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Abstract

In this paper we analyse a method for triangulating the sphere sphere originally proposed by Baumgardner and Frederickson in 1985. The method is essentially a refinement procedure for arbitrary spherical triangles that fit into a hemisphere. Refinement is carried out by dividing each triangle into four by introducing the midpoints of the sides as new vertices and connecting then in the usual 'red' way. We show that this process can be described by a sequence of piecewise smooth mappings from a reference triangle onto the spherical triangle. We then prove that the whole sequence of mappings is uniformly bi–Lipschitz and converges uniformly to a non–smooth parameterization of the spherical triangle, recovering the Baumgardner and Frederickson spherical barycentric coordinates. We also prove that the sequence of triangulations is quasi–uniform, that is, areas of triangles and lengths of the sides are roughly the same at each refinement level. Some numerical experiments confirm the theoretical results.

Keywords: Spherical triangulations, quasi-uniform triangulations.

AMS Classification: 65N50, 65D17

1 Introduction

In 1985 an icosahedral triangulation of the sphere was introduced by Baumgardner and Frederickson (see [2]). This kind of triangulation has been extensively employed in the oceanographical community, since it gives very uniform–looking partitions of the sphere. In the original paper, the method consisted essentially in defining curved barycentric coordinates for arbitrary spherical triangles by doing a sequence of dyadic partition of the triangles and assuming that the process led to a limit by a continuity argument. The proof, which was simply sketched, proceeded by giving image to points with dyadic coordinates. Nothing was done to show that this process defined indeed a continuous

function on the dense set of points with dyadic coordinates, which is a necessary point to be able to extend the map to the whole triangle. The fact that the resulting triangulations are quasi–uniform, that is, areas of triangles and lengths of sides are roughly the same at each level, was only mentioned.

In this paper we work on several results related to this triangulation, that confirm in theory what was numerically checked originally. First of all, we prove that the sequence of transformations leads to a continuous Lipschitz but non–smooth parameterization of any spherical triangle. We also prove that in the sequence of inscribed polyhedra resulting from the successive application of the partition procedure all the triangles have roughly the same area, that is, the minimum area bounds the maximum area from above (with an adequate constant independent of the level of the partition) and that this quasi–uniformity applies also to the lengths of the sides. All these results are proved with asymptotics in mind: we are not measuring the quality of the quasi–uniformity of the partitions, only its non–degeneracy as the refinement increases.

At this point, it is worth mentioning a very different starting point that can be derived from the work of Alfeld, Neamtu and Schumaker in [1]. One fixes a flat triangle and maps a uniform triangulation from it to the surface of the sphere by simply moving vertices in the radial direction. The procedure has trivially good asymptotic properties but has two drawbacks. The main one is the fact that when one refines some triangles and not others, one has still to keep the original triangulation in memory, since the map was defined from it and the process does not begin afresh in each refinement, unlike in the method we are studying here. The second disadvantage is the fact that the level of quasi–uniformity is far worse from the numerical point of view in most examples. Again, we insist that we are not going to computer the quasi–uniformity constants, so we are not able to compare both methods in theory, but a simple numerical inspection (which we will perform in the last section) shows the great advantage of the Baumgardner and Frederickson approach from this point of view.

Besides the inherent interest of showing that a method that is being used in practice has good asymptotic properties, there is an additional application of the work developed in this paper. In [3] we proposed and numerically checked an automatic quadrature method on spherical triangles based on compound formulas with extrapolation as error control. As tools to suggest a proof of the good properties of the extrapolation scheme (a proper proof is the object of ongoing research), the quasi–uniformity of the refinement process was of great relevance.

The structure of the paper is as follows. In Section 2 we explain the refinement process and state the two main results of the paper: (a) quasi–uniformity in diameters, areas and sides for the different levels of refinement; (b) the existence of a limit for the process. In Section 3 we will prove the main results on the quasi–uniformity of the triangulations by looking at the sequence of inscribed polyhedra. As a consequence the first main theorem will have been proved. In Section 4 we will derive an important result on the fact that the sequence of deformations which we see in the zenithal projection is uniformly Lipschitz with uniformly Lipschitz inverse. With this and some additional technical tools, we will prove the existence of a limit to the sequence of triangulations that defines a Lipschitz parameterization of the spherical triangle. We will also derive a result on the equivalence of fractional Sobolev norms between the sphere and a sequence of inscribed polyhedra.

Finally, Section 5 is devoted to checking numerically some of the statements proven in the preceding sections.

2 The partition process

Since the whole process is done triangle–by–triangle, we will work with successive partitions of a single spherical triangle that fits into a hemisphere.

Let us take three points on the unit sphere $\mathcal{S}$ not lying on a maximum circle and consider the spherical triangle defined by taking the shortest arcs of maximum circles that connect the points. The planar triangle defined by these points (and inscribed in the sphere) will be generically denoted by T and the corresponding spherical triangle will be denoted by $\widetilde{T}$. Notice that the map

$$\psi(\mathbf{x}) := \frac{1}{|\mathbf{x}|}\,\mathbf{x} \tag{1}$$

(where $|\cdot|$ denotes the Euclidean norm) maps T bijectively onto $\widetilde{T}$. The maximum distance between points of T will be called the diameter of T and denoted $h(T)$. The area of T will be denoted $a(T)$. Similarly we define $h(\widetilde{T})$ and $a(\widetilde{T})$. Let us now rotate the sphere so that $\widetilde{T}$ has its three vertices on the same horizontal plane intersecting the upper hemisphere, $z = \delta$, where T lies. This quantity $\delta(T) = \delta(\widetilde{T}) > 0$ is also the height of the triangular pyramid with basis on T (it is therefore a tetrahedron) and vertex on the origin (the center of the sphere).

