Every Random Variable Satisfies a Certain Nontrivial Integrability Condition
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Abstract

We show that every random variable $X$ fulfils a certain nontrivial integrability condition, in the sense that there always exists a nonnegative function $g$—depending on $X$—growing to $\infty$ as $x \to \infty$, and such that $Eg(|X|) < \infty$. Refinements of this universal property allow us to give some simple but striking statements in connection with Markov’s inequality and the central limit theorem.

Key words: Integrability, Markov’s inequality, central limit theorem.

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1 Introduction and main result

It is an obvious fact that the integrability requirements imposed on a certain random variable $X$ have a great influence in most of its properties. In many occasions, such integrability conditions refer to the finiteness of the moments or the moment generating function of $X$. On the other hand, there are many simple ways of constructing random variables having no finite moments of any order, or even satisfying extremely poor integrability conditions. For instance, the random variable $X$ whose probability law is given by $P(X = \exp(e^n) - 1) = 6/(\pi n)^2$, $n = 1, 2, \ldots$, satisfies that $E(\log(1 + X))^\alpha = \infty$ for any $\alpha > 0$.

Let $\mathcal{I}$ be the set of nonnegative and nondecreasing functions $f$ defined on $[0, \infty)$. Denote by $\mathcal{J}$ the subset of $\mathcal{I}$ consisting of those functions $g \geq 1$ such

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that \( g(x) \to \infty \) as \( x \to \infty \), and \( g(x)/x \) is nonincreasing in \((0, \infty)\). Despite the preceding example and many others one can have in mind, it is the aim of this note to point out that every random variable \( X \) fulfils a certain nontrivial integrability condition in the sense of the following.

**Theorem 1** Let \( X \) be an arbitrary real-valued random variable and let \( f \in \mathcal{I} \). Assume that \( 0 < \mu := Ef(|X|) < \infty \). Then, there exists \( g \in \mathcal{J} \) such that \( Ef(|X|) \leq Ef(|X|)g(|X|) < \infty \).

Setting \( f := 1 \), Theorem 1 says that for any random variable \( X \) there always exists a function \( g \in \mathcal{J} \) with \( Ef(|X|)g(|X|) < \infty \).

Some interesting consequences in various contexts may be drawn from the preceding result. We mention here those concerning Markov’s inequality and the classical version of the central limit theorem. Markov’s inequality is one of the most widely used tools in estimating the tail probabilities of \( X \) (see, for instance, Petrov [5, pp. 54–58], Ghosh [2], and Csiszár et al. [1]). In its general form, this inequality provides the upper bound

\[
P(|X| \geq r) \leq \frac{Ef(|X|)}{f(r)}, \quad r > 0,
\]

whenever \( f \in \mathcal{I} \) satisfies that \( Ef(|X|) < \infty \) and \( f(r) > 0, r > 0 \). It follows from Theorem 1 that, on the one hand, every random variable \( X \) satisfies Markov’s inequality and, on the other, that the only use of Markov’s inequality cannot give us the optimal rate of convergence for \( P(|X| \geq r) \) as \( r \to \infty \). Indeed, if inequality (1) has been established for a certain \( f \) as above, there is a further function \( g \in \mathcal{J} \) with \( Ef(|X|)g(|X|) < \infty \) which gives us the estimate

\[
P(|X| \geq r) \leq \frac{Ef(|X|)g(|X|)}{f(r)g(r)}, \quad r > 0.
\]

Since \( g(r) \) grows to \( \infty \) as \( r \to \infty \), the rate of convergence in (2) is strictly better than that in (1).

Let \((X_n)_{n \geq 1}\) be a sequence of independent identically distributed random variables with zero mean and unit variance. Denote by \( F_n \) the distribution function of \( S_n := n^{-1/2}(X_1 + \ldots X_n) \), by \( \Phi \) the standard normal distribution function, and by \( \| \cdot \| \) the usual supremum norm. Then, the Berry–Esseen theorem (cf. Petrov [5, p. 150]) states that \( \|F_n - \Phi\| \) converges to 0 as \( n \to \infty \) at the rate \( n^{-1/2} \), whenever \( E|X_1|^3 < \infty \). If, on the contrary, \( E|X_1|^3 = \infty \), the corresponding rate of convergence is worse than \( n^{-1/2} \) and depends upon some truncated moments of \( X_1 \) (cf. Osipov [4] and Hall [3]). In such circumstances,
a nice result is achieved under the following assumption (see Petrov [5, p. 151]). If \( g \in J \) satisfies that
\[
E X_1^2 g(|X_1|) < \infty,
\]
then
\[
\| F_n - \Phi \| \leq \frac{C}{g(\sqrt{n})}, \quad n = 1, 2, \ldots,
\]
where \( C \) is a positive constant depending on \( g \). In particular, if \( E |X_1|^{2+\alpha} < \infty \) for some \( 0 < \alpha \leq 1 \), then \( \| F_n - \Phi \| \) converges to 0 as \( n \to \infty \) at the rate \( n^{-\alpha/2} \), at least. Surprisingly, condition (3) is not restrictive, in the sense that there always exists a function \( g \in J \) satisfying (3) and, therefore, estimate (4) holds for such a particular \( g \).

