}_3 & 0 & -\text{u}_1 & 0 & -\text{u}_2 \\
\text{u}_1 & -\text{u}_2 & 0 & \text{v}_1 & 0 & -\text{v}_3 & 0 & 0 & -\text{e}_1 \\
\text{v}_1 & 0 & -\text{v}_2 & 0 & \text{u}_1 & 0 & -\text{u}_3 & -\text{e}_2 & 0 \\
\text{u}_2 & -\text{u}_3 & 0 & 0 & -\text{e}_1 & \text{v}_2 & 0 & -\text{v}_1 & 0 \\
\text{v}_2 & 0 & -\text{v}_3 & -\text{e}_2 & 0 & 0 & \text{u}_2 & 0 & -\text{u}_1 \\
\text{u}_3 & -\text{u}_1 & 0 & -\text{v}_2 & 0 & 0 & -\text{e}_1 & \text{v}_3 & 0 \\
\text{v}_3 & 0 & -\text{v}_1 & 0 & -\text{u}_2 & -\text{e}_2 & 0 & 0 & \text{u}_3 \\
\end{array}
\]

Multiplication table of the Okubo algebra
If the characteristic of the ground field is $\neq 3$, then the Okubo algebra $(\mathcal{O}, *)$ is isomorphic to $(\mathfrak{sl}_3(\mathbb{F}), *)$, with

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy) 1,$$

for a primitive cubic root of unity $\omega$. 
If the characteristic of the ground field is \( \neq 3 \), then the Okubo algebra \((\mathcal{O}, \ast)\) is isomorphic to \((\mathfrak{sl}_3(\mathbb{F}), \ast)\), with

\[
x \ast y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \text{tr}(xy)1,
\]

for a primitive cubic root of unity \( \omega \).

The \(\mathbb{Z}_3^2\)-grading on the Okubo algebra is the restriction of the \(\mathbb{Z}_3^2\)-grading on \(\text{Mat}_3(\mathbb{F})\) induced by the Pauli matrices.
Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathcal{O}$. Then the multiplication in the Albert algebra

$$A = \bigoplus_{i=1}^{3} (\mathbb{F}E_i \oplus \tilde{\iota}_i(\mathcal{O}))$$

becomes:

$$E_i \circ E_{i+1} = 0,$$

$$E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2} \tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),$$

$$\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x \ast y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),$$

for $i = 1, 2, 3$ and $x, y \in \mathcal{O}$. 
Assume now \( \text{char } \mathbb{F} \neq 3 \). Then the \( \mathbb{Z}_3^3 \)-grading on the Okubo algebra is determined by two commuting order 3 automorphisms \( \varphi_1, \varphi_2 \in \text{Aut}(\mathbb{O}, \ast) \):

\[
\begin{align*}
\varphi_1(e_1) &= \omega e_1, & \varphi_1(u_1) &= u_1, \\
\varphi_2(e_1) &= e_1, & \varphi_2(u_1) &= \omega u_1,
\end{align*}
\]

where \( \omega \) is a primitive cubic root of unity in \( \mathbb{F} \).
The commuting order 3 automorphisms \( \varphi_1, \varphi_2 \) of \((\mathcal{O}, \ast)\) extend to commuting order 3 automorphisms of \( \mathcal{A} \):

\[
\varphi_j(E_i) = E_i, \quad \varphi_j(\tilde{\iota}_i(x)) = \tilde{\iota}_i(\varphi_j(x)).
\]

On the other hand, the linear map \( \varphi_3 \in \text{End}(\mathcal{A}) \) defined by

\[
\varphi_3(E_i) = E_{i+1}, \quad \varphi_3(\tilde{\iota}_i(x)) = \tilde{\iota}_{i+1}(x),
\]

is another order 3 automorphism, which commutes with \( \varphi_1 \) and \( \varphi_2 \).
The commuting order 3 automorphisms $\varphi_1, \varphi_2$ of $(\mathcal{O}, \ast)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j(\tilde{i}_i(x)) = \tilde{i}_i(\varphi_j(x)).$$

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$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3(\tilde{i}_i(x)) = \tilde{i}_{i+1}(x),$$

is another order 3 automorphism, which commutes with $\varphi_1$ and $\varphi_2$.

The subgroup of $\text{Aut}(\mathbb{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$. 
The commuting order 3 automorphisms $\varphi_1$, $\varphi_2$ of $(\mathcal{O}, \ast)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

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The subgroup of $\text{Aut}(\mathbb{A})$ generated by $\varphi_1$, $\varphi_2$, $\varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$.