Let us fix a spherical triangle $\widetilde{T}_0$ in the form just given. We divide this triangle into four by taking as vertices the original vertices plus the midpoints of the sides and joining these nodes in the usual *red* refinement: each vertex with the two closest midpoints and the midpoints among themselves. The resulting triangles form a first refinement level:

$$\widetilde{\mathcal{T}}_1 := \{\widetilde{T}_1^{(1)}, \widetilde{T}_2^{(1)}, \widetilde{T}_3^{(1)}, \widetilde{T}_4^{(1)}\}$$

(any numbering works). If we divide similarly each of the resulting triangles we arrive at a second triangulation $\widetilde{\mathcal{T}}_2$, formed by sixteen spherical triangles. Similarly, if this process is repeated m times we obtain a triangulation formed of 4^m triangles. The first main result of this work can be stated already at this point. The proof will be given in Section 3, under the additional hypothesis that

$$h(T_0) < \sqrt{3}.$$

This is a purely technical hypothesis related to the way we make our estimates. It allows still for very large triangles to begin the partition. For larger triangles, numerical experiments seem to confirm that the restriction is not necessary.

Theorem 1 *There exist constants C_1, C_2 and C_3, depending only on $\widetilde{T}_0$, such that for all m*

$$\max\{a(T) \,|\, T \in \widetilde{\mathcal{T}}_m\} \le C_1 \,\min\{a(T) \,|\, T \in \widetilde{\mathcal{T}}_m\},$$

$$\max\{h(T)\,|\,T\in\widetilde{\mathcal{T}}_m\}\le C_2\,\min\{h(T)\,|\,T\in\widetilde{\mathcal{T}}_m\}$$
$$\max\{\text{length}\,(e)\,|\,e\in\widetilde{\mathcal{E}}_m\}\le C_3\min\{\text{length}\,(e)\,|\,e\in\widetilde{\mathcal{E}}_m\},$$
where $\widetilde{\mathcal{E}}_m$ denotes the set of the sides of the triangles that compose $\widetilde{\mathcal{T}}_m$.

Notice that the partition process can be done by looking at the corresponding inscribed polyhedra. By looking at the $m-$th level of refinement $\widetilde{\mathcal{T}}_m$ we can define an open polyhedron formed of 4^m triangular faces obtained by considering the planar triangles whose vertices are the same as those of the spherical triangles. The map ψ defined in (1) defines a bijection of this polyhedron onto $\widetilde{T}_0$. By taking midpoints of the sides of the polyhedron, moving them up to the sphere with ψ, and joining them as before, we obtain a new polyhedron which corresponds to the following triangulation $\widetilde{\mathcal{T}}_{m+1}$.

To each spherical triangle $\widetilde{T}_j^{(m)}$, we associate the corresponding planar triangle $\widehat{T}_j^{(m)}$. The polyhedron formed by the union of these triangles will be denoted $\widehat{\mathcal{S}}_m$. As already mentioned, $\psi(\widehat{\mathcal{S}}_m)=\widetilde{T}_0$.

Using these polyhedra we can define a sequence of parameterizations of the spherical triangle $\widetilde{T}_0$ from T_0. We consider a uniform triangulation of T_0 into 4^m equal–sized triangles $\{T_j^{(m)}\}$ ordered in the same way as we did in $\widetilde{\mathcal{T}}_m$. Then we consider a continuous piecewise affine map $\phi_m : T_0\to\widehat{\mathcal{S}}_m$ that maps each $T_j^{(m)}$ affinely and bijectively onto $\widehat{T}_j^{(m)}$. Finally, $\psi_m:=\psi\circ\phi_m : T_0\to\widetilde{T}_0$ is a continuous piecewise smooth parameterization of $\widetilde{T}_0$.

Theorem 2 *The sequence of maps ψ_m converges in the maximum norm to a function $\psi_{\lim} : T_0\to\widetilde{T}_0$ which is Lipschitz with Lipschitz inverse.*

The proof of this result is left for Section 4, where we will work with a zenithal view of the whole division process to move the whole problem the plane.

A graphical explanation of the different geometrical elements at the second level of refinement in shown in Figure 1.

3 Quasi–uniformity of the triangulations

Proposition 3 *For all m and $1\le j\le 4^m$,*

$$\delta(\widehat{T}_j^{(m)})\,a(\widehat{T}_j^{(m)})>4^{-m}\delta(T_0)a(T_0).$$

Proof. The volume of the tetrahedron defined by the triangle T_0 and the origin is $\delta(T_0)\,a(T_0)/3$. This tetrahedron is divided into four of equal volume by uniformly dividing T_0 into four triangles $\{T_j^{(1)}\,|\,j=1,2,3,4\}$.

The tetrahedron with basis on $T_j^{(1)}$ and vertex on the origin is contained in the one with basis on $\widehat{T}_j^{(1)}$ and the same vertex. Therefore, comparing volumes we have that

$$\delta(\widehat{T}_j^{(1)})\,a(\widehat{T}_j^{(1)})/3>\delta(T_0)a(T_0)/12.$$

Because of the way the partitions are defined by successive divisions, this inequality proves the result. □

Consider the function $g : [0, 2) \to \mathbb{R}$ given by

$$g(h) := \frac{1}{2\sqrt{1 - h^2/4}}.$$

This function is continuous, strictly increasing and $g(h) < 1$ if and only if $h < \sqrt{3}$.

Let

$$h_m := \max_j h(\widehat{T}_j^{(m)})$$

be the maximum of the diameters of the faces of the polyhedron $\widehat{\mathcal{S}}_m$.

