Geometric characterization and generalized principal lattices

Jesús Carnicer
Carmen Godés

Universidad de Zaragoza
Geometric characterization and generalized principal lattices
Jesús Carnicer* and Carmen Godés*
Departamento de Matemática Aplicada. Universidad de Zaragoza. Spain.

Abstract. The sets of nodes in the plane such that the $n$-th degree Lagrange polynomials can be factored as a product of first degree polynomials satisfy a geometric characterization: for each node there exist a set of $n$ lines containing the other nodes. Generalized principal lattices are sets of nodes defined by 3 families of lines. Generalized principal lattices are sets of nodes satisfying the geometric characterization containing exactly 3 alignments of $n+1$ nodes. In this paper we show a converse, valid for degrees $n \leq 7$, if a set of nodes satisfies the geometric characterization and there exist exactly three lines containing more nodes than the degree, then it is is a generalized principal lattice.

1. Introduction
The geometric characterization introduced in [8] identifies unisolvent sets of nodes in the plane such that the Lagrange polynomials can be expressed as a product of first degree polynomials, leading to simple Lagrange interpolation formulae.

Definition 1.1. A set of $(n+2)(n+1)/2$ nodes satisfies the geometric characterization $GC_n$ if for each node $x \in X$, there exist $n$ lines containing all nodes in $X \setminus \{x\}$ but not $x$.

Natural lattices in the plane, which can be defined as the set of all intersection points of $n+2$ lines in general position are configurations satisfying the $GC_n$ condition. Another typical example of node configurations satisfying $GC_n$ are planar principal lattices,

$$\left\{ \frac{i}{n}a_0 + \frac{j}{n}a_1 + \frac{k}{n}a_2 \mid i,j,k \in \{0,1,\ldots,n\}, \quad i+j+k = n \right\},$$

where $a_0, a_1, a_2$ are the vertices of a triangle. A generalization of principal lattices was described in [11], considering lattices generated by three linear pencils. Recently, in [6,7], a wider class of lattices has been described using algebraic cubic pencils. This new set of examples of nodal sets satisfying the $GC_n$ condition motivated a definition of generalized principal lattices. Generalized principal lattices are nodal sets generated from three families of lines, retaining some incidence properties from principal lattices: nodes are the points of concurrence of three lines each belonging to a different family. However, we do not require that lines of the same family are parallel or concurrent. Figure 1 shows a generalized principal lattice defined from a cubic pencil. The envelope of the cubic pencil is a triscuspidal quartic.

In section 2, generalized principal lattices are defined and their properties are analyzed. Sublattices of a generalized principal lattice are defined and analyzed in Proposition 2.4. A characterization of the lines of the families in terms of the nodes is provided in Proposition 2.5. Finally Theorem 2.7 and Corollary 2.8 show that a part of the lattice determines the whole set of lines and nodes. This means that generalized principal lattices are rigid

* Partially supported by the Spanish Research Grant BFM2003-03510, by Gobierno de Aragón and Fondo Social Europeo.
structures in the sense that a change of a node cannot be local and leads to a restructuring of the whole lattice. These results are the main tools because they allow to prove by induction further properties of generalized principal lattices. Section 3 is devoted to show that, generalized principal lattices are sets of nodes satisfying the geometric characterization (GC\(_n\)) containing exactly three alignements of \(n + 1\) nodes. This is shown for degrees up to 7. The proof uses the Cayley-Bacharach theorem. The general case depends on the verification of a conjecture by Gasca and Maeztu (see \([10], [1], [3]\)). The result shows how the GC\(_n\) condition implies a highly structured configuration of nodes and lines.

