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Alberto Elduque  
Alicia Labra

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**ON THE CLASSIFICATION OF COMMUTATIVE  
RIGHT-NILALGEBRAS OF NILINDEX FIVE AND  
DIMENSION FOUR<sup>1</sup>**

**Alberto Elduque<sup>2</sup>, and Alicia Labra<sup>3</sup>**

<sup>2</sup> Departamento de Matemáticas, Facultad de Ciencias,  
Universidad de Zaragoza, 50009 Zaragoza, Spain  
e-mail: elduque@unizar.es

<sup>3</sup> Departamento de Matemáticas, Facultad de Ciencias  
Universidad de Chile, Casilla 653, Santiago-Chile  
e-mail: alimat@uchile.cl

Abstract

Gerstenhaber and Myung in [10] classified all commutative-power associative nilalgebras of dimension 4. In [7] Gerstenhaber and Myung's results are generalized by giving a classification of commutative right-nilalgebras of right-nilindex four and dimension at most four, without assuming power-associativity. In this paper we complete this research and give a classification of commutative right-nilalgebras of right-nilindex five and dimension four, without assuming power-associativity, thus completing the classification of commutative right-nilalgebras of dimension at most four.

**Key words:** nilpotency, right-nilalgebras, power-associative.

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# 1 Introduction

The problem of nilpotency in a commutative power-associative algebra is known as *Albert's problem* [1] (1948): Is every commutative finite dimensional power-associative nilalgebra nilpotent?

Suttles [14] in 1972, gave an example of a commutative five dimensional power-associative nilalgebra of nilindex 4 which is solvable and not nilpotent, thus showing Albert's conjecture to be false and forcing a new reformulation of the conjecture. Therefore what is now really known by the Albert conjecture is whether any commutative finite dimensional power associative nilalgebra is solvable.

Gerstenhaber and Myung [10] proved that any commutative and power-associative nilalgebra of nilindex 4 and dimension 4 over a field of characteristic  $\neq 2$  is nilpotent. There are many others papers dealing with Albert's problem (see for instance [2], [3], [4], [5], [6], [8], [9], [11] and [12]).

Moreover, in [10] Gerstenhaber and Myung determined the isomorphism classes of commutative power-associative nilalgebras of nilindex four and dimension four. They found one family of algebras parameterized by  $F^\times/(F^\times)^2$  and 5 individual algebras, two of them being associative.

More recently, Elduque and Labra in [7] determined the isomorphism classes of commutative right-nilalgebras of nilindex four and dimension four. They do not assume power-associativity. They found 7 individual non isomorphic algebras.

In this paper we complete this study by giving a classification of commu-

tative right-nilalgebras of right-nilindex five and dimension four, over fields of characteristic  $\neq 2$  without assuming power-associativity. Extending thus the classification given by Gerstenhaber and Myung.

## 2 Right-nilpotency

Let  $A$  be a nonassociative algebra and let  $x \in A$ . We define the *right-principal powers of  $x$*  by  $x^1 = x$  and  $x^{n+1} = x^n x$  for all  $n \geq 1$ . An element  $x \in A$  is called *right-nilpotent with right-nilindex  $n > 1$*  if  $x^n = 0$  and  $x^{n-1} \neq 0$ .  $A$  is called *right-nilalgebra with right-nilindex  $n$*  if  $x^n = 0$  for all  $x \in A$  and there exists some  $a$  in  $A$  with  $a^{n-1} \neq 0$ .

The algebra  $A$  is called *nilpotent* (respectively, *right-nilpotent*) in case the descending chain of ideals (respectively right ideals) defined by  $A^1 = A$ ,  $A^n = \sum_{r+s=n} A^r A^s = \sum_{i=1}^{n-1} A^i A^{n-i}$  for  $n > 1$  ( or  $A^{<1>} = A$  and  $A^{<n>} = A^{<n-1>} A$  for  $n > 1$ ) ends up in zero. The smallest  $r$  such that  $A^r = 0$  (respectively  $A^{<r>} = 0$ ) is called the *index of nilpotency* (respectively *index of right-nilpotency*) of  $A$ . Clearly, if  $A$  is nilpotent then  $A$  is right-nilpotent. Moreover, if  $A$  is commutative or anticommutative, then  $A^5 = A^3 A^2 + A^4 A \subseteq A^3 A = A^3 A = A^{<4>}$ . Thus, if  $A^{<4>} = 0$ , then  $A$  is nilpotent with nilpotent index at most 5.

More generally, it is known (see [15, Proposition 1]) that if  $A$  is a commutative or anti-commutative algebra, then  $A^{2^n} \subseteq A^{<n>}$ . Therefore if  $A$  is right-nilpotent, it is nilpotent too.

*Throughout the paper, all the algebras considered will be commutative and defined over a ground field  $F$  of characteristic  $\neq 2$ .*

## 2.1 Nilindex 5

In what follows  $A$  will be a commutative nonassociative algebra that satisfies strictly the identity  $x^5 = (((xx)x)x)x = 0$ , and contains an element  $a$  in  $A$  such that  $a^4 \neq 0$ . This means that  $A$  satisfies the identity  $x^5 = 0$  and all its linearizations.

The following notation will be used. Given a set  $S$ ,  $\langle S \rangle$  will denote the subspace generated by  $S$ , while  $\text{alg}\langle S \rangle$  will be the subalgebra generated by  $S$ .