Proposition 4 *We have that $h_1 \leq g(h_0)\, h_0$. Therefore, diameters are strictly decreasing if $h_0 < \sqrt{3}$.*

Proof. Let $\mathbf{v}_i$ $(i = 1, 2, 3)$ be the three vertices of T_0 and $\{\mathbf{m}_i\}$ be the midpoints of the sides, numbered by opposition to the corresponding vertex, so that $\mathbf{m}_3$ is the midpoint of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so on. Then the points $\widetilde{\mathbf{m}}_i := \psi(\mathbf{m}_i)$ are the additional vertices that conform $\widehat{\mathcal{S}}_1$. If ℓ_i is the length of the side whose midpoint is $\mathbf{m}_i$, then by the Pythagoras theorem

$$|\mathbf{m}_i| = \sqrt{1 - \ell_i^2/4} \tag{2}$$

and thus

$$|\widetilde{\mathbf{m}}_i - \mathbf{m}_i| = 1 - |\mathbf{m}_i| = 1 - \sqrt{1 - \ell_i^2/4}. \tag{3}$$

Since the diameter of a plane triangle is the maximum length of its sides, then we just have to compare sides of the faces of $\widehat{\mathcal{S}}_1$ with h_0.

Consider the side formed by $\mathbf{v}_i$ and $\widetilde{\mathbf{m}}_j$ $(j \neq i)$. Without loss of generality we take $\mathbf{v}_1$ and $\widetilde{\mathbf{m}}_3$. By looking at the rectangle triangle with vertices on $\widetilde{\mathbf{m}}_3$, $\mathbf{m}_3$ and $\mathbf{v}_1$ (the right angle is at $\mathbf{m}_3$), we obtain from (3) that

$$\begin{aligned} |\widetilde{\mathbf{m}}_3 - \mathbf{v}_1|^2 &= |\widetilde{\mathbf{m}}_3 - \mathbf{m}_3|^2 + |\mathbf{v}_1 - \mathbf{v}_2|^2/4 \\ &= \left(1 - \sqrt{1 - \ell_3^2/4}\right)^2 + \ell_3^2/4 = 2 - 2\sqrt{1 - \ell_3^2/4} \leq (\ell_3\, g(\ell_3))^2. \end{aligned}$$

The last bound is basic calculus. Since g is increasing, we have bounded six of the sides of $\widehat{\mathcal{S}}_1$ by $h_0\, g(h_0)$.

Let us now consider the side formed by $\widetilde{\mathbf{m}}_1$ and $\widetilde{\mathbf{m}}_2$ (as a particular case of the three remaining sides). Since $\mathbf{m}_i = |\mathbf{m}_i|\, \widetilde{\mathbf{m}}_i$ and $|\widetilde{\mathbf{m}}_i| = 1$, then

$$|\mathbf{m}_1 - \mathbf{m}_2|^2 = |\mathbf{m}_1|^2 + |\mathbf{m}_2|^2 - 2\, |\mathbf{m}_1|\, |\mathbf{m}_2|\, \widetilde{\mathbf{m}}_1 \cdot \widetilde{\mathbf{m}}_2$$

and thus

$$\begin{aligned} |\widetilde{\mathbf{m}}_1 - \widetilde{\mathbf{m}}_2|^2 &= 2 - 2\, \widetilde{\mathbf{m}}_1 \cdot \widetilde{\mathbf{m}}_2 = 2 + \frac{|\mathbf{m}_1 - \mathbf{m}_2|^2 - |\mathbf{m}_1|^2 - |\mathbf{m}_2|^2}{|\mathbf{m}_1|\, |\mathbf{m}_2|} \\ &= 2 + \frac{(\ell_1^2 + \ell_2^2 + \ell_3^2)/4 - 2}{\sqrt{1 - \ell_1^2/4}\, \sqrt{1 - \ell_2^2/4}}, \end{aligned}$$

where we have applied (2) and the fact that $|\mathbf{m}_1 - \mathbf{m}_2| = |\mathbf{v}_1 - \mathbf{v}_2|/2 = \ell_3/2$. Bounding $\ell_i \leq h_0$, we trivially obtain that

$$|\widetilde{\mathbf{m}}_1 - \widetilde{\mathbf{m}}_2|^2 \leq 2 + \frac{3h_0^2/4 - 2}{1 - h_0^2/4} = (h_0\, g(h_0))^2,$$

which finishes the proof. □

Consider now the increasing function $f : [0, \sqrt{3}) \to \mathbb{R}$

$$f(h) := \prod_{m=0}^{\infty} (1 + (2\, g(h) - 1)\, g(h)^{2m}).$$

Proposition 5 *If $h_0 < \sqrt{3}$, then*

$$h_m \leq f(h_0)\, h_0\, 2^{-m}.$$

Proof. To simplify some forthcoming expressions, let us introduce the function $\alpha(h) := h^{-2}(2g(h) - 1)$, which is continuous and increasing in $[0, 2)$ and satisfies $g(h) = (1 + \alpha(h)\, h^2)/2$.

By Proposition 4, the sequence h_m is stritcly decreasing. Repeated applications of this result prove that

$$h_m \leq (1 + \alpha(h_{m-1})\, h_{m-1}^2) h_{m-1}/2 \leq \ldots \leq \left(\prod_{j=0}^{m-1} (1 + \alpha(h_j)\, h_j^2) \right) h_0\, 2^{-m}.$$

Since $\alpha(h_j) \leq \alpha(h_0)$ and

$$h_j \leq g(h_{j-1})\, h_{j-1} \leq g(h_0)\, h_{j-1} \leq \ldots \leq g(h_0)^j h_0,$$

we have the bound

$$\prod_{j=0}^{m-1} (1 + \alpha(h_j)\, h_j^2) \leq \prod_{j=0}^{m-1} (1 + \alpha(h_0)\, h_0^2\, g(h_0)^{2j}) \leq \prod_{j=0}^{\infty} (1 + \alpha(h_0)\, h_0^2\, g(h_0)^{2j}) = f(h_0)$$

and the proposition is proven. □

The last direct bound on quantities related to the faces of the sequence of inscribed polyhedra involves a new magnitude. The inradius of a plane triangle $\rho(T)$ is the radius of the inscribed circumference, and bounds from below the length of all sides.