2 The proof

Let \( X \) and \( T \) be two independent random variables such that \( X \) is nonnegative with distribution function \( F \) and \( T \) has the exponential distribution with unit mean. Denote by \( G \) the distribution function of \( X + T \). The following auxiliary result will be useful.

**Lemma 2** Let \( a := \sup\{ x \in \mathbb{R} : F(x) = 0 \} \). Then, \( G \) is continuous and strictly increasing in \([a, \infty)\) with \( G(a) = 0 \).

**PROOF.** Observe that for any \( x \geq 0 \) we have that
\[
G(x) = e^{-x} \int_0^x F(u) e^u \, du \leq F(x)(1 - e^{-x}),
\]
thus implying that \( G'(x) = F(x) - G(x) \geq e^{-x}F(x) \). In other words, \( G \) is absolutely continuous with density \( G'(x) = 0 \) if \( x < a \) and \( G'(x) > 0 \) if \( x > a \). The conclusion follows. \( \square \)

We are in a position to prove Theorem 1. No generality is lost if we assume that \( X \) is nonnegative with distribution function \( F \) (otherwise, we would simply replace \( X \) by \(|X|\)). For any \( a \geq 0 \), denote by \( C_a \) the set of distribution functions \( K \) such that \( K \) is continuous and strictly increasing in \([a, \infty)\) with \( K(a) = 0 \). We consider the following two steps.

**Step 1.** Assume that \( f \geq 1 \) and that \( F \in C_a \), for some \( a \geq 0 \). If \( EX f(X) < \infty \), it suffices to choose \( g(x) := x + 1, \ x \geq 0 \). Suppose that \( EX f(X) = \infty \). Since
If \( f \geq 1 \) and \( F \in C_a \), the function \( H \) defined by
\[
H(x) := \int_{[0,x]} f(u) \, dF(u), \quad x \geq 0,
\]
is continuous and strictly increasing in \([a, \infty)\), \( H(a) = 0 \), and \( H(x) \to \mu \) as \( x \to \infty \). We define recursively the strictly increasing sequence \((b_n)_{n \geq 0}\) by setting \( b_0 := 0 \) and
\[
H(b_{n+1}) - H(b_n) := \mu 2^{-(n+1)}, \quad n = 0, 1, \ldots.
\]
Note that \( b_n \to \infty \) as \( n \to \infty \). We define the function \( g \) as follows: \( g(x) := 1 \) if \( b_0 \leq x \leq b_1 \), \( g(b_{n+1}) := \rho_n g(b_n) \), \( n = 1, 2, \ldots \), where
\[
\rho_n := \min \left( b_{n+1}/b_n, 3/2 \right) > 1, \quad n = 1, 2, \ldots, \quad (5)
\]
whereas, if \( x \in (b_n, b_{n+1}) \), then \( g(x) \) linearly interpolates \( g(b_n) \) and \( g(b_{n+1}) \), that is,
\[
g(x) := g(b_n) \left( 1 + \frac{\rho_n - 1}{b_{n+1} - b_n} (x - b_n) \right), \quad b_n < x < b_{n+1}, \quad n = 1, 2, \ldots. \quad (6)
\]
By construction, \( g \in \mathcal{I} \) with \( g \geq 1 \). From (5), we see that \( g(b_{n+1}) \leq (3/2)^n \), \( n = 0, 1, \ldots \) and therefore that
\[
Ef(X)g(X) = \sum_{n=0}^{\infty} \int_{[b_n,b_{n+1})} f(x)g(x) \, dF(x)
\]
\[
\leq \sum_{n=0}^{\infty} (3/2)^n (H(b_{n+1}) - H(b_n)) = \frac{\mu}{2} \sum_{n=0}^{\infty} (3/4)^n < \infty.
\]
By assumption, we have
\[
\infty = EXf(X) = \sum_{n=0}^{\infty} \int_{[b_n,b_{n+1})} xf(x) \, dF(x) \leq \mu \sum_{n=0}^{\infty} b_{n+1}/2^{n+1},
\]
which entails that \( \lim_n (b_{n+1}/b_n) \geq 2 \). By (5), this means that \( g(b_{n+1}) = 3g(b_n)/2 \) infinitely often and therefore that \( g(x) \to \infty \) as \( x \to \infty \).

Finally, the function \( h(x) := g(x)/x \) is nonincreasing in \((0, \infty)\). In fact, in any of the intervals \((b_n, b_{n+1})\), \( n = 1, 2, \ldots \), it follows from (6) that the inequality \( h'(x) \leq 0 \) is equivalent to \( \rho_n \leq b_{n+1}/b_n \), and this is true by virtue of (5). This shows that \( g \in \mathcal{J} \) and completes the proof of Theorem 1 under the additional assumptions in Step 1.

**Step 2.** Let \( G \) be the distribution function of \( X + T \) as in Lemma 2, and denote by \( f^*(x) := f(x) + 1 \), \( x \geq 0 \). By Lemma 2 and Step 1, we can find a
function $g \in J$ such that $Ef^*(X + T)g(X + T) < \infty$. Since $g \geq 1$ and the functions $f$, $f^*$, and $g$ belong to $\mathcal{I}$, we have

$$Ef(X) \leq Ef(X)g(X) \leq Ef^*(X + T)g(X + T) < \infty.$$  

This concludes the proof. □

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References