**Theorem 1.** *Let  $A$  be a commutative algebra of dimension 4 over a field  $F$ ,  $\text{char}(F) \neq 2$ , which satisfies the identity  $x^5 = 0$  strictly and contains elements  $y$  with  $y^4 \neq 0$ . Then  $A$  is nilpotent.*

**Proof:** Extending scalars if necessary to get an infinite ground field, the set  $\mathcal{G} = \{x \in A \mid x^4 \neq 0\} = \{x \in A \mid \{x, x^2, x^3, x^4\} \text{ is a basis of } A\}$  is Zariski-open and not empty in  $A$ , so it is dense and we conclude that  $L_x^4 = 0$ , for any  $x \in \mathcal{G}$  where  $L_x(y) = xy$  is the left multiplication operator and, by Zariski density, we have that this is true for any  $x \in A$ . That is

$$L_x^4 = 0 \quad \forall x \in A. \tag{1}$$

All references to density will refer to density in Zariski topology (for its definition and main features on not necessarily finite dimensional spaces one may consult [13]).

Using the nilpotency of the operator  $L_x$ , the first linearization of the identity  $x^5 = 0$  becomes  $((x^2y)x)x + (x^3y)x + x^4y = 0$  for every  $x, y \in A$ .

Expressing this identity in terms of multiplication operators we obtain:

$$L_x^2 L_{x^2} + L_x L_{x^3} + L_{x^4} = 0 \quad \forall x \in A. \quad (2)$$

Fix an element  $x \in \mathcal{G}$ , then for every  $y \in A$ , there exist  $\alpha_{y,x^i} \in F$ ,  $i = 1, \dots, 4$  such that  $y = \alpha_{y,x} x + \alpha_{y,x^2} x^2 + \alpha_{y,x^3} x^3 + \alpha_{y,x^4} x^4$ . The equation (2) is equivalent to the following conditions for every  $y \in A$ ,

$$\alpha_{x^4 y, x} = 0, \quad (3)$$

$$\alpha_{x^4 y, x^2} + \alpha_{x^3 y, x} = 0, \quad (4)$$

$$\alpha_{x^4 y, x^3} + \alpha_{x^3 y, x^2} + \alpha_{x^2 y, x} = 0, \quad (5)$$

$$\alpha_{x^4 y, x^4} + \alpha_{x^3 y, x^3} + \alpha_{x^2 y, x^2} = 0. \quad (6)$$

In particular we have that for  $a, b, c, d, e, f \in F$ ,

$$x^4 x^4 = ax^3 + bx^4, \quad x^3 x^4 = -ax^2 + cx^3 + dx^4, \quad x^2 x^4 = -(b+c)x^2 + ex^3 + fx^4$$

If  $a = 0$ , then the nilpotency of the operator  $L_{x^4}$  implies that  $b = 0$ . Since  $x^3(x^4 - dx) = cx^3$ , the nilpotency of  $L_{x^4-dx}$  implies that  $c = 0$  and  $x^3 x^4 = dx^4$ . Therefore  $d = 0$ , as  $L_{x^3}$  is nilpotent. In this way we have that  $x^3 x^4 = 0 = x^4 x^4$ . Now (4) implies that  $\alpha_{x^3 x^2, x} = 0 = \alpha_{x^3 x^3, x}$  and then (5) implies that  $\alpha_{x^3 x^3, x^2} = 0$ . Therefore  $x^3 x^3 = gx^3 + hx^4$ ,  $g, h \in F$ , that is,  $x^3(x^3 - hx) = gx^3$  and the nilpotency of  $L_{x^3-hx}$  implies that  $g = \alpha_{x^3 x^3, x^3} = 0$ . Now (6) implies that  $\alpha_{x^2 x^3, x^2} = 0$ . Therefore,  $I = \langle x^3, x^4 \rangle$  is an ideal of  $A$  and  $A/I$  is a two-dimensional right-nilalgebra, then  $A$  is nilpotent. In particular, we have that  $x^2 x^2 \in \langle x^3, x^4 \rangle$  and (5) implies that  $\alpha_{x^4 x^2, x^3} = 0$  so  $e = 0$  and  $\langle x^4 \rangle$  is an ideal of  $A$ . Since  $L_y$  is nilpotent for every,  $y \in A$ ,

we have that  $x^4A = 0$ ,  $A / \langle x^4 \rangle$  is a three-dimensional right-nilalgebra, hence nilpotent, and therefore the whole  $A$  is nilpotent.

Now we will see that the case  $a \neq 0$  is not possible. If  $a \neq 0$ , extending scalars if necessary, we can take  $a = 1$ . The equations (3), (4), (5) and (6) give the following matrices  $A_i$  corresponding to  $L_{x^i}$ ,  $i = 1, \dots, 4$ .

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & d+k-e & b+c & 0 \\ 0 & -(i+f) & -(d+k) & -(b+c) \\ 1 & g & i & e \\ 0 & h & j & f \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & b+c & 1 & 0 \\ 0 & -(d+k) & -(b+2c) & -1 \\ 0 & i & k & c \\ 1 & j & l & d \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(b+c) & -1 & 0 \\ 0 & e & c & 1 \\ 0 & f & d & b \end{pmatrix}.$$

with  $b, c, d, e, f, g, h, i, j, k, l \in F$ .