Proposition 6 *If $h_0 < \sqrt{3}$, then for all m and j*

$$\delta(\widehat{T}_j^{(m)})\, \rho(\widehat{T}_j^{(m)}) > \left(\frac{2}{3} \frac{\delta_0\, \rho_0}{f(h_0)} \right) 2^{-m},$$

where $\delta_0 := \delta(T_0)$ and $\rho_0 := \rho(T_0)$.

Proof. By the fact that the inradius of a triangle times its perimeter is twice its area (see [5, Chapter 1]), then for any triangle

$$h(T)\,\rho(T) < a(T) < (3/2)\,h(T)\,\rho(T).$$

These inequalities and Propositions 3 and 5 (notice that $h(\widehat{T}_j^{(m)}) \leq h_m$ by definition), prove that

$$\delta(\widehat{T}_j^{(m)})\,\rho(\widehat{T}_j^{(m)}) \geq \frac{2}{3}\,\frac{\delta(\widehat{T}_j^{(m)})\,a(\widehat{T}_j^{(m)})}{h(\widehat{T}_j^{(m)})} > \frac{2}{3}\,\frac{\delta(T_0)\,a(T_0)}{2^m\,f(h_0)\,h_0} > \frac{2}{3}\,\frac{\delta_0\,\rho_0}{2^m\,f(h_0)}.$$

□

We are now in position to prove a theorem similar to Theorem 1, although defined on the triangles of the polyhedron $\widehat{\mathcal{S}}_m$.

Theorem 7 *There exist constants depending only on h_0, ρ_0 and δ_0 such that*

$$\begin{aligned}
\max_j a(\widehat{T}_j^{(m)}) &\leq \widehat{C}_1 \min_j a(\widehat{T}_j^{(m)}) \\
\max_j h(\widehat{T}_j^{(m)}) &\leq \widehat{C}_2 \min_j h(\widehat{T}_j^{(m)}) \\
\max_j \rho(\widehat{T}_j^{(m)}) &\leq \widehat{C}_3 \min_j \rho(\widehat{T}_j^{(m)})
\end{aligned}$$

Proof. By Propositions 5 and 6 (notice that $\delta(T) < 1$), we have that

$$\left(\frac{2}{3}\,\frac{\delta_0\,\rho_0}{f(h_0)}\right) 2^{-m} < \rho(\widehat{T}_j^{(m)}) \leq h(\widehat{T}_j^{(m)}) \leq f(h_0)\,h_0 2^{-m}.$$

These inequalities and Proposition 3 prove that

$$4^{-m}\delta_0 a(T_0) < a(\widehat{T}_j^{(m)}) \leq (3/2) h(\widehat{T}_j^{(m)})\,\rho(\widehat{T}_j^{(m)}),$$

so we have constants depending uniquely on $\delta_0, a(T_0), \rho_0$ and h_0 such that

$$\begin{aligned}
\widehat{A}_1 \leq\ & 4^m\,a(\widehat{T}_j^{(m)}) \leq \widehat{A}_2, \\
\widehat{H}_1 \leq\ & 2^m\,h(\widehat{T}_j^{(m)}) \leq \widehat{H}_2, \\
\widehat{R}_1 \leq\ & 2^m\,\rho(\widehat{T}_j^{(m)}) \leq \widehat{R}_2.
\end{aligned} \tag{4}$$

A rearrangement of these constants proves the result. □

Proposition 8 *There exists a constant independent of m such that*

$$|\mathbf{x} - \mathbf{y}| \leq L_0 |\psi(\mathbf{x}) - \psi(\mathbf{y})|, \qquad \forall \mathbf{x}, \mathbf{y} \in \widehat{\mathcal{S}}_m.$$

Proof. Notice first that if $\mathbf{x} \in \widehat{\mathcal{S}}_m$, then there exists j such that $\mathbf{x} \in \widehat{T}_j^{(m)}$. By Proposition 6 and (4), it follows that there exists $c_0 > 0$ such that

$$|\mathbf{x}| \geq \delta(\widehat{T}_j^{(m)}) \geq c_0.$$

Consider then the maps $\psi_\delta : \{(x_1, x_2, \delta) \,|\, x_1^2 + x_2^2 \leq 1 - \delta^2\} \to \{\mathbf{y} \in \mathcal{S} \,|\, y_3 \geq \delta\}$, given by $\psi_\delta(\mathbf{x}) := \psi(\mathbf{x})$. Their inverses are $\psi_\delta^{-1}(\mathbf{y}) = (\delta/y_3)\,\mathbf{y}$. If $\delta \geq c_0$ we can obtain a uniform bound, independent of δ, for the derivatives of ψ_δ and its inverse. Upon adequate rotations, the mappings $\psi|_{\widehat{\mathcal{S}}_m} : \widehat{\mathcal{S}}_m \to \widetilde{T}_0$ and their inverses are piecewise formed by elements of this family of functions. The result is then straightforward. □

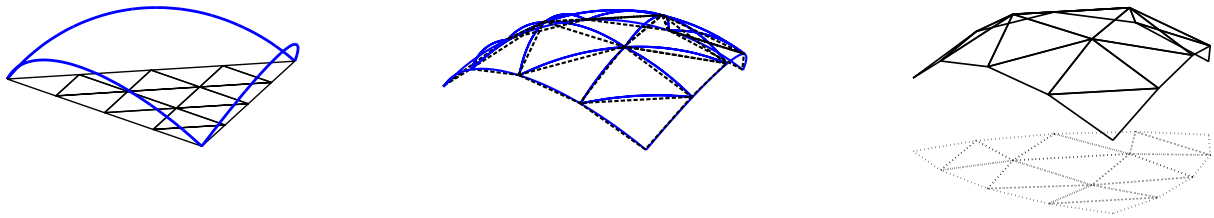

Figure 1: The second level of refinement. On the left, $\widetilde{T}_0$ and the uniform triangulation of T_0. In the center, the corresponding triangulation of the spherical triangle, $\widetilde{\mathcal{T}}_2$, and the inscribed polyhedron $\widehat{\mathcal{S}}_2$. On the right, $\widehat{\mathcal{S}}_2$ and its zenithal view Q_2.