2. Generalized principal lattices

Generalized principal lattices were introduced in \([6,7]\) and can be described as a certain set of intersections of three families of lines. In order to show Theorem 3.6, we need to weaken some properties of the families of lines making the definition more general. Let us first introduce the following notation where \(N_n := \{0, 1, \ldots, n\} \subset \mathbb{Z}\) and

\[
S_n := \{(i, j, k) \mid i, j, k \in N_n, i + j + k = n\} \subseteq \mathbb{Z}^3.
\]

**Definition 2.1.** Let

\[
L_{0,n}^r := (L_i^r)_{i \in N_n}, \quad r = 0, 1, 2,
\]  \hspace{1cm} (2.1)
be 3 families of lines each containing \( n + 1 \) lines such that the \( 3n + 3 \) lines are distinct, that is,
\[
\left| \bigcup_{r=0}^{2} \{ L_i^r : i \in \mathbb{N}_n \} \right| = 3n + 3,
\] (2.2)
and such that,
\[
L_i^0, L_j^1, L_k^2 \text{ are concurrent, for all } (i, j, k) \in S_n
\] (2.3)
The set of points
\[
X = \{ x_{ijk} \mid x_{ijk} := L_i^0 \cap L_j^1 \cap L_k^2, (i, j, k) \in S_n \},
\] (2.4)
is a \emph{generalized principal lattice} of degree \( n \), if
\[
L_i^0 \cap L_j^1 \cap L_k^2 \cap X \neq \emptyset \implies (i, j, k) \in S_n.
\] (2.5)

**Proposition 2.2.** Let \( X \) be a generalized principal lattice (2.4) defined by the families of lines (2.1).

(a) If a node \( x \in X \) belongs to a line of one of the families \( x \in L_i^r, r \in \{0, 1, 2\} \), then it cannot belong to any other line \( L_j^r, j \in \mathbb{N}_n \setminus \{i\} \) of this family.

(b) The mapping \( (i, j, k) \in S_n \mapsto x_{ijk} \in X \) is a bijection and the cardinal of \( X \) is \(|X| = \binom{n+2}{2}\).

(c) \( X \) is a \( GC_n \) set.

(d) If \( i, j, k \in \mathbb{N}_n \), \( \min(i+j, i+k, j+k) \leq n \), then
\[
(i, j, k) \in S_n \iff L_i^0, L_j^1, L_k^2 \text{ are concurrent.}
\] (2.6)

(e) If \( i, j \in \mathbb{N}_n, r, s \in \{0, 1, 2\}, L_i^r \cap L_j^s \cap X \neq \emptyset \), then \( r \neq s, i+j \leq n \).

**Proof:** (a) Let \( (i, j, k) \in S_n, x_{ijk} = L_i^0 \cap L_j^1 \cap L_k^2 \in X \). If \( x_{ijk} \) lies on \( L_i^r \), then \( L_i^r \cap L_j^1 \cap L_k^2 \cap X \neq \emptyset \). By (2.5), \( (i', j', k') \in S_n \). Then we have \( i' + j + k = n = i + j + k \) and so, \( i' = i \). Analogously, if \( x_{ijk} \) lies on \( L_j^1 \), (resp., \( L_k^2 \)), then \( j' = j \) (resp., \( k' = k \)).

(b) From (a) it follows that points corresponding to different indices in \( S_n \) are distinct. In fact, if \( x_{ijk} = x_{i'j'k'} \), with \( (i, j, k), (i', j', k') \in S_n \), we have by (a) that \( i' = i, j' = j \) and \( k' = k \). Therefore the mapping \( (i, j, k) \in S_n \mapsto x_{ijk} \in X \) is a bijection and the cardinal of the set of points \( X \) defined in (2.4) is \(|S_n| = \binom{n+2}{2}\).

(c) For each \( (i, j, k) \in S_n \), the set of \( n \) lines
\[
L_i^{0'}, \quad i' < i, \quad L_j^{1'} \quad j' < j, \quad L_k^{2'}, \quad k' < k,
\]
contain all nodes of \( X \setminus \{x_{ijk}\} \). By (a), this set of lines does not contain \( x_{ijk} \).