For every  $x_1, x_2, x_3, x_4 \in F$ , the matrix  $A = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$  is nilpotent,  $A^4 = 0$  and the  $(1, 1)$  entry of  $A^4$  is a polynomial  $p_{(1,1)}(x_1, x_2, x_3, x_4)$  of degree  $\leq 4$  which is zero for any value of  $x_1, x_2, x_3, x_4$ . Since  $F$  is an infinite field, all the coefficient are zero. Using a program of symbolic calculus we obtain that:

The coefficient of  $x_1^2x_3x_4$  in  $p_{(1,1)}$  is  $-c - 2b$ , then  $c = -2b$ .

Replacing this value of  $c$ , the coefficient of  $x_1x_3x_4^2$  is  $f - b^3$ , those of  $x_2x_3x_4^2$  is  $-k + 4b^2$  and those of  $x_1^2x_3^2$  is  $-d - 2b^2$ . Therefore,  $f = b^3$ ,  $k = 4b^2$  and  $d = -2b^2$ .

Substituting the above values, the coefficient of  $x_3^4$  is  $-i - 2b^3$ , those of  $x_3^3x_4$  is  $b^2 - e$  and  $i = -2b^2$ ,  $e = b^2$ .

Finally substituting the new values, the coefficient of  $x_1x_3^3$  is  $-2b^5 - bj - 1$  and  $b \neq 0$ . On the other hand, the coefficient of  $x_1x_2^3$  is  $b^8 - b^4g$ , so  $g = b^4$ . But the coefficient of  $x_2^4$  is  $b^2$  then  $b = 0$ . A contradiction. This finish the proof of the Theorem.  $\square$

**Remark 1.** *In the proof of the above Theorem it was proved that  $x^4A = 0$  and that  $x^2x^2 \in \langle x^3, x^4 \rangle$  which is an ideal of  $A$ . Using (6) this fact prove that  $x^2x^3 \in \langle x^4 \rangle$  and the multiplication table of  $A$  in the basis  $\{x, x^2, x^3, x^4\}$  is:*

	$x$	$x^2$	$x^3$	$x^4$
$x$	$x^2$	$x^3$	$x^4$	$0$
$x^2$	$x^3$	$\epsilon x^3 + \alpha x^4$	$\beta x^4$	$0$
$x^3$	$x^4$	$\beta x^4$	$\gamma x^4$	$0$
$x^4$	$0$	$0$	$0$	$0$

for suitable scalars  $\epsilon, \alpha, \beta, \gamma \in F$ . Moreover, scaling  $x$  if necessary,  $\epsilon$  may be taken to be either 0 or 1, with  $\epsilon = 0$  if and only if  $(A^2A^2)A = 0$ .

**Remark 2.** *Correa, Hentzel and Peresi in [2] prove the above Theorem in  $\text{char}(F) \neq 2, 3$  using the complete linearization of the identity  $x^5 = 0$  and evaluating there several elements of  $A$ . The proof that we give is valid in  $\text{char}(F) = 3$  and use elementary linear algebra instead of the complete linearization.*

### 3 Classification

Our aim in this section is to classify the algebras in Remark 1.

**CASE**  $\epsilon = 0$ . Here we have that  $A^2A^2 \subseteq A^{<4>} = A^4$ .

**Table I**

	$x$	$x^2$	$x^3$	$x^4$
$x$	$x^2$	$x^3$	$x^4$	0
$x^2$	$x^3$	$\alpha x^4$	$\beta x^4$	0
$x^3$	$x^4$	$\beta x^4$	$\gamma x^4$	0
$x^4$	0	0	0	0

Denote by  $A^0(\alpha, \beta, \gamma)$  the algebra with multiplication given by Table I.

Let  $y \in \mathcal{G} (= \{z \in A : z^4 \neq 0\})$  then  $y = \mu x + \nu x^2 + \rho x^3 + \delta x^4$ , with  $\mu \neq 0$ .

$$y^2 = \mu^2 x^2 + 2\mu\nu x^3 + (2\mu\rho + \alpha\nu^2 + 2\nu\rho\beta + \rho^2\gamma)x^4,$$

$$y^3 = \mu^3 x^3 + (2\mu^2\nu + \mu^2\rho\beta + \mu^2\nu\alpha + 2\mu\nu^2\beta + 2\mu\nu\rho\gamma)x^4,$$

$$y^4 = (\mu^4 + \mu^3\nu\beta + \mu^3\rho\gamma)x^4 = \mu^3(\mu + \beta\nu + \gamma\rho)x^4,$$

$$y^2y^2 = (\mu^4\alpha + 4\mu^3\nu\beta + 4\mu^2\nu^2\gamma)x^4 = \mu^2(\mu^2\alpha + 4\mu\nu\beta + 4\nu^2\gamma)x^4,$$

$$y^2y^3 = (\mu^5\beta + 2\mu^4\nu\gamma)x^4 = \mu^4(\mu\beta + 2\nu\gamma)x^4,$$

$$y^3y^3 = \mu^6\gamma x^4.$$

We have three sub-cases:

**Sub-case (i):**  $A^2A^3 = 0$ . Then  $\beta = \gamma = 0$ . The multiplication table depends on  $\alpha$ . We have that  $y^4 = \mu^4 x^4$  and  $y^2y^2 = \mu^4 \alpha x^4 = \alpha y^4$ . Therefore:

$$A^0(\alpha, 0, 0) \simeq A^0(\alpha', 0, 0) \Leftrightarrow \alpha = \alpha'.$$

Moreover  $A^0(\alpha, 0, 0)$  is associative if and only if  $\alpha = 1$ . Note that  $A^0(\alpha, 0, 0)$  satisfies  $z^2z^2 = \alpha z^4$  for any  $z \in A$ .