Proof of Theorem 1. Parameterizing triangles and sides of $\widetilde{\mathcal{T}}_m$ from $\widehat{\mathcal{S}}_m$ and using (4) and Proposition 8, it is simple to prove that there exist constants such that for all m

$$\begin{array}{lllll} \widetilde{A}_1 \leq & 4^m a(\widetilde{T}_j^{(m)}) & \leq \widetilde{A}_2, & \forall j, \\ \widetilde{E}_1 \leq & 2^m \text{length}\,(e) & \leq \widetilde{E}_2, & \forall e \in \widetilde{\mathcal{E}}_m. \end{array}$$

This proves the theorem. □

Remark. From Proposition 8 it is simple to see that if $\widehat{\mathcal{S}}_m$ is $m-$th element of the sequence of polyhedra defined as above from an original one $\widehat{\mathcal{S}}_0$, then the maps $\psi : \widehat{\mathcal{S}}_m \to \mathcal{S}$ define $m-$independently bounded isomorphisms between the Sobolev spaces $H^r(\widehat{\mathcal{S}}_m)$ and $H^r(\mathcal{S})$ for $-1 \leq r \leq 1$. (For a precise definition of these spaces on Lipschitz closed surfaces see [7, Chapter 3]). □

4 Mappings and limits

Let us rotate the sphere so that T_0 lies in the plane $z = \delta_0 = \delta(T_0)$. Then we define the zenithal projection

$$\varphi(x_1, x_2, x_3) := (x_1, x_2) \equiv (x_1, x_2, \delta_0).$$

The function

$$\varphi^{-1}(x_1, x_2) := (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$$

is a right inverse of φ. Restricted to the circle $\{x_1^2 + x_2^2 \leq 1 - \delta_0^2\}$, which is where the zenithal projections of $\widetilde{T}_0$ and of all the polyhedra are, φ^{-1} is a $\mathcal{C}^\infty$ function.

If we compose the continuous piecewise affine map $\phi_m : T_0 \to \widehat{\mathcal{S}}_m$ defined in Section 2 with φ we obtain another continuous piecewise affine map

$$\xi_m := \varphi \circ \phi_m : T_0 \subset \mathbb{R}^2 \times \{\delta_0\} \equiv \mathbb{R}^2 \to \mathbb{R}^2.$$

Let then $Q_m := \varphi(\widehat{\mathcal{S}}_m) = \xi_m(T_0)$ be the polygon obtained by viewing $\widehat{\mathcal{S}}_m$ from above. The map $\xi_m : T_0 \to Q_m$ is an invertible deformation of a uniform triangulation of T_0 (with 4^m triangles) into Q_m. This map is a convex combination mapping [6].

Let $\{T_j^{(m)}\}$ be the uniform triangulation of T_0 into 4^m triangles, numbered respecting the ordering we have given to the faces of the polyhedron $\{\widehat{T}_j^{(m)}\}$. The triangles $K_j^{(m)} := \varphi(\widehat{T}_j^{(m)}) = \xi_m(T_j^{(m)})$ tile the polygon $Q_m = \varphi(\widehat{\mathcal{S}}_m)$.

The vertices of $\widehat{\mathcal{S}}_m$ lie on $\mathcal{S}$ and are images by φ^{-1} of the vertices of the triangles that compose Q_m. Since φ and φ^{-1} are $\mathcal{C}^\infty$, it is easy to derive from the inequalities (4) that the set of triangles $\{K_j^{(m)}\}$ satisfies

$$\begin{aligned} A_1 \leq \ & 4^m\, a(K_j^{(m)}) \ \leq A_2, \\ H_1 \leq \ & 2^m\, h(K_j^{(m)}) \ \leq H_2, \\ R_1 \leq \ & 2^m\, \rho(K_j^{(m)}) \ \leq R_2, \end{aligned} \tag{5}$$

for some constants independent of the discretization level.

Proposition 9 *The maps $\xi_m : T_0 \to Q_m$ and $\xi_m^{-1} : Q_m \to T_0$ are uniformly Lipschitz, that is, there exist $L_1, L_2 > 0$ such that for all m*

$$L_1\, |\mathbf{x} - \mathbf{y}| \leq |\xi_m(\mathbf{x}) - \xi_m(\mathbf{y})| \leq L_2\, |\mathbf{x} - \mathbf{y}|, \qquad \forall \mathbf{x}, \mathbf{y} \in T_0.$$

Proof. Let us consider the collection of affine maps that compose ξ_m, i.e.,

$$\xi_j^{(m)} := \xi_m|_{T_j^{(m)}} : T_j^{(m)} \to K_j^{(m)}.$$

Each $D\xi_j^{(m)}$ is constant. By a very well–known bound in finite element theory (see [4, Theorem 3.1.3]), we have (the matrix norm is the one induced by the Eculidean norm)

$$|D\xi_j^{(m)}| \leq h(K_j^{(m)})/\rho(T_j^{(m)}), \qquad |(D\xi_j^{(m)})^{-1}| \leq h(T_j^{(m)})/\rho(K_j^{(m)}).$$