All the homogeneous components have dimension 1.
The $\mathbb{Z}_3^3$-grading is determined by

\[
\begin{align*}
\deg \left( \sum_{i=1}^{3} \tilde{\iota}_i(e_1) \right) &= (\bar{1}, \bar{0}, \bar{0}), \\
\deg \left( \sum_{i=1}^{3} \tilde{\iota}_i(u_1) \right) &= (\bar{0}, \bar{1}, \bar{0}), \\
\deg \left( \sum_{i=1}^{3} \omega^{-i} E_i \right) &= (\bar{0}, \bar{0}, \bar{1}),
\end{align*}
\]
The $\mathbb{Z}_3^3$-grading is determined by

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\begin{align*}
\deg(\sum_{i=1}^3 \tilde{\iota}_i(e_1)) &= (\bar{1}, \bar{0}, \bar{0}), \\
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\deg(\sum_{i=1}^3 \omega^{-i} E_i) &= (\bar{0}, \bar{0}, \bar{1}),
\end{align*}
\]

**Theorem**

Let $\Gamma$ be the $\mathbb{Z}_3^3$-grading on the Albert algebra. Then $W(\Gamma)$ is the commutator subgroup of $\text{Aut}(\mathbb{Z}_3^3)$, i.e.,

$W(\Gamma) \cong SL_3(3)$. 

Why $SL_3(3)$ and not $GL_3(3)$?
Why $SL_3(3)$ and not $GL_3(3)$?

Consider the $\mathbb{Z}_3^3$-grading $\Gamma^-$ determined by

$$
\begin{align*}
\deg\left(\sum_{i=1}^{3} \tilde{i}_i(e_1)\right) &= (\bar{0}, \bar{1}, \bar{0}), \\
\deg\left(\sum_{i=1}^{3} \tilde{i}_i(u_1)\right) &= (\bar{1}, \bar{0}, \bar{0}), \\
\deg\left(\sum_{i=1}^{3} \omega^{-i}E_i\right) &= (\bar{0}, \bar{0}, \bar{1}).
\end{align*}
$$
Why $SL_3(3)$ and not $GL_3(3)$?

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$$\deg\left(\sum_{i=1}^3 \tilde{i}_i(u_1)\right) = (\bar{1}, \bar{0}, \bar{0}),$$
$$\deg\left(\sum_{i=1}^3 \omega^{-i} E_i\right) = (\bar{0}, \bar{0}, \bar{1}),$$

Then, for $X_1 \in A_{(\bar{1}, \bar{0}, \bar{0})}$, $X_2 \in A_{(\bar{0}, \bar{1}, \bar{0})}$, $X_3 \in A_{(\bar{0}, \bar{0}, \bar{1})}$, we have:

$$\begin{align*}
(X_1 \circ X_2) \circ X_3 &= \begin{cases}
\omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma, \\
\omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-.
\end{cases}
\end{align*}$$
Why $SL_3(3)$ and not $GL_3(3)$?

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\deg(\sum_{i=1}^3 \tilde{\iota}_i(e_1)) &= (\bar{0}, \bar{1}, \bar{0}), \\
\deg(\sum_{i=1}^3 \tilde{\iota}_i(u_1)) &= (\bar{1}, \bar{0}, \bar{0}), \\
\deg(\sum_{i=1}^3 \omega^{-i}E_i) &= (\bar{0}, \bar{0}, \bar{1}),
\end{align*}
\]

Then, for $X_1 \in \mathbb{A}_{(\bar{1},\bar{0},\bar{0})}$, $X_2 \in \mathbb{A}_{(\bar{0},\bar{1},\bar{0})}$, $X_3 \in \mathbb{A}_{(\bar{0},\bar{0},\bar{1})}$, we have:

\[
(X_1 \circ X_2) \circ X_3 = \begin{cases} \\
\omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma, \\
\omega^{-1}X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-.
\end{cases}
\]

Hence $\Gamma$ and $\Gamma^-$ are equivalent, but NOT isomorphic, gradings on $\mathbb{A}$ by $\mathbb{Z}_3^3$. 
Why $SL_3(3)$ and not $GL_3(3)$?