(d) By (2.3), if \( (i, j, k) \in S_n \) then \( L_i^0, L_j^1, L_k^2 \) are concurrent. Conversely, assume that \( L_i^0, L_j^1, L_k^2 \) are concurrent and that \( i + j + k \leq n \). Without loss of generality we may assume that \( j + k \leq n \). By (2.3), (2.4), \( x_{n-j-k,j,k} = L_{n-j-k}^0 \cap L_j^1 \cap L_k^2 \in X \) and, by (2.2), \( L_{n-j-k}^0 \cap L_j^1 \cap L_k^2 = L_j^1 \cap L_k^2 = L_i^0 \cap L_j^1 \cap L_k^2 \). Therefore
\[
L_i^0 \cap L_j^1 \cap L_k^2 \cap X \neq \emptyset,
\]
which implies by (2.5) that \((i, j, k) \in S_n\). The cases \(i + j \leq n\) and \(i + k \leq n\) are completely analogous.

(e) By (a), \(r \neq s\). Without loss of generality, we take \(r = 0\) and \(s = 1\). Assume that 
\[ x = L^0_0 \cap L^1_1 \cap X \neq \emptyset. \]
Since \(X \subseteq \bigcup_{i=0}^{n} L^k_k\), we would have that 
\[ L^0_i \cap L^1_j \cap L^k_k \cap X \neq \emptyset \]
for some \(k \in \mathbb{N}_n\). By (2.5), \(i + j + k = n\), which implies that \(i + j \leq n\). ■

**Remark 2.3.** In [6,7], a generalized principal lattice was defined by families of lines (2.1), (2.2) such that (2.6) hold for each \(i, j, k \in \mathbb{N}_n\). Let us observe that (2.3) and (2.5) are weaker that condition (2.6) and therefore Definition 2.1 is more general than the one given in [6,7]. According to Proposition 2.2 (d), Definition 2.1 implies that (2.6) holds only for all indices \(i, j, k \in \mathbb{N}_n\) satisfying \(\min(i + j, i + k, j + k) \leq n\). However, it might be possible that the lines \(L^0_i \cap L^1_j \cap L^k_k\) are concurrent for \((i, j, k) \notin S_n\), if \(i + j, i + k, j + k > n\), provided that the point of concurrence does not belong to the node set (2.4).

It is convenient to deal with subfamilies of the families (2.1) which we shall denote by

\[ L^r_{i_0:i_1} := (L^r_i)_{i_0 \leq i \leq i_1}, \quad r = 0, 1, 2. \]

We can also define sublattices of a generalized principal lattice \(X\), associated with any triple \((i_0, j_0, k_0) \in S_n\).

**Proposition 2.4.** Let \(X\) be a generalized principal lattice (2.4) defined by the families of lines (2.1). Let \((i_0, j_0, k_0) \in S_{n-m}, 0 \leq m \leq n\). Then

\[ X_{i_0,j_0,k_0} := \{x_{ijk} \mid (i, j, k) \in S_n, \ i \geq i_0, j \geq j_0, k \geq k_0\} \subseteq X. \tag{2.7} \]

is a generalized principal lattice of degree \(m = n - (i_0 + j_0 + k_0)\), defined by the three families of lines

\[ L^0_{0:m}(X_{i_0,j_0,k_0}) := L^0_{i_0:i_0+m}, \ L^1_{0:m}(X_{i_0,j_0,k_0}) := L^1_{j_0:j_0+m}, \ L^2_{0:m}(X_{i_0,j_0,k_0}) := L^2_{k_0:k_0+m}. \tag{2.8} \]

Furthermore we have

\[ X_{i_0,j_0,k_0} = X \setminus \left( \bigcup_{i < i_0} L^0_i \cup \bigcup_{j < j_0} L^1_j \cup \bigcup_{k < k_0} L^2_k \right). \tag{2.9} \]

