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A depolarization criterion in Mueller matrices

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Abstract. A necessary and sufficient scalar condition for a Mueller matrix **M** to describe a non-depolarizing optical system is obtained. By computing only one scalar parameter, it can be determined whether a given Mueller matrix **M** describes a non-depolarizing, or a depolarizing optical system. The following theorem is stated: a necessary and sufficient condition for a given Mueller matrix **M** to describe a non-depolarizing optical system is $\mathrm{Tr}\,(\mathbf{M}^{\mathrm{T}}\mathbf{M}) = 4m_{00}^2$.

1. Introduction

The nature and mathematical expression of the relationships between the 16 elements of a Mueller matrix have been the subject of a number of previous papers [1, 2]. Abhyankar and Fymat [3] proved some relationships between the elements of the matrix that describes a non-depolarizing system in several matricial representations of polarization phenomena. Fry and Kattawar [4] performed a more detailed study of these relationships in the Stokes–Mueller formalism (SMF), which included some inequalities for depolarizing systems. A system of equations equivalent to the one given by the above authors [1, 2, 3], but with a different mathematical expression, was obtained by Barakat [5], and Simon [6] later obtained a matrix condition for a Mueller matrix, **M**, to describe a totally polarized system. In this paper we show that, because of the peculiar structure of Mueller matrices, which satisfy a series of constraining inequalities [2, 7], the set of nine scalar equalities involved in Simon's matrix condition can be replaced by only one set of scalar conditions, namely $\mathrm{Tr}\,(\mathbf{M}^{\mathrm{T}}\mathbf{M}) = 4m_{00}^2$.

2. Condition of the norm in Mueller matrices

For every Mueller matrix **M** we can define a positive-semidefinite norm $\Gamma(\mathbf{M})$, given by

$$\Gamma(\mathbf{M}) = [\mathrm{Tr}\,(\mathbf{M}^{\mathrm{T}}\mathbf{M})]^{1/2} = \left[\sum_{i,j=0}^{3} m_{ij}^2\right]^{1/2} \qquad (1)$$

where $m_{ij} (i, j = 0, 1, 2, 3)$ are the elements of **M**, Tr represents the trace, and $\mathbf{M}^{\mathrm{T}}$ is the transposed matrix of **M**.

Fry and Kattawar [4] proved that for every non-depolarizing Mueller matrix **M** we can write

$$\sum_{i,j=0}^{3} m_{ij}^2 = 4m_{00}^2. \tag{2}$$

This equation can also be written as

$$\Gamma(\mathbf{M}) = 2m_{00}. \tag{3}$$

So, equation (3) is a necessary condition for **M** to describe a non-depolarizing system. However, we are now going to prove that equation (3) is also a sufficient condition.

For an incident, totally polarized light beam, a depolarizing optical system produces an incoherent superposition of totally polarized outgoing light beams with different polarizations. Taking into account the principle of optical equivalence of polarization states [1], the above observation implies that a depolarizing optical system is optically equivalent to a system composed of a parallel combination of several non-depolarizing optical systems. Then, the outgoing light beam is optically equivalent to the one given by the superposition of the light beams emergent from each of the non-depolarizing optical systems of the equivalent parallel combination.

The Mueller matrix of a parallel combination of optical systems is given by the sum of the Mueller matrices of the systems which form the parallel combination [1]. Therefore, a depolarizing Mueller matrix can be written as the sum of various non-depolarizing Mueller matrices. Since a depolarizing Mueller matrix depends, in general, on 16 independent parameters [1], and a non-depolarizing Mueller matrix depends, in general, on 7 independent parameters [1], we see that a depolarizing Mueller matrix can be written as the sum of at least 3 non-depolarizing Mueller matrices. Therefore we can write

$$\mathbf{M} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \dots \tag{4}$$

where **A**, **B**, **C**, . . . , are different non-depolarizing Mueller matrices. They satisfy the following relations:

$$\left.\begin{array}{l} \mathrm{Tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \mathrm{Tr}\,(\mathbf{A}\mathbf{A}^{\mathrm{T}}) = 4a_{00}^2 \\ \mathrm{Tr}\,(\mathbf{B}^{\mathrm{T}}\mathbf{B}) = \mathrm{Tr}\,(\mathbf{B}\mathbf{B}^{\mathrm{T}}) = 4b_{00}^2 \\ \mathrm{Tr}\,(\mathbf{C}^{\mathrm{T}}\mathbf{C}) = \mathrm{Tr}\,(\mathbf{C}\mathbf{C}^{\mathrm{T}}) = 4c_{00}^2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{array}\right\}. \tag{5}$$

The decomposition given by equation (4) is not unique, but this fact does not affect our demonstration.

For simplicity, we will consider the case in which **M** is obtained from only two matrices, **A** and **B**, so that

$$\mathbf{M} = \mathbf{A} + \mathbf{B}. \tag{6}$$

In order to prove that equation (3) is a sufficient condition, we assume that it is satisfied by **M**. Equation (6) allows us to write

$$\mathrm{Tr}\,(\mathbf{M}^{\mathrm{T}}\mathbf{M}) = \mathrm{Tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{B}^{\mathrm{T}}\mathbf{B}), \tag{7}$$

and

$$m_{00} = a_{00} + b_{00}. \tag{8}$$

Then

$$4m_{00}^2 = 4a_{00}^2 + 4b_{00}^2 + 8a_{00}b_{00}. \quad (9)$$

Since $\mathrm{Tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \mathrm{Tr}\,(\mathbf{B}^{\mathrm{T}}\mathbf{A})$, we can see that

$$\mathrm{Tr}\,(\mathbf{M}^{\mathrm{T}}\mathbf{M}) = 4a_{00}^2 + 4b_{00}^2 + 2\mathrm{Tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{B}). \quad (10)$$

Thus, equations (2), (9) and (10) lead to

$$\mathrm{Tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = \mathrm{Tr}\,(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = 4a_{00}b_{00}. \quad (11)$$

From equations (5), and squaring in equation (11), we can write

$$16a_{00}^2b_{00}^2 = \mathrm{Tr}\,(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathrm{Tr}\,(\mathbf{B}^{\mathrm{T}}\mathbf{B}) = (\mathrm{Tr}\,(\mathbf{B}^{\mathrm{T}}\mathbf{A}))^2. \quad (12)$$

However, taking into account the relation

$$\mathrm{Tr}\,(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = \sum_{i,j=0}^{3} a_{ij}b_{ij}, \quad (13)$$

equation (2) becomes

$$\left(\sum_{i,j=0}^{3} a_{ij}^2\right)\left(\sum_{i,j=0}^{3} b_{ij}^2\right) = \left(\sum_{i,j=0}^{3} a_{ij}b_{ij}\right)^2. \quad (14)$$

This relation is only satisfied if $\mathbf{A} = \mathbf{B}$, so

$$\mathbf{M} = 2\mathbf{A} = 2\mathbf{B}. \quad (15)$$

In the general case of equation (3) we would have obtained

$$\mathbf{A} = \mathbf{B} = \mathbf{C} = \ldots, \quad (16)$$

Therefore, since **A** is a non-depolarizing Mueller matrix, and **M** is proportional to it, we can say that **M** is also non-depolarizing.

We can summarize the above results with the following statement: a Mueller matrix **M** describes a non-depolarizing optical system if, and only if, it satisfies equation (3). This statement allows to to discover quickly, and through a simple criterion, whether or not a sample, whose Mueller matrix has been obtained experimentally, can produce depolarization of light.

References

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