Consider the $\mathbb{Z}_3^3$-grading $\Gamma^-$ determined by

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\begin{align*}
\deg \left( \sum_{i=1}^{3} \tilde{\iota}_i(e_1) \right) &= (\bar{0}, \bar{1}, \bar{0}), \\
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\deg \left( \sum_{i=1}^{3} \omega^{-i} E_i \right) &= (\bar{0}, \bar{0}, \bar{1}),
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Then, for $X_1 \in \mathbb{A}_{(\bar{1}, \bar{0}, \bar{0})}$, $X_2 \in \mathbb{A}_{(\bar{0}, \bar{1}, \bar{0})}$, $X_3 \in \mathbb{A}_{(\bar{0}, \bar{0}, \bar{1})}$, we have:

$$(X_1 \circ X_2) \circ X_3 = \begin{cases} 
\omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma, \\
\omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-.
\end{cases}$$

Hence $\Gamma$ and $\Gamma^-$ are equivalent, but NOT isomorphic, gradings on $\mathbb{A}$ by $\mathbb{Z}_3^3$. Besides, any fine $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ is isomorphic to either $\Gamma$ or $\Gamma^-$, so $W(\Gamma)$ has index two in $\text{Aut}(U(\Gamma)) \cong GL_3(3)$. 
$\mathbb{Z}_3^3$-grading and the Tits construction

Let $R = \text{Mat}_3(F)$. Then

$$A = R_0 \oplus R_1 \oplus R_2,$$

with $R_0, R_1, R_2$ copies of $R$. 

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Let $\mathcal{R} = \text{Mat}_3(\mathbb{F})$. Then

$$\mathbb{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2,$$

with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$. The product in $\mathbb{A}$ satisfies $\mathcal{R}_i \circ \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$ (mod 3) and it is given by:

\[
\begin{array}{c|ccc}
\circ & a'_0 & b'_1 & c'_2 \\
\hline
a_0 & (a \circ a')_0 & (\bar{a}b')_1 & (c'\bar{a})_2 \\
b_1 & (\bar{a}'b)_1 & (b \times b')_2 & (bc')_2 \\
c_2 & (c\bar{a}')_2 & (\bar{b}'c)_0 & (c \times c')_1 \\
\end{array}
\]

where

- $a \circ a' = \frac{1}{2}(aa' + a'a)$,
- $a \times b = a \circ b - \frac{1}{2} (\text{tr}(a)b + \text{tr}(b)a) + \frac{1}{2} (\text{tr}(a) \text{tr}(b) - \text{tr}(ab))1$,
- $\bar{a} = a \times 1 = \frac{1}{2} (\text{tr}(a)1 - a)$. 
Assume $\text{char } \mathbb{F} \neq 3$. Take Pauli matrices in $\mathcal{R}$:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\omega, \omega^2$ are the primitive cubic roots of 1, which satisfy

$$x^3 = 1 = y^3, \quad xy = \omega xy.$$
Assume \( \text{char } \mathbb{F} \neq 3 \). Take Pauli matrices in \( \mathcal{R} \):

\[
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\]

where \( \omega, \omega^2 \) are the primitive cubic roots of 1, which satisfy

\[
x^3 = 1 = y^3, \quad yx = \omega xy.
\]

These Pauli matrices give a grading over \( \mathbb{Z}_3^2 \) on \( \mathcal{R} \), with

\[
\mathcal{R}(\alpha_1, \alpha_2) = \mathbb{F} x^{\alpha_1} y^{\alpha_2}.
\]
**$\mathbb{Z}_3^3$-grading and the Tits construction**

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$$
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\end{align*}
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where $\omega, \omega^2$ are the primitive cubic roots of 1, which satisfy

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These Pauli matrices give a grading over $\mathbb{Z}_3^2$ on $\mathcal{R}$, with

$$
    \mathcal{R}(\alpha_1, \alpha_2) = \mathbb{F}x^{\alpha_1}y^{\alpha_2}.
$$

This grading combines with the $\mathbb{Z}_3$-grading on $\mathfrak{A}$ induced by Tits construction, to give the unique, up to equivalence, fine grading over $\mathbb{Z}_3^3$ of the Albert algebra.
For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1}y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq \mathbb{A}.$$
For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq A.$$ 