**Sub-case (ii):**  $A^3A^3 = 0 \neq A^2A^3$ . Then  $\gamma = 0 \neq \beta$ .

In this case we have

$$y^4 = \mu^3(\mu + \beta\nu)x^4, \quad y^2y^2 = \mu^3(\mu\alpha + 4\nu\beta), \quad y^2y^3 = \mu^5\beta x^4, \quad y^3y^3 = 0.$$

Since  $y^4 \neq 0$  then  $\mu \neq 0 \neq \mu + \beta\nu$ , and

$$y^2y^2 = \alpha'y^4, \quad \alpha' = \frac{\mu\alpha + 4\nu\beta}{\mu + \nu\beta},$$

$$y^2y^3 = \beta'y^4, \quad \beta' = \frac{\mu^2\beta}{\mu + \nu\beta}.$$

If we take  $\nu = \frac{\mu(\mu\beta-1)}{\beta}$ , we get  $\beta' = 1$  and  $\alpha' = \frac{\alpha-4+4\mu\beta}{\mu\beta}$ .

Two possibilities appear:

(a)  $\alpha \neq 4$ , then taking  $\mu = \frac{4-\alpha}{\beta}$ , we have the algebra  $A^0(0, 1, 0)$ .

(b)  $\alpha = 4$ , and we have the algebra  $A^0(4, 1, 0)$ . This algebra satisfies  $y^2y^2 = 4y^4$ , for any  $y \in \mathcal{G}$ , since  $y^2y^2 = \mu^3(4\mu + 4\nu\beta)x^4 = 4\mu^3(\mu + \nu\beta)x^4 = 4y^4$  and, since  $\mathcal{G}$  is Zariski open,  $A^0(4, 1, 0)$  satisfies  $y^2y^2 = 4y^4$  for any  $y \in A$ .

Therefore both algebras are not isomorphic.

**Sub-case (iii):**  $A^3A^3 \neq 0$ . Then  $\gamma \neq 0$ .

In this case we have  $y^4 = \mu^3(\mu + \nu\beta + \rho\gamma)x^4$  and  $y^2y^3 = \mu^4(\mu\beta + 2\nu\gamma)x^4$ .

Since  $y^4 \neq 0$  then  $0 \neq \mu + \nu\beta + \rho\gamma$ , and we have  $y^2y^3 = \beta'y^4$ , where  $\beta' = \frac{\mu(\mu\beta+2\nu\gamma)}{\mu+\nu\beta+\rho\gamma}$ .

Taking  $\nu = -\frac{\mu\beta}{2\gamma}$  and  $\rho$  such that  $\mu + \nu\beta + \rho\gamma \neq 0$ , we have  $\beta' = 0$ . Hence we may assume that  $\beta = 0 \neq \gamma$ . In this case:

$$y^4 = \mu^3(\mu + \rho\gamma)x^4, \quad y^2y^2 = \mu^2(\mu^2\alpha + 4\nu^2\gamma)x^4, \quad y^2y^3 = 2\mu^4\nu\gamma x^4, \quad y^3y^3 = \mu^6\gamma x^4.$$

Therefore if  $\mu \neq 0 \neq \mu + \gamma\rho$ ,  $y^4 \neq 0$  and we obtain that

$$\begin{aligned} y^2 y^2 &= \alpha' y^4, \text{ where } \alpha' = \frac{\mu^2 \alpha + 4\nu^2 \gamma}{\mu(\mu + \rho\gamma)} \\ y^2 y^3 &= \beta' y^4, \text{ where } \beta' = \frac{2\mu\nu\gamma}{\mu + \rho\gamma} \\ y^3 y^3 &= \gamma' y^4, \text{ where } \gamma' = \frac{\mu^3 \gamma}{\mu + \rho\gamma}. \end{aligned}$$

If we take  $\nu = 0$ ,  $\rho = \frac{\mu^3 \gamma - \mu}{\gamma}$  this imply that  $\beta' = 0$ ,  $\gamma' = 1$  and

$$\alpha' = \frac{\mu\alpha}{\mu + \rho\gamma} = \frac{\mu\alpha}{\mu^3 \gamma} = \frac{\alpha}{\mu^2 \gamma}$$

With  $\mu = \frac{1}{\eta^2 \gamma}$ , we have  $\alpha' = \alpha\gamma\eta^2 = \theta\eta^2$ , with  $\theta = \alpha\gamma$  and the algebra is (isomorphic to)  $A^0(\theta\eta^2, 0, 1)$ , for any  $\eta \in F^\times$ .

We can observe that for the algebra  $A = A^0(\theta, 0, 1)$ , the matrix of the bilinear form

$$A^2/A^{\langle 4 \rangle} \times A^2/A^{\langle 4 \rangle} \rightarrow A^{\langle 4 \rangle} = Fx^4 \simeq F$$

in the basis  $\{x^2 + A^{\langle 4 \rangle}, x^3 + A^{\langle 4 \rangle}\}$  is  $\begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$ .