The fact that $\{T_j^{(m)}\}$ is a uniform partition of T_0 and the inequalities (5) prove that

$$|D\xi_j^{(m)}| \leq M_1, \qquad |D(\xi_j^{(m)})^{-1}| = |(D\xi_j^{(m)})^{-1}| \leq M_2,$$

for all m and j. Applying the mean value theorem, it is simple to prove that

$$|\xi_m(\mathbf{x}) - \xi_m(\mathbf{y})| \leq \left(\max_j |D\xi_j^{(m)}| \right) |\mathbf{x} - \mathbf{y}|,$$

which gives the right–most inequality of the statement with $L_2 = M_1$. By working on the inverse, we obtain the other one with $L_1 = 1/M_2$. □

Proposition 10 *The sequence of continuous piecewise affine maps $\xi_m : T_0 \to \mathbb{R}^2$ converges in maximum norm to a function $\xi_{\lim} : T_0 \to \mathbb{R}^2$ that satisfies*

$$L_1\, |\mathbf{x} - \mathbf{y}| \leq |\xi_{\lim}(\mathbf{x}) - \xi_{\lim}(\mathbf{y})| \leq L_2\, |\mathbf{x} - \mathbf{y}|, \qquad \forall \mathbf{x}, \mathbf{y} \in T_0. \tag{6}$$

Proof. Fix m and j. Consider the continuous piecewise affine map

$$\eta_j^{(m)} := \xi_{m+1} \circ (\xi_j^{(m)})^{-1} : K_j^{(m)} \to \mathbb{R}^2.$$

This mapping deforms $K_j^{(m)}$ into a set of four triangles (four of the $K_i^{(m+1)}$), keeping the three vertices of $K_j^{(m)}$ fixed. If $\mathbf{m}_\alpha$ ($\alpha = 1, 2, 3$) are the midpoints of the sides of $K_j^{(m)}$), then

$$|\eta_j^{(m)}(\mathbf{y}) - \mathbf{y}| \le \max_\alpha |\eta_j^{(m)}(\mathbf{m}_\alpha) - \mathbf{m}_\alpha|, \qquad \forall \mathbf{y} \in K_j^{(m)}.$$

The point $\mathbf{m}_\alpha$ is the zenithal view of a midpoint of a side of $\widehat{T}_j^{(m)}$, say $\widehat{\mathbf{m}}_\alpha$. The point $\widetilde{\mathbf{m}}_\alpha := \psi(\widehat{\mathbf{m}}_\alpha)$ is the new vertex generated from $\widehat{\mathbf{m}}_\alpha$ in the $(m+1)-$th step of the partition process. Hence $\varphi(\widetilde{\mathbf{m}}_\alpha) = \eta_j^{(m)}(\mathbf{m}_\alpha)$. Thus

$$|\eta_j^{(m)}(\mathbf{m}_\alpha) - \mathbf{m}_\alpha| = |\varphi(\widetilde{\mathbf{m}}_\alpha) - \varphi(\widehat{\mathbf{m}}_\alpha))| \le |\widetilde{\mathbf{m}}_\alpha - \widehat{\mathbf{m}}_\alpha|.$$

By (3),

$$|\widetilde{\mathbf{m}}_\alpha - \widehat{\mathbf{m}}_\alpha| \le 1 - \sqrt{1 - h_m^2/4} \le h_m^2/4.$$

Let then $\mathbf{x} \in T_0$. For a given m, there exists j such that $\mathbf{x} \in T_j^{(m)}$ and thus $\mathbf{y} := \xi_m(\mathbf{x}) \in K_j^{(m)}$. The foregoing argument shows that

$$|\xi_{m+1}(\mathbf{x}) - \xi_m(\mathbf{x})| = |\eta_j^{(m)}(\mathbf{y}) - \mathbf{y}| \le h_m^2/4.$$

Thus, for any $n \ge 1$,

$$\begin{aligned} |\xi_{m+n}(\mathbf{x}) - \xi_m(\mathbf{x})| &\le \frac{1}{4}\sum_{k=0}^{n-1} h_{m+k}^2 \\ &\le \frac{h_m^2}{4}\sum_{k=0}^{n-1} g(h_m)^{2k} \le \frac{h_m^2}{4}\sum_{k=0}^{\infty} g(h_0)^{2k} = C_0 h_m^2. \end{aligned}$$

where we have applied Proposition 4 (taking m as zero level), the fact that g is increasing and that $g(h_0) < 1$. Since the sequence h_m converges to zero (see Proposition 5), this proves that the sequence ξ_m is Cauchy in $\mathcal{C}(T_0; \mathbb{R}^2)$ and therefore converges. The Lipschitz bound follows trivially from Proposition 9. □

Proof of Theorem 2. Let L_3 be such that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le |\mathbf{x} - \mathbf{y}| \le L_3 |\varphi(\mathbf{x}) - \varphi(\mathbf{y})|, \qquad \mathbf{x}, \mathbf{y} \in \{\mathbf{x} \in \mathcal{S} \,|\, x_3 \ge \delta_0\}. \tag{7}$$

Fix now m, take $\mathbf{x} \in T_0$ and choose j such that $\mathbf{x} \in T_j^{(m)}$ and $\mathbf{v}$ any vertex of that triangle. Notice that

$$\mathcal{S} \ni \phi_m(\mathbf{v}) = \varphi^{-1} \circ \xi_m(\mathbf{v}) = \varphi^{-1} \circ \xi_{\lim}(\mathbf{v}),$$

since $\mathbf{v}$ is a vertex for all the subdivisions and therefore is already fixed. Notice also the points $\phi_m(\mathbf{v})$ and $\phi_m(\mathbf{x})$ belong to the same triangle $\widehat{T}_j^{(m)}$. Then by (6) and (7)

$$\begin{aligned} |\phi_m(\mathbf{x}) - \varphi^{-1} \circ \xi_{\lim}(\mathbf{x})| &\le |\phi_m(\mathbf{x}) - \phi_m(\mathbf{v})| + |\varphi^{-1} \circ \xi_{\lim}(\mathbf{v}) - \varphi^{-1} \circ \xi_{\lim}(\mathbf{x})| \\ &\le h(\widehat{T}_j^{(m)}) + L_2\, L_3 h(T_j^{(m)}) \le (1 + L_2\, L_3)\, h_m. \end{aligned}$$