Then, for any $\alpha, \beta \in \mathbb{Z}_3^3$:

$$Z^\alpha \circ Z^\beta = \begin{cases} \omega \tilde{\psi}(\alpha, \beta) Z^{\alpha + \beta} & \text{if } \dim_{\mathbb{Z}_3} (\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\ -\frac{1}{2} \omega \tilde{\psi}(\alpha, \beta) Z^{\alpha + \beta} & \text{otherwise,} \end{cases}$$

where

$$\tilde{\psi}(\alpha, \beta) = (\alpha_2 \beta_1 - \alpha_1 \beta_2)(\alpha_3 - \beta_3) - (\alpha_1 \beta_2 + \alpha_2 \beta_1).$$
Consider now the elements (Racine 1990, unpublished)

\[ W^\alpha := \omega^{-\alpha_1 \alpha_2} Z^\alpha. \]
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\[ W^\alpha := \omega^{-\alpha_1 \alpha_2} Z^\alpha. \]

\[ W^\alpha \circ W^\beta = \omega^{-\alpha_1 \alpha_2 - \beta_1 \beta_2} Z^\alpha \circ Z^\beta \]

\[
\begin{cases}
\omega \tilde{\psi}(\alpha,\beta) - (\alpha_1 \alpha_2 + \beta_1 \beta_2) Z^\alpha + \beta & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\
-\frac{1}{2} \omega \tilde{\psi}(\alpha,\beta) - (\alpha_1 \alpha_2 + \beta_1 \beta_2) Z^\alpha + \beta & \text{otherwise,}
\end{cases}
\]

\[
\begin{cases}
\omega \tilde{\psi}(\alpha,\beta) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) W^\alpha + \beta & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\
-\frac{1}{2} \omega \tilde{\psi}(\alpha,\beta) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) W^\alpha + \beta & \text{otherwise.}
\end{cases}
\]
The Albert algebra as a twisted group algebra

**Theorem (Griess 1990)**

The Albert algebra is, up to isomorphism, the twisted group algebra

\[ \mathbb{A} = \mathbb{F}_\sigma [\mathbb{Z}_3^3], \]

with

\[ \sigma(\alpha, \beta) = \begin{cases} 
\omega \psi(\alpha, \beta) & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3 \alpha + \mathbb{Z}_3 \beta) \leq 1, \\
-\frac{1}{2} \omega \psi(\alpha, \beta) & \text{otherwise},
\end{cases} \]

where

\[ \psi(\alpha, \beta) = (\alpha_2 \beta_1 - \alpha_1 \beta_2)(\alpha_3 - \beta_3). \]
Theorem (Draper–Martín-González 2009 (char = 0), E.–Kochetov 2010)

Up to equivalence, the fine gradings of the Albert algebra are:

1. The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \text{Der}(A)$).
2. The $\mathbb{Z}_2^5$-grading obtained by combining the natural $\mathbb{Z}_2^2$-grading on $3 \times 3$ hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathfrak{O}$.
3. The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathfrak{O}$.
4. The $\mathbb{Z}_3^3$-grading with $\dim A_g = 1 \ \forall g$ (char $F \neq 3$).
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4. The $\mathbb{Z}_3^3$-grading with $\text{dim } \mathbb{A}_g = 1 \ \forall g$ (char $\mathbb{F} \neq 3$).

All the gradings up to isomorphism on $\mathbb{A}$ have been classified too (E.–Kochetov).
1 Gradings

2 The algebra of Octonions

3 The Albert algebra

4 $G_2$ and $F_4$
Gradings and comodule algebras

$G$-grading $\leftrightarrow$ comodule algebra over the group algebra $\mathbb{F}G$
Gradings and comodule algebras

$G$-grading $\iff$ comodule algebra over the group algebra $\mathbb{F}G$

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \Rightarrow \quad \rho_{\Gamma} : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathbb{F}G$$

$$x_g \mapsto x_g \otimes g$$

(algebra morphism and comodule structure)
Gradings and comodule algebras

\[ \Gamma : A = \bigoplus_{g \in G} A_g \quad \Rightarrow \quad \rho_\Gamma : A \rightarrow A \otimes \mathbb{F}G \]
\[ x_g \mapsto x_g \otimes g \]
(algebra morphism and comodule structure)

\[ \Gamma_\rho : A = \bigoplus_{g \in G} A_g \quad \Leftarrow \quad \rho : A \rightarrow A \otimes \mathbb{F}G \]
\[ (A_g = \{ x \in A : \rho(x) = x \otimes g \}) \]
A comodule algebra map

\[ \rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \]

induces a \textit{generic automorphism} of $\mathbb{F}G$-algebras

\[ \mathcal{A} \otimes \mathbb{F}G \longrightarrow \mathcal{A} \otimes \mathbb{F}G \]
\[ x \otimes h \mapsto \rho(x)h. \]
A comodule algebra map

$$\rho : A \to A \otimes FG$$

induces a *generic automorphism* of $FG$-algebras

$$A \otimes FG \longrightarrow A \otimes FG$$

$$x \otimes h \mapsto \rho(x)h.$$ 

All the information on the grading $\Gamma$ attached to $\rho$ is contained in this single automorphism!
Gradings and affine group schemes