Then for this matrix we have that  $\theta(F^\times)^2$  is the discriminant of this bilinear form. Therefore, denoting the multiplicative group of  $F$  by  $F^\times$  we have

$$A^0(\theta, 0, 1) \simeq A^0(\theta', 0, 1) \Leftrightarrow \theta(F^\times)^2 = \theta'(F^\times)^2.$$

**Theorem 2.** *Let  $A$  be a four dimensional commutative and not power associative algebra satisfying the identity  $x^5 = 0$  but not  $x^4 = 0$ , and such that  $(A^2 A^2)A = 0$ . Then  $A$  is isomorphic to one and only one of the following algebras:  $A^0(\alpha, 0, 0)$  ( $\alpha \in F$ ),  $A^0(0, 1, 0)$ ,  $A^0(4, 1, 0)$ ,  $A^0(0, 0, 1)$ , or  $A^0(\alpha, 0, 1)$  ( $\alpha \in \mathcal{H}$ ), where  $\mathcal{H}$  is a set of representatives of  $F^\times / (F^\times)^2$ .*

**Corollary 1.** *Let  $F$  be a quadratically closed field. Then any four dimensional commutative and not power-associative algebra satisfying the identity  $x^5 = 0$  and with multiplication given by table I is isomorphic to one and only one of the following algebras:  $A^0(\alpha, 0, 0)$  ( $\alpha \in F$ ),  $A^0(0, 1, 0)$ ,  $A^0(4, 1, 0)$ ,  $A^0(0, 0, 1)$  or  $A^0(1, 0, 1)$ .*

**CASE**  $\epsilon = 1$ . Here we have that  $A^2A^2 \subseteq A^3 = A^4$ .

**Table II**

	$x$	$x^2$	$x^3$	$x^4$
$x$	$x^2$	$x^3$	$x^4$	0
$x^2$	$x^3$	$x^3 + \alpha x^4$	$\beta x^4$	0
$x^3$	$x^4$	$\beta x^4$	$\gamma x^4$	0
$x^4$	0	0	0	0

Denote by  $A^1(\alpha, \beta, \gamma)$  the algebra with this multiplication table.

Let  $y \in A \setminus A^2$  then  $y = \mu x + \nu x^2 + \rho x^3 + \delta x^4$ , with  $\mu \neq 0$ .

$$y^2 = \mu^2 x^2 + (2\mu\nu + \nu^2)x^3 + (2\mu\rho + \alpha\nu^2 + 2\nu\rho\beta + \rho^2\gamma)x^4,$$

$$y^3 = \mu^2(\mu + \nu)x^3 + (\mu^2\nu\alpha + \mu(2\mu\nu + \nu^2) + \mu^2\rho\beta + (2\mu\nu + \nu^2)\nu\beta + (2\mu\nu + \nu^2)\rho\gamma)x^4,$$

$$y^4 = \mu^2(\mu + \nu)[\mu + \nu\beta + \rho\gamma]x^4$$

$$y^2y^2 = \mu^4x^3 + [\mu^4\alpha + 2\mu^2(2\mu\nu + \nu^2)\beta + (2\mu\nu + \nu^2)^2\gamma]x^4$$

$$y^2y^3 = [\mu^4(\mu + \nu)\beta + \mu^2(\mu + \nu)(2\mu\nu + \nu^2)\gamma]x^4,$$

$$y^3y^3 = \mu^4(\mu + \nu)^2\gamma x^4.$$

Therefore  $y^4 \neq 0$  (that is,  $y \in \mathcal{G}$ ) if and only if  $\mu, \mu + \nu, \mu + \nu\beta + \rho\gamma \neq 0$ .

Putting  $\Delta = \mu + \nu\beta + \rho\gamma$  we have that  $\Delta \neq 0$  for  $y \in \mathcal{G}$ . On the other hand

$y^2y^2 - y^3 \in \langle y^4 \rangle = \langle x^4 \rangle$  if and only if  $\mu^4 = \mu^2(\mu + \nu)$ . Since  $\mu \neq 0$ , then

$$\mu^2 = \mu + \nu, \Delta = \beta\mu^2 + (1 - \beta)\mu + \rho\gamma, y^4 = \mu^4\Delta x^4, \text{ that is } x^4 = \frac{1}{\mu^4\Delta}y^4. \quad (7)$$

and

$$2\mu\nu + \nu^2 = \nu(\mu + (\mu + \nu)) = (\mu^2 - \mu)(\mu + \mu^2) = \mu(\mu - 1)\mu(\mu + 1) = \mu^2(\mu^2 - 1). \quad (8)$$

Since  $\Delta \neq 0$ , using (7) and (8) we have that:

$$\begin{aligned} y^3y^3 &= \mu^8\gamma x^4 = \frac{\mu^4\gamma}{\Delta} y^4 \\ y^2y^3 &= (\mu^6\beta + \mu^6(\mu^2 - 1)\gamma)x^4 = \frac{\mu^2(\beta + (\mu^2 - 1)\gamma)}{\Delta} y^4 \\ y^2y^2 &= \mu^4x^3 + [\mu^4\alpha + 2\mu^4(\mu^2 - 1)\beta + (\mu^2(\mu^2 - 1))^2\gamma]x^4 \end{aligned}$$

On the other hand,

$$\begin{aligned} y^3 &= \mu^4x^3 + [\mu^3(\mu - 1)\alpha + \mu^3(\mu^2 - 1) + \mu^2\rho\beta + \mu^3(\mu^2 - 1)(\mu - 1)\beta \\ &\quad + \mu^2(\mu^2 - 1)\rho\gamma]x^4 \\ &= \mu^4x^3 + \mu^2[\mu(\mu - 1)\alpha + \mu(\mu^2 - 1) + (\rho(\beta + (\mu^2 - 1)\gamma) + \mu(\mu^2 - 1)(\mu - 1)\beta)]x^4, \end{aligned}$$

Therefore

$$y^3 = \mu^4x^3 + \mu^2\Gamma x^4 = \mu^4x^3 + \frac{\Gamma}{\mu^2\Delta}y^4,$$

where  $\Gamma = \mu(\mu - 1)\alpha + \mu(\mu^2 - 1) + \rho(\beta + (\mu^2 - 1)\gamma) + \mu(\mu^2 - 1)(\mu - 1)\beta$ .