This proves that $\phi_m \to \varphi^{-1} \circ \xi_{\lim}$ in maximum norm. Therefore $\psi_m = \psi \circ \phi_m \to \psi \circ \varphi^{-1} \circ \xi_{\lim} = \varphi^{-1} \circ \xi_{\lim} =: \psi_{\lim}$ in maximum norm. Taking the limit in the expression

$$L_1|\mathbf{x}-\mathbf{y}| \le |\varphi^{-1} \circ \xi_m(\mathbf{x}) - \varphi^{-1} \circ \xi_m(\mathbf{y})| \le L_2\, L_3|\mathbf{x}-\mathbf{y}|, \qquad \forall \mathbf{x}, \mathbf{y} \in T_0,$$

we prove the bi–Lipschitz character of $\psi_{\lim}$. □

5 Numerical experiments

We are going to illustrate some of the properties related to the quasi–uniformity of this kind of triangulations by means of some numerical experiments. We first inscribe a regular octahedron inside the unit sphere and begin the subdivision process. We compute

$$\frac{\max_j a(\widehat{T}_j^{(m)})}{\min_j a(\widehat{T}_j^{(m)})}, \qquad \frac{\max_j h(\widehat{T}_j^{(m)})}{\min_j h(\widehat{T}_j^{(m)})}, \qquad \frac{\max_j \rho(\widehat{T}_j^{(m)})}{\min_j \rho(\widehat{T}_j^{(m)})}$$

for $m = 0, \ldots, 10$. We are thus computing on the icosahedron instead of on the sphere. Notice that in the finer triangulation we have over eight million triangles. The asymptotics of this sequence shows the behaviour of the constants $\widehat{C}_i$ $(i = 1, 2, 3)$ in Theorem 7. We compare this behaviour with that generated by the uniform refinement that derives from [1], i.e., the uniform triangulation is defined on each face of the octahedron and mapped to $\mathcal{S}$ using ψ. The results are plotted on Figure 2, where it is clear that the Baumgardner–Frederickson triangulation provides a very close to uniform partition.

The same process is then shown on a icosahedral partition of the sphere, as in the original paper [2]. We compare again with the mapping of a uniform triangulation on the icosahedron. These data are plotted in Figure 3. Notice that compared to the octahedral partition, the quality of the constants is improved here, i.e., they are closer to the unit.

In the third experiment we show how the constants degenerate when we take an original triangle which is close to being equatorial. To do that, we consider the equilateral triangles defined by three equispaced points on the parallel with radius $r_n := 1 - 2^{-n}$. We compute the constants of Theorem 7 up to the point where we see them stagnated (notice that all of these triangles satisfy the technical requirement of $h_0 < \sqrt{3}$). In Figure 4 we show how the constants behave as n increases (from $n = 3$ to $n = 9$) and the triangles become larger and larger. The limiting case is degenerate, since the triangles tend to the half–sphere, which is not a spherical triangle, and the points would define instead a closed maximum circle by joining with geodesics.

A final example is set with a set of very large triangles. As T_0, we take five different right triangles disposed on the circle with radius 0.9999. Therefore one side of the triangle is very close to being half a meridian circle. The third point determining T_0 (the point where the right angle is located) is randomly taken and we average the results for five different choices of this point. We only show the way the ratios of areas evolve as the refinement level increases. This is done in Figure 5: the horizontal axis ticks the refinement level.

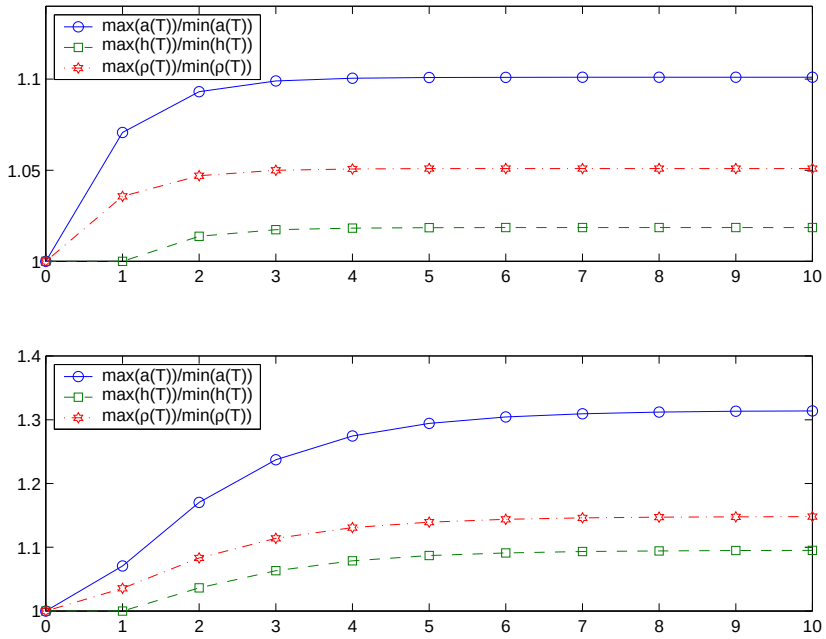

Figure 2: The quasi–uniformity constants for the octahedral partition. Above, for the method of Baumgardner and Frederickson [2] studied in this work. Below, by a simple mapping of a uniform partition of the octahedron.