**Proof:** Clearly all lines (2.8) are distinct and each family contains \(m\) lines. Take \((i, j, k) \in S_n\), with \(i \geq i_0, j \geq j_0\) and \(k \geq k_0\). Since \(X\) is a generalized principal lattice, the lines \(L^0_i, L^1_j, L^2_k\) are concurrent. So, we can define the set (2.7) which is a subset of \(X\). On the other hand, they are lines belonging to each of the families (2.8), corresponding to indices \(i_1 := i - i_0, j_1 := j - j_0, k_1 := k - k_0\) with \(i_1 + j_1 + k_1 = n - (i_0 + j_0 + k_0) = m\), that is \((i_1, j_1, k_1) \in S_m\). Equality (2.9) follows from (2.7) and Proposition 2.2 (a).

For any \(i \geq i_0, j \geq j_0, k \geq k_0, i + j + k \neq n\), we have by (2.5) that

\[ L^0_i \cap L^1_j \cap L^2_k \cap X_{i_0,j_0,k_0} \subseteq L^0_i \cap L^1_j \cap L^2_k \cap X = \emptyset. \]

So, if \(L^0_i \cap L^1_j \cap L^2_k \cap X_{i_0,j_0,k_0} \neq \emptyset\), we must have that \(i + j + k = n\), that is, \((i - i_0, j - j_0, k - k_0) \in S_m\). Therefore (2.7) is a generalized principal lattice defined by the families of lines (2.8). ■
Figure 2 illustrates the sublattices $X_{100}$, $X_{010}$ and $X_{001}$ of a generalized principal lattice.

![Sublattices of a generalized principal lattice](image)

**Figure 2. Sublattices of a generalized principal lattice**

**Proposition 2.5.** Let $X$ be a generalized principal lattice (2.4) defined by the families of lines (2.1).

(a) If $n \geq 1$, there exist exactly 3 lines containing $n + 1$ nodes of $X$, which are $L^r_0$, $r = 0, 1, 2$.

(b) Each of the lines $L^r_i$, $i \in \mathbb{N}_{n-1}$, $r = 0, 1, 2$, is characterized by the following property

$$|L^r_i \cap (X \setminus \bigcup_{i' < i} L^r_{i'})| = n + 1 - i, \quad L^r_i \neq L^s_0, \quad \forall s \in \{0, 1, 2\} \setminus \{r\}.$$  

(c) The families $L^r_{0:n-1} = (L^r_i)_{i \in \mathbb{N}_{n-1}}$, $r = 0, 1, 2$, are uniquely determined by the set $X$, up to permutation of the indices $r \in \{0, 1, 2\}$. Conversely, the set $X$ is determined uniquely by the reduced families $L^r_{0:n-1}$, $r = 0, 1, 2$.

**Proof:** (a) Let us show by induction on $n$ that $L^r_0$, $r = 0, 1, 2$, are the only lines containing $n + 1$ nodes. For $n = 1$, the lattice $X$ consists of three noncollinear points forming a triangle whose sides are $L^0_0$, $L^1_0$, $L^2_0$, and (a) follows. Let us now assume that (a) holds for all lattices up to degree $n - 1$. Clearly,

$$L^0_0 \cap X = \{x_{0jk} \mid j, k \in \mathbb{N}_n, j + k = n\},$$
$$L^1_0 \cap X = \{x_{iok} \mid i, k \in \mathbb{N}_n, i + k = n\},$$
$$L^2_0 \cap X = \{x_{ij0} \mid i, j \in \mathbb{N}_n, i + j = n\},$$

and each of the lines $L^r_0$, $r = 0, 1, 2$ contains exactly $n + 1$ nodes.