Since  $\mu \neq 0$  and  $\Delta \neq 0$ , we obtain  $x^3 = \frac{1}{\mu^4}y^3 - \frac{\Gamma}{\mu^6\Delta}y^4$

Replacing this value in  $y^2y^2 = \mu^4x^3 + \mu^4[\alpha + 2(\mu^2 - 1)\beta + (\mu^2 - 1)^2\gamma]x^4$  and putting  $\Phi = \alpha + 2(\mu^2 - 1)\beta + (\mu^2 - 1)^2\gamma$  we obtain

$$y^2y^2 = \mu^4(x^3 + \Phi x^4) = y^3 - \frac{\Gamma}{\mu^2\Delta}y^4 + \frac{\Phi}{\Delta}y^4 = y^3 + \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta}y^4. \quad (9)$$

Summarizing, if  $y$  is any element of  $A$  with  $y^4 \neq 0$  and  $y^2y^2 - y^3 \in \langle y^4 \rangle$ , then there are elements  $\mu \neq 0$  and  $\rho$  in  $F$  such that:

$$y^2y^2 = y^3 + \alpha'y^4$$

$$y^2y^3 = \beta'y^4$$

$$y^3y^3 = \gamma'y^4$$

where

$$\alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta}, \quad \beta' = \frac{\mu^2(\beta + (\mu^2 - 1)\gamma)}{\Delta}, \quad \gamma' = \frac{\mu^4\gamma}{\Delta},$$

with

$$\Delta = \mu + \mu(\mu - 1)\beta + \rho\gamma,$$

$$\Phi = \alpha + 2(\mu^2 - 1)\beta + (\mu^2 - 1)^2\gamma,$$

$$\Gamma = \mu(\mu^2 - 1) + \mu(\mu - 1)\alpha + (\rho + \mu(\mu^2 - 1)(\mu - 1))\beta + (\mu^2 - 1)\rho\gamma.$$

We have three sub-cases:

**Sub-case (i):**  $A^2A^3 = 0$ . Then  $\beta = \gamma = 0$ . So we have that

$$\Delta = \mu, \quad \Gamma = \mu(\mu - 1)\alpha + \mu(\mu^2 - 1), \quad \Phi = \alpha.$$

Then  $y^2y^3 = 0$ ,  $y^3y^3 = 0$  and

$$y^2y^2 = y^3 + \alpha'y^4, \text{ where } \alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta} = \frac{\mu^2\alpha - \mu(\mu - 1)\alpha - \mu(\mu^2 - 1)}{\mu^3} = \frac{\alpha - \mu^2 + 1}{\mu^2}, \text{ that is,}$$

$$(1 + \alpha')\mu^2 = 1 + \alpha.$$

Therefore:

$$A^1(\alpha, 0, 0) \simeq A^1(\alpha', 0, 0) \Leftrightarrow (1 + \alpha)(F^\times)^2 = (1 + \alpha')(F^\times)^2.$$

**Sub-case (ii):**  $A^3A^3 = 0 \neq A^2A^3$ . Then  $\gamma = 0 \neq \beta$ .

In this case we have

$$\Delta = \beta\mu^2 + (1 - \beta)\mu = \mu((\mu - 1)\beta + 1), \quad \Phi = \alpha + 2(\mu^2 - 1)\beta,$$

$$\Gamma = \mu(\mu - 1)\alpha + \mu(\mu^2 - 1) + \mu(\mu^2 - 1)(\mu - 1)\beta + \rho\beta.$$

Therefore, using (7) and (9) we obtain that

$$y^3y^3 = 0, \quad y^2y^2 = y^3 + \alpha'y^4, \quad \text{where } \alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta}.$$

$$y^2y^3 = \beta'y^4, \quad \text{where } \beta' = \frac{\mu^2\beta}{\Delta} = \frac{\mu\beta}{\beta\mu + (1-\beta)}.$$

We have two possibilities:

(a)  $\beta = 1$ . Then  $\beta' = 1$  and taking  $\mu = 1$  and  $\rho$  such that  $\Gamma = \mu^2\Phi$  we have that  $\alpha' = 0$ , so that the algebra is (isomorphic to)  $A^1(0, 1, 0)$ .

(b)  $\beta \neq 1$ . Then taking  $\mu = \frac{2(\beta-1)}{\beta}$ , we obtain  $\beta' = 2$ , and then taking  $\rho$  such that  $\mu^2\Phi = \Gamma$ , we have that  $\alpha' = 0$ , so that the algebra is  $A^1(0, 2, 0)$ .