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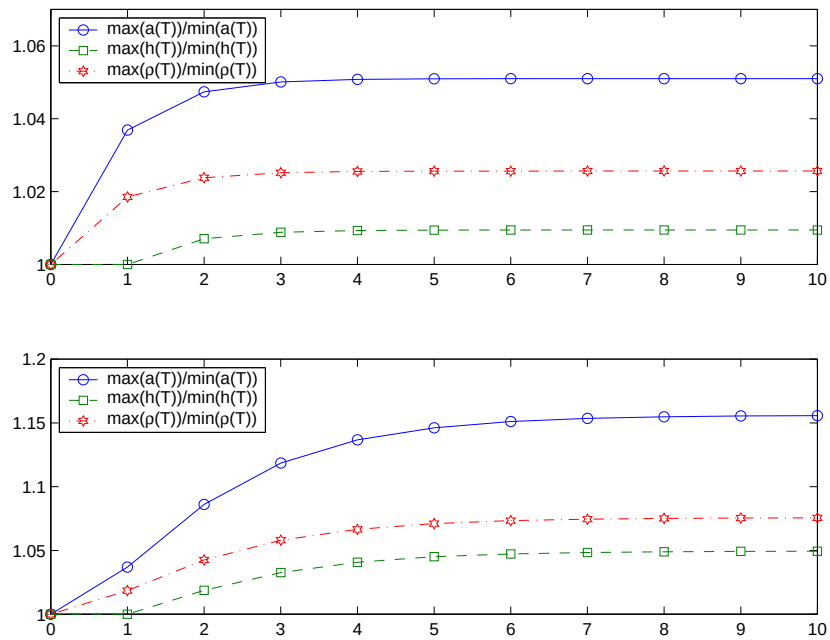

Figure 3: The quasi–uniformity constants for the icosahedral partition. Above, for the method of Baumgardner and Frederickson [2]. Below, by a simple mapping of a uniform partition of the icosahedron.

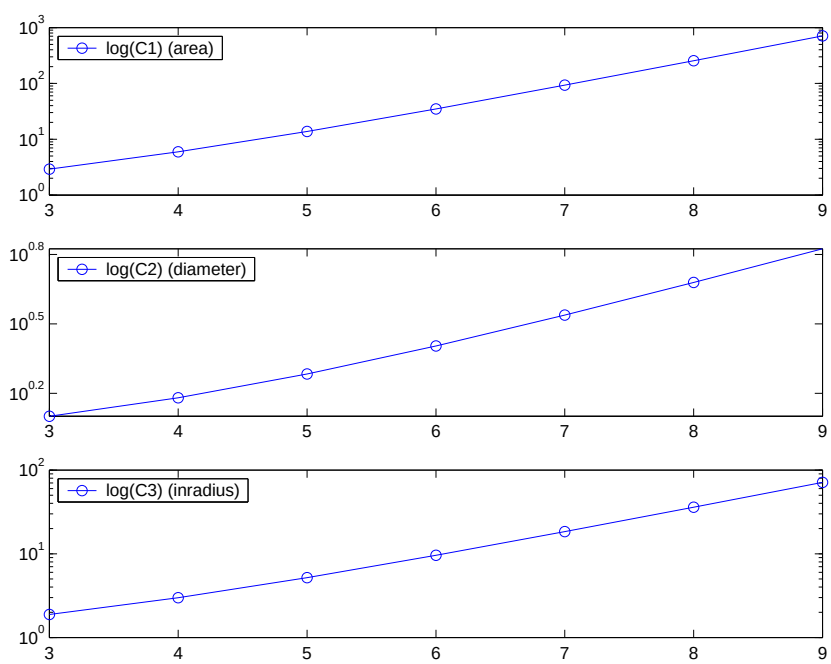

Figure 4: Behaviour of the constants (in logarithmic scale) for equilateral triangles as they degenerate to a half–sphere.

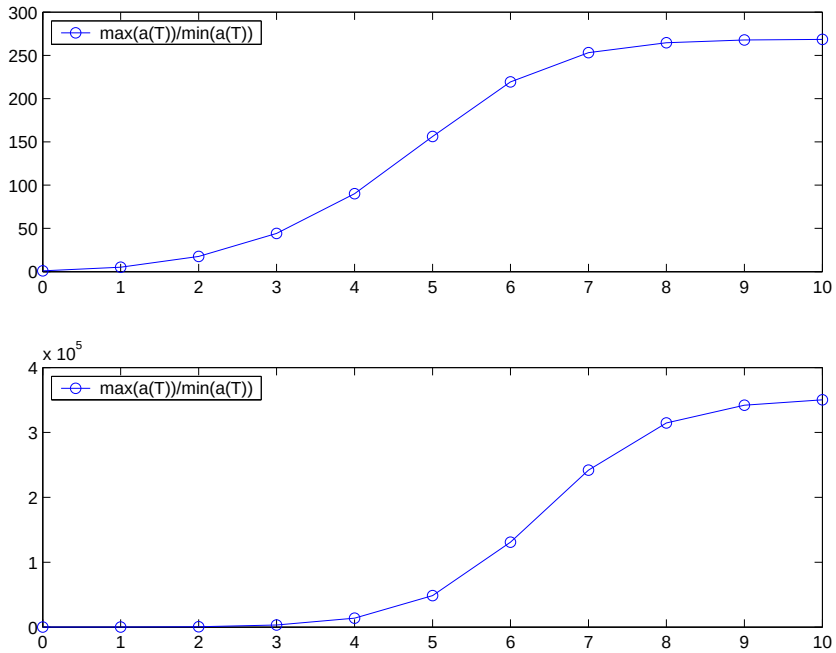

Figure 5: Averaged behaviour for areas of some nearly equatorial triangles: comparison of the Baumgardner–Frederickson triangulation [2] with mapping uniform triangulation [1], which gives extremely non–uniform triangulations.

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