Let $L$ be a line containing $n + 1$ nodes, $L \neq L^r_0$. By Proposition 2.2 (c) $X$ is a GC$_n$ set and, by Proposition 2.1 (vi) of [2], lines containing $n + 1$ nodes must intersect at a node, that is,

$$L \cap L^0_0 \cap X \neq \emptyset. \quad (2.10)$$

On the other hand, by Proposition 2.4, $X \setminus L^0_0 = X_{100}$ is a generalized principal lattice of degree $n - 1$ and, by (2.10), $L$ is a line containing $n$ nodes of $X_{100}$. By the induction hypothesis $L = L^0_1$ or $L = L^1_0$ or $L = L^2_0$. By Proposition 2.2 (a),

$$L^0_1 \cap L^0_0 \cap X = \emptyset. \quad (2.11)$$

Comparing (2.10) and (2.11) we deduce that $L \neq L^0_1$ and so, $L$ must be either $L^0_1$ or $L^2_0$.

(b) We can take without loss of generality $r = 0$, and observe that, by Proposition 2.4, $X_{i00} = X \setminus \bigcup_{i' < i} L^r_{i'}$ is a generalized principal lattice of degree $n - i \geq 1$. By (a),
the only lines containing \( n - i + 1 \) nodes of \( X_{i00} \) are the first lines in each of the families \( L_i^0 = L_i^0(X_{i00}), \) \( L_0^1 = L_0^1(X_{100}) \) and \( L_0^2 = L_0^2(X_{i00}) \). Therefore \( L_i^0 \) is the unique line containing \( n + 1 - i \) nodes of \( X_{i00} \) and distinct from \( L_0^1 \) and \( L_0^2 \).

(c) By (a), the lines \( L_i^0, L_0^1, L_0^2 \) are the only lines containing \( n + 1 \) nodes and so they are uniquely determined, up to a permutation of the indices \( r \in \{0, 1, 2\} \). Once we have fixed the indices, we can use (b) to show that the families \( L_{0,n-1}^r \) are uniquely determined. For the converse, we see that each node in \( X \) except \( x_{n00}, x_{0n0}, x_{00n} \) can be determined as the intersection of three lines of the three reduced families and the three remaining ones are determined by the intersection of two of them \( x_{n00} = L_0^0 \cap L_0^1, x_{0n0} = L_0^0 \cap L_0^2, x_{00n} = L_0^1 \cap L_0^2 \). ■

**Remark 2.6.** According to Proposition 2.2 (a), if a node lies on a line, it cannot lie on any other line of the same family. From the characterization of Proposition 2.5 (b), it follows that the lines \( L_i^r, r = 0, 1, 2 \), contain exactly \( n + 1 - i \) nodes. However, this fact does not exclude the existence of lines \( L \) not belonging to any of the families with the same number of nodes.

The following result shows that a sublattice of degree 5, determines in some sense the rest of the generalized principal lattice.

**Theorem 2.7.** Let \( X \) and \( \tilde{X} \) two generalized principal lattices of degree \( n \geq 5 \). Let \( L_{0,n}^r(X) \) and \( L_{0,n}^r(\tilde{X}) \), \( r = 0, 1, 2 \), be the families of lines associated to each of the sets. If

\[
X \setminus L_0^0(X) = \tilde{X} \setminus L_0^0(\tilde{X}), \quad L_0^1(X) = L_0^1(\tilde{X}), \quad L_0^2(X) = L_0^2(\tilde{X})
\]

then \( X = \tilde{X} \).

**Proof:** Taking into account that \( X_{100} = X \setminus L_0^0(X) = \tilde{X} \setminus L_0^0(\tilde{X}) = \tilde{X}_{100} \), we can use Proposition 2.4 and 2.5 to obtain

\[
L_{1:n-1}^0(X) = L_{0:n-2}^0(X_{100}) = L_{1:n-1}^0(\tilde{X}),
\]

\[
L_{1:n-2}^1(X) = L_{0:n-2}^1(X_{100}) = L_{1:n-2}^1(\tilde{X}),
\]

\[
L_{2:n-2}^2(X) = L_{0:n-2}^2(X_{100}) = L_{2:n-2}^2(\tilde{X}).
\]