Both algebras are not isomorphic, because the first algebra satisfies  $z^2z^2 \cdot z = z^2z^3$ ,  $\forall z \in A$  which is not satisfied by the second one. In fact, for  $\beta = 1$ , if  $z = ax + bx^2 + cx^3 + dx^4$ , then  $z^2 = a^2x^2 + (b^2 + 2ab)x^3 + 2(ac + bc)x^4$ ,  $z^3 = (a^3 + ba^2)x^3 + (ab^2 + 2a^2b)x^4 + (b^3 + 2ab^2 + a^2c)x^4$ ,  $z^2z^2 = a^4x^3 + 2(a^2b^2 + 2a^3b)x^4$ ,  $(z^2z^2)z = a^5x^4 + ba^4x^4$ ,  $z^2z^3 = a^2(a^3 + ba^2)x^4$ .

**Sub-case (iii):**  $A^3A^3 \neq 0$ . Then  $\gamma \neq 0$ .

Since  $\gamma' = \frac{\mu^4\gamma}{\Delta} = \frac{\mu^4\gamma}{\mu + \mu(\mu-1)\beta + \rho\gamma}$ ,  $\rho$  can be taken so that  $\gamma' = 1$ , and we have to deal with the algebra  $A^1(\alpha, \beta, 1)$ .

Thus, assume from now on that  $\gamma = 1$ . Observe that for  $y = \mu x + \nu x^2 + \rho x^3 + \delta x^4$  one gets:

$$\begin{aligned}(y^2)^3 &= \mu^4 x^3 (\mu^2 x^2 + (2\mu\nu + \nu^2)x^3) \\ &= (\mu^6 \beta + \mu^4(2\mu\nu + \nu^2))x^4 \\ &= \mu^4(\mu^2 \beta + 2\mu\nu + \nu^2)x^4,\end{aligned}$$

while  $y^3 y^3 = \mu^4(\mu + \nu)^2 x^4$ , so  $(y^2)^3 = (y^3)^2$  for any  $y$  if and only if  $\beta = 1$ .

Now, with  $\gamma = 1$ , we look for elements  $y \in \mathcal{G}$  such that  $y^2 y^2 - y^3 \in \langle y^4 \rangle$  and  $y^3 y^3 = y^4$ .

This gives  $\mu^4 = \Delta$ , so  $\rho = \mu^4 - \mu - \mu(\mu - 1)\beta = \mu(\mu - 1)(\mu^2 + \mu + 1 - \beta)$ , and

$$\beta' = \frac{\mu^2(\beta + (\mu^2 - 1))}{\Delta} = \frac{\beta + (\mu^2 - 1)}{\mu^2} = \frac{\beta - 1}{\mu^2} + 1,$$

that is,

$$\beta - 1 = \mu^2(\beta' - 1).$$

Therefore, if  $A^1(\alpha, \beta, 1)$  is isomorphic to  $A^1(\alpha', \beta', 1)$ , then  $(\beta - 1)(F^\times)^2 = (\beta' - 1)(F^\times)^2$ .

In this case,

$$\begin{aligned}\Gamma &= \mu(\mu^2 - 1) + \mu(\mu - 1)\alpha + (\rho + \mu(\mu^2 - 1)(\mu - 1))\beta + (\mu^2 - 1)\rho \\ &= \mu(\mu^2 - 1) + \mu(\mu - 1)\alpha + \mu(\mu - 1)(2\mu^2 + \mu - \beta)\beta \\ &\quad + (\mu^2 - 1)\mu(\mu - 1)(\mu^2 + \mu + 1 - \beta) \\ &= \mu(\mu - 1)\left(\alpha + \mu + 1 + (2\mu^2 + \mu - \beta)\beta + (\mu^2 - 1)(\mu^2 + \mu + 1 - \beta)\right).\end{aligned}$$

We are left with two possibilities:

(a)  $\beta = 1$ . Here

$$\begin{aligned}
\rho &= \mu(\mu - 1)(\mu^2 + \mu) = \mu^2(\mu^2 - 1), \\
\Phi &= \alpha + 2(\mu^2 - 1) + (\mu^2 - 1)^2 = \alpha + \mu^4 - 1, \\
\Gamma &= \mu(\mu - 1)(\alpha + \mu + 1 + 2\mu^2 + \mu - 1 + (\mu^2 - 1)(\mu^2 + \mu)) \\
&= \mu(\mu - 1)(\alpha + 2\mu(\mu + 1) + \mu(\mu + 1)(\mu^2 - 1)) \\
&= \mu(\mu - 1)(\alpha + \mu(\mu + 1)(\mu^2 + 1)) \\
&= \mu(\mu - 1)\alpha + \mu^2(\mu^4 - 1).
\end{aligned}$$

Hence  $\mu^2\Phi - \Gamma = \mu\alpha$ , and thus

$$\alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^2\Delta} = \frac{\mu\alpha}{\mu^6} = \frac{\alpha}{\mu^5}$$

and, therefore,

$$A^1(\alpha, 1, 1) \simeq A^1(\alpha', 1, 1) \Leftrightarrow \alpha(F^\times)^5 = \alpha'(F^\times)^5.$$

(b)  $\beta \neq 1$ . Here  $A^1(\alpha, \beta, 1)$  is isomorphic to  $A^1(\alpha', \beta', 1)$  if and only if there is a scalar  $\mu \in F^\times$  such that  $\beta - 1 = \mu^2(\beta' - 1)$  and  $\alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^6}$ , with  $\Phi$  and  $\Gamma$  as above.

Once  $\beta \neq 1$  is fixed,  $A^1(\alpha, \beta, 1)$  is isomorphic to  $A^1(\alpha', \beta, 1)$  if and only if  $\alpha' = \frac{\mu^2\Phi - \Gamma}{\mu^6}$  for  $\mu = \pm 1$ , if and only if either  $\alpha' = \alpha$  ( $\mu = 1$ ) or  $\alpha' = -\alpha + (\beta - 1)\beta$  ( $\mu = -1$ ).