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \Leftrightarrow \quad \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \]

(comodule algebra structure)
\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \Leftrightarrow \quad \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{FG} \]

(comodule algebra structure)

Now,

\[ \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{FG} \quad \Leftrightarrow \quad \eta_{\Gamma} : G^D \to \textbf{Aut} \mathcal{A} \]

(comodule algebra) \quad (morphism of affine group schemes)
Gradings and affine group schemes

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \iff \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \]

(comodule algebra structure)

Now,

\[ \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \iff \eta_{\Gamma} : G^D \to \text{Aut} \mathcal{A} \]

(comodule algebra) (morphism of affine group schemes)

For any \( \varphi \in G^D(\mathcal{R}) \), \( \eta_{\Gamma}(\varphi) \in \text{Aut}_\mathcal{R}(\mathcal{A} \otimes \mathcal{R}) \) is given by:

\[ \eta_{\Gamma}(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g)r. \]
Gradings and affine group schemes

\[ \Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \iff \quad \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \]

(comodule algebra structure)

Now,

\[ \rho_{\Gamma} : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \quad \iff \quad \eta_{\Gamma} : G^D \to \text{Aut} \mathcal{A} \]

(comodule algebra) \quad (morphism of affine group schemes)

For any \( \varphi \in G^D(\mathcal{R}) \), \( \eta_{\Gamma}(\varphi) \in \text{Aut}_\mathcal{R}(\mathcal{A} \otimes \mathcal{R}) \) is given by:

\[ \eta_{\Gamma}(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g)r. \]

and \( \rho_{\Gamma} \) is recovered as

\[ \rho_{\Gamma}(x) = \eta_{\Gamma}(id_{\mathbb{F}G})(x \otimes 1) \quad \left( \eta_{\Gamma}(id_{\mathbb{F}G}) \in \text{Aut}_{\mathbb{F}G}(\mathcal{A} \otimes \mathbb{F}G) \right) \]
Consider a homomorphism $\Phi : \text{Aut} \mathcal{A} \longrightarrow \text{Aut} \mathcal{A}'$ of affine group schemes.

Then any grading $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ induces a grading $\Gamma' : \mathcal{A}' = \bigoplus_{g \in G} \mathcal{A}'_g$ by means of:

$$\eta_{\Gamma'} = \Phi \circ \eta_{\Gamma} : G \rightarrow \text{Aut} \mathcal{A}'$$

If $\Gamma_1 : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma_2 : \mathcal{A} = \bigoplus_{h \in H} \mathcal{A}_h$ are weakly isomorphic, then the induced gradings $\Gamma'_1$ and $\Gamma'_2$ on $\mathcal{A}'$ are weakly isomorphic too through the automorphism $\Phi_F(\psi) \in \text{Aut} \mathcal{A}'$ and $\phi : G \rightarrow H$. 
Consider a homomorphism $\Phi : \text{Aut} \mathcal{A} \rightarrow \text{Aut} \mathcal{A}'$ of affine group schemes. Then any grading $\Gamma : \mathcal{A} = \bigoplus_{g \in \mathcal{G}} \mathcal{A}_g$ induces a grading $\Gamma' : \mathcal{A}' = \bigoplus_{g \in \mathcal{G}} \mathcal{A}'_g$ by means of:

$$\eta_{\Gamma'} = \Phi \circ \eta_\Gamma : \mathcal{G}^D \rightarrow \text{Aut} \mathcal{A}' .$$
Consider a homomorphism $\Phi : \text{Aut } A \longrightarrow \text{Aut } A'$ of affine group schemes. Then any grading $\Gamma : A = \oplus_{g \in G} A_g$ induces a grading $\Gamma' : A' = \oplus_{g \in G} A'_g$ by means of:

$$\eta_{\Gamma'} = \Phi \circ \eta_{\Gamma} : G^D \rightarrow \text{Aut } A'.$$

If $\Gamma_1 : A = \oplus_{g \in G} A_g$ and $\Gamma_2 : A = \oplus_{h \in H} A'_h$ are weakly isomorphic, then the induced gradings $\Gamma'_1$ and $\Gamma'_2$ on $A'$ are weakly isomorphic too through the automorphism $\Phi_{F}(\psi) \in \text{Aut } A'$ and $\varphi : G \rightarrow H.$
For $G = \text{Aut} \, A$, $\text{Lie}(G) = \text{Der}(A)$, so

$$\text{Ad} : \text{Aut} \, A \to \text{Aut}(\text{Der}(A))$$

is a homomorphism, and any grading $\Gamma : A = \bigoplus_{g \in G} A_g$ induces a grading

$$\Gamma' : \text{Der}(A) = \bigoplus_{g \in G} \text{Der}(A)_g,$$

$$\text{Der}(A)_g = \{ d \in \text{Der}(A) : d(A_h) \subseteq A_{gh} \, \forall h \in G \}.$$

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If $\text{Aut} \ A \cong \text{Aut} \ B$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $A$ and $B$ are equivalent.
If $\text{Aut } A \cong \text{Aut } B$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $A$ and $B$ are equivalent.

If the characteristic of the ground field $F$ is $\neq 2, 3$, then

$$\text{Ad} : \text{Aut } \mathcal{O} \to \text{Aut } g_2$$

is an isomorphism, and (assuming just $\text{char } F \neq 2$),

$$\text{Ad} : \text{Aut } A \to \text{Aut } f_4$$

is an isomorphism too.
Up to equivalence, the fine gradings on \( g_2 \) are

- the Cartan grading, and

- a \( \mathbb{Z}_2^3 \)-grading with \( (g_2)_0 = 0 \) and where \( (g_2)_g \) is a Cartan subalgebra of \( g_2 \) for any \( 0 \neq g \in \mathbb{Z}_2^3 \).
Theorem

Up to equivalence, the fine gradings on $\mathfrak{f}_4$ are

- the Cartan grading,
- a grading by $\mathbb{Z}_2^5$, obtained by combining the $\mathbb{Z}_2^2$-grading given by the decomposition $\mathfrak{f}_4 = \mathfrak{o}_4 \oplus \text{natural} \oplus \text{spin} \oplus \text{spin}$, with the $\mathbb{Z}_3^2$-grading on the octonions (which is the vector space behind the natural and spin representations of $\mathfrak{o}_4$).
- a grading by $\mathbb{Z} \times \mathbb{Z}_2^3$, obtained by looking at $\mathfrak{f}_4$ as the Kantor Lie algebra of a structurable algebra: $\mathfrak{f}_4 = \mathcal{K}(\mathbb{O}, -)$, and combining the natural 5-grading on $\mathcal{K}(\mathbb{O}, -)$ and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.
- a $\mathbb{Z}_3^3$-grading (only if $\text{char } \mathbb{F} \neq 3$), with $(\mathfrak{f}_4)_0 = 0$ and where $(\mathfrak{f}_4)_g \oplus (\mathfrak{f}_4)_{-g}$ is a Cartan subalgebra of $\mathfrak{f}_4$ for any $0 \neq g \in \mathbb{Z}_3^3$. 
Gradings on $F_4$

**Theorem**

Up to equivalence, the fine gradings on $f_4$ are

1. the Cartan grading,
2. a grading by $\mathbb{Z}_2^5$, obtained by combining the $\mathbb{Z}_2^2$-grading given by the decomposition $f_4 = \mathfrak{so}_4 \oplus \text{natural} \oplus \text{spin} \oplus \text{spin}$, with the $\mathbb{Z}_2^3$-grading on the octonions (which is the vector space behind the natural and spin representations of $\mathfrak{so}_4$).
3. a grading by $\mathbb{Z} \times \mathbb{Z}_2^3$, obtained by looking at $f_4$ as the Kantor Lie algebra of a structurable algebra: $f_4 = \mathcal{K}(\mathbb{O}, -)$, and combining the natural 5-grading on $\mathcal{K}(\mathbb{O}, -)$ and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.
4. a $\mathbb{Z}_3^3$-grading (only if $\text{char } \mathbb{F} \neq 3$), with $(f_4)_0 = 0$ and where $(f_4)_g \oplus (f_4)_{-g}$ is a Cartan subalgebra of $f_4$ for any $0 \neq g \in \mathbb{Z}_3^3$.

That’s all. Thanks

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