From the fact that \( L_0^0(X) = L_0^0(\tilde{X}) \) and \( L_0^2(X) = L_0^2(\tilde{X}) \), it follows that \( r_1 = 1, r_2 = 2 \) and \( r_0 = 0 \). Then we have

\[
L_i^0(X) = L_i^0(\tilde{X}), \quad 1 \leq i \leq n - 1,
\]

\[
L_j^1(X) = L_j^1(\tilde{X}), \quad L_k^2(X) = L_k^2(\tilde{X}), \quad 0 \leq j, k \leq n - 2.
\]

Therefore \( x_{ijk} = \tilde{x}_{ijk} \) if and only if \((i, j, k) \in S_n \) and at least two of the lines determining \( x_{ijk} \) coincide. We then deduce that \( x_{ijk} = \tilde{x}_{ijk} \) for all triples \((i, j, k) \in S_n \) such that at least two of the following inequalities hold

\[
1 \leq i \leq n - 1, \quad 0 \leq j \leq n - 2, \quad 0 \leq k \leq n - 2.
\]

Taking into account that \( i = n - j - k \), the above inequalities the problem are reduced to

\[
0 \leq j \leq n - 2, \quad 0 \leq k \leq n - 2, \quad 1 \leq j + k \leq n - 1.
\]
Corresponding nodes of $X$ and $\hat{X}$ might be different if and only if the indices are $(0, 0, n)$, $(0, 1, n - 1)$, $(0, n - 1, 1)$, $(0, n, 1)$. Since $n \geq 5$, we conclude that there are at least two nodes $x_{0,n-k,k} = \hat{x}_{0,n-k,k}$, $2 \leq k \leq n - 2$ determining the line $L_0^0(X) = L_0^0(\hat{X})$. Now, it is straightforward to show that the remaining nodes coincide.

**Corollary 2.8.** Let $X$ and $\hat{X}$ two generalized principal lattices of degree $n \geq 5$. Let $L_{0,n}^r(X)$ and $L_{0,n}^r(\hat{X})$, $r = 0, 1, 2$, be the families of lines associated to each of the sets. If

$$X \setminus \bigcup_{i=0}^{n-4} L_i^0(X) = \hat{X} \setminus \bigcup_{i=0}^{n-4} L_i^0(\hat{X}), \quad L_0^1(X) = L_0^1(\hat{X}), \quad L_0^2(X) = L_0^2(\hat{X})$$

then $X = \hat{X}$.

**Proof:** It follows from Theorem 2.7 by induction on $n \geq 5$.

---

3. **Generalized principal lattices and geometric characterization.**

In the previous section, it has been shown that all principal lattices satisfy the geometric characterization. However the converse is not true for $n \geq 2$, since the natural lattices are not generalized principal lattices. On the other hand the complete classification of sets of nodes satisfying the $GC_n$ (see [5]) depends on the verification of the conjecture established by Gasca and Maeztu [10] up to degree $n$, which has been verified only for degrees up to 4 [1, 3].

**Conjecture 3.1.** Let $X$ be a set satisfying the $GC_n$ condition, then there exist at least one line $L$ such that $|X \cap L| = n + 1$.

The following consequence of the verification of Conjecture 3.1, was proved in Theorem 4.1 of [4]. Let us state that result for the sake of completeness.

**Theorem 3.2.** Assume that Conjecture 3.1 holds for all degrees up to $\nu$. If $X$ is a set satisfying the $GC_n$ condition and $n \leq \nu$, then there exist at least three lines containing $n + 1$ nodes of $X$.

In order to classify the $GC_n$ configurations of nodes, the **defect** of a $GC_n$ configurations was introduced in [2] (there we used the name “default”).

**Definition 3.3.** Let $X$ be a set satisfying the $GC_n$ and let $\mathcal{K}$ the set of all lines of $X$ containing exactly $n + 1$ nodes. Then the **defect** of $X$ is the number $d = n + 2 - |\mathcal{K}|$. We say that $X$ is a $GC_{n,d}$ set to indicate that $X$ satisfies the $GC_n$ condition and that the number of lines containing $n + 1$ nodes is $n + 2 - d$.