In particular, if  $F$  is quadratically closed, then one can always take  $\beta' = 0$ , and then we get the algebras  $A^1(\alpha, 0, 1)$  with

$$A^1(\alpha, 0, 1) \simeq A^1(\alpha', 0, 1) \Leftrightarrow \alpha' = \pm\alpha.$$

**Theorem 3.** *Let  $A$  be a four dimensional commutative and not associative algebra satisfying the identity  $x^5 = 0$  but not  $x^4 = 0$ , and such that  $(A^2A^2)A \neq 0$ . Then  $A$  is isomorphic to one of the following algebras:*

- (i)  $A^1(-1, 0, 0)$ ,
- (ii)  $A^1(\alpha, 0, 0)$ , with  $\alpha \in -1 + \mathcal{H}$ , where  $\mathcal{H}$  is a set of representatives of  $F^\times / (F^\times)^2$ ,
- (iii)  $A^1(0, 1, 0)$ ,
- (iv)  $A^1(0, 2, 0)$ ,
- (v)  $A^1(0, 1, 1)$ ,
- (vi)  $A^1(\alpha, 1, 1)$ , with  $\alpha \in \mathcal{I}$ , where  $\mathcal{I}$  is a set of representatives of  $F^\times / (F^\times)^5$ ,
- (vii)  $A^1(\alpha, \beta, 1)$ , where  $\beta \in 1 + \mathcal{H}$  ( $\mathcal{H}$  as in (ii)).

Moreover, algebras in different items are not isomorphic, and so are algebras in the same item, with the exception of item (vii), where  $A^1(\alpha, \beta, 1)$  is isomorphic to  $A^1(\alpha', \beta', 1)$  ( $\beta, \beta' \in 1 + \mathcal{H}$ ) if and only if  $\beta = \beta'$  and either  $\alpha' = \alpha$  or  $\alpha' = -\alpha + (\beta - 1)\beta$ .

**Corollary 2.** *Let  $F$  be an algebraically closed field. Then any four dimensional commutative and not power-associative algebra satisfying the identity  $x^5 = 0$  and the multiplication given by table II is isomorphic to one and only one of the following algebras:  $A^1(0, 0, 0)$ ,  $A^1(-1, 0, 0)$ ,  $A^1(0, 1, 0)$ ,  $A^1(0, 2, 0)$ ,  $A^1(0, 1, 1)$ ,  $A^1(1, 1, 1)$ ,  $A^1(0, 0, 1)$ , or  $A^1(\alpha, 0, 1)$  for  $\alpha \in \mathcal{F}$ , where  $\mathcal{F}$  is a subset of  $F$  satisfying  $\mathcal{F} \cup -\mathcal{F} = F^\times$  and  $\mathcal{F} \cap -\mathcal{F} = \emptyset$ .*

Summarizing, in the case of an algebraically closed field  $F$ , Table III display all the non isomorphic four dimensional commutative and not power-associative algebras satisfying the identity  $x^5 = 0$ , but not satisfying the identity  $x^4 = 0$ .

**Table III**

$A^4 = A^{\langle 4 \rangle}$ , $\dim(A^4) = 1$ . $(\epsilon = 0)$ ,	$A^2A^3 = 0$	$A^0(\alpha, 0, 0)$	$y^2y^2 = \alpha y \ \forall y$
	$A^3A^3 = 0 \neq A^2A^3$	$A^0(4, 1, 0)$	$y^2y^2 = 4y^4 \ \forall y \in A$
		$A^0(0, 1, 0)$	$\exists y, y^2y^2 \neq 4y^4$
	$A^3A^3 \neq 0$	$A^0(0, 0, 1)$	rank( $A^2 \times A^2 \rightarrow A^4$ : $(u, v) \mapsto uv$ ) = 1
		$A^0(1, 0, 1)$	rank( $A^2 \times A^2 \rightarrow A^4$ : $(u, v) \mapsto uv$ ) = 2
$A^4 = A^3$ , $\dim(A^4) = 2$ , $(\epsilon = 1)$ ,	$A^2A^3 = 0$	$A^1(0, 0, 0)$	$\exists y \in A \setminus A^2 : y^2y^2 = y^3$
		$A^1(-1, 1, 0)$	$\nexists y \in A \setminus A^2 : y^2y^2 = y^3$
	$A^3A^3 = 0 \neq A^2A^3$	$A^1(0, 1, 0)$	$y^2y^2 \cdot y = y^2y^3 \ \forall y \in A$
		$A^1(0, 2, 0)$	$\exists y \in A, y^2y^2 \cdot y \neq y^2y^3$
	$A^3A^3 \neq 0$	$A^1(0, 1, 1)$	$(y^2)^3 = (y^3)^2 \ \forall y$ $\exists z \ z^3z^3 = z^4 \neq 0, z^2z^2 = z^3$
		$A^1(1, 1, 1)$	$(y^2)^3 = (y^3)^2 \ \forall y$ $\nexists z \ z^3z^3 = z^4 \neq 0, z^2z^2 = z^3$
		$A^1(\alpha, 0, 1)$	$\exists y \ (y^2)^3 \neq (y^3)^2$ $A^1(\alpha, 0, 1) \simeq A^1(\alpha', 0, 1)$ $\Leftrightarrow \alpha' = \pm\alpha$

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