The defect was used in [5] to provide a complete classification of $GC_n$ configurations for degree less than 4.

Let us observe that, since the number of lines containing $n + 1$ nodes is at most $n + 2$, the defect is always a nonnegative number. Conjecture 3.1 means that the defect of a $GC_n$ set is less than $n + 2$. Furthermore, according to Theorem 3.2, if Conjecture 3.1 holds for all degrees less than $n$, the defect of any $GC_n$ configuration must be less than or equal to $n - 2$. 

7
We want to show that if Conjecture 3.1 holds for degrees less that \( n \), generalized principal lattices are just \( GC_{n,n-1} \) sets, that is, sets of nodes satisfying the \( GC_n \) condition with exactly 3 lines containing \( n+1 \) nodes.

The next lemma contains results which will be used in this section. Part of these results have been discussed in [4].

**Lemma 3.4.** Let \( X \) be a \( GC_n \) set and let \( L \) be a line such that \( |X \cap L| = n+1 \).

(a) If \( X \) has defect \( d \), then \( X \setminus L \) is a \( GC_{n-1,d'} \) set with \( d' \leq d \).

(b) If \( X \setminus L \) has defect \( d' \leq n-2 \) then \( X \) is a \( GC_{n,d} \) set with \( d \leq d' + 1 \).

(c) Assume that Conjecture 3.1 holds for all degrees up to \( \nu \) and \( n \leq \nu + 3 \). If \( X \) is \( GC_{n,n-1} \), then \( X \setminus L \) is a \( GC_{n-1,n-2} \) set.

(d) If \( 1 \leq n \leq 7 \) and \( X \) is a \( GC_{n,n-1} \) set, then \( X \setminus L \) is a \( GC_{n-1,n-2} \) set.

**Proof:** (a) follows directly from Proposition 2.5 (a) of [4]. (b) is discussed in Remark 3.6 of [4] and it is a consequence of Corollary 3.5 of [4].

(c) The cases \( n = 1 \) and \( n = 2 \) are trivial. If \( n = 3 \) and \( X \) is \( GC_{3,2} \), then we have by (a) that \( X \) is a \( GC_{2,d'} \), with \( d' \leq 2 \). From Proposition 3 (b) of [5], we conclude that \( d' = 0 \) or \( d' = 1 \). Finally (b) implies that \( 2 \leq d' + 1 \), that is \( d' \geq 1 \). Therefore \( d' = 1 \) and \( X \setminus L \) is a \( GC_{2,1} \) set.

Now, let us assume that Conjecture 3.1 holds for all degrees up to \( \nu \). Let \( X \) be a \( GC_{n,n-1} \) set with \( 3 \leq n \leq \nu + 3 \) and let \( L \) be any line such that \( |X \cap L| = n+1 \). By (a), we have that \( X \setminus L \) is a \( GC_{n-1,d} \) set with \( d_1 \leq n-1 \) Since \( d_1 \leq n-1 \), we have that \( X \setminus L \) has \( n+1-d_1 \geq 2 \) lines containing \( n \) nodes. Let \( L_1 \) be a line containing \( n \) nodes. Applying (a) again, we deduce that \( X \setminus (L \cup L_1) \) is a \( GC_{n-2,d_2} \) set, where \( d_2 \leq d_1 \leq n-1 \). Since \( n-d_2 \geq n-d_1 \geq 1 \), there exists at least one line \( L_2 \) containing \( n-1 \) nodes of \( X \setminus (L \cup L_1) \).

Again by (a), we deduce that \( X \setminus (L \cup L_1 \cup L_2) \) is a \( GC_{n-3,d_3} \) set with \( d_3 \leq d_2 \leq d_1 \leq n-1 \).

By hypothesis, Conjecture 3.1 holds for all degrees up to \( n-3 \), and using Theorem 3.2, we can deduce that \( X \setminus (L \cup L_1 \cup L_2) \) is a \( GC_{n-3,d_3} \) set with \( d_3 \leq n-4 \). Applying (b), we obtain successively that \( d_2 \leq n-3 \) and that \( d_1 \leq n-2 \). Finally we apply again (b) to deduce that \( n-1 \leq d_1 + 1 \) obtaining \( d_1 \geq n-2 \). So we have obtained \( d_1 = n-2 \). In the same way follows that \( d_2 = n-3 \) and \( d_3 = n-4 \).

(d) Follows from (c), taking into account that Conjecture 3.1 has been proved in [1] (see also [3]) for degrees up to \( \nu = 4 \). ■

We also use Theorem CB4 of [9], a version of the Cayley-Bacharach Theorem which we restate below for the sake of completeness

**Theorem 3.5.** (Cayley-Bacharach) Let \( \Gamma_1, \Gamma_2 \) be plane curves in the projective plane of degree \( d \) and \( e \) respectively meeting in \( de \) distinct points. If \( \Gamma \) is any plane curve of degree \( d+e-3 \) containing all but one point of \( Y = \Gamma_1 \cap \Gamma_2 \), then \( \Gamma \) contains all of \( Y \).

**Theorem 3.6.** Assume that Conjecture 3.1 holds for all degrees up to \( \nu \). For any \( n \leq \nu + 3 \), the following statements are equivalent

(a) \( X \) is a generalized principal lattice of degree \( n \)

(b) \( X \) is a \( GC_{n,n-1} \) set.

**Proof:** From Proposition 2.2 (c) and Proposition 2.5 (a) we see that (a) implies (b). Let us show by induction on \( n \) that (b) implies (a). The cases \( n = 1 \) and \( n = 2 \) are
straightforward. Let $3 \leq n \leq \nu + 3$ and assume that all GC$_{n-1,n-2}$ sets are generalized principal lattices. Let $X$ be a GC$_{n,n-1}$ set. By Definition 3.3, there exist at least three lines $L_0^r$, $L_1^r$ and $L_2^r$ such that $|L_r^r \cap X| = n + 1$, $r = 0, 1, 2$. Let us define the sets

$$X_{100} := X \setminus L_0^0, \quad X_{010} := X \setminus L_1^1, \quad X_{000} := X \setminus L_2^2.$$  

By Lemma 3.4 (c), $X_{100}$, $X_{010}$ and $X_{001}$ are GC$_{n-1,n-2}$ sets and by the induction hypothesis, these sets are generalized principal lattices of degree $n - 1$. Now we apply Proposition 2.5 (c) and deduce that the lines $L_{0,n-2}^r(X_{100})$, $L_{0,n-2}^r(X_{010})$ and $L_{0,n-2}^r(X_{001})$, $r = 0, 1, 2$, are determined by the set $X$ up to a permutation of the indices. Since $L_0^r$ and $L_2^r$ contain $n$ nodes of the set $X_{100}$, the indices can be rearranged in order to have

$$L_0^0(X_{100}) = L_0^1, \quad L_0^2(X_{100}) = L_0^2. \quad (3.1)$$

Analogously we can reorder the indices in the other families

$$L_0^0(X_{010}) = L_0^0, \quad L_0^2(X_{010}) = L_0^2, \quad L_0^0(X_{001}) = L_0^0, \quad L_0^1(X_{001}) = L_0^1. \quad (3.2)$$

Let us now define

$$L_i^0 := L_{i-1}^0(X_{100}), \quad L_i^1 := L_{i-1}^1(X_{010}), \quad L_i^2 := L_{i-1}^2(X_{001}), \quad i = 1, \ldots, n - 1. \quad (3.3)$$

We also choose $L_n^0$ (resp., $L_n^1$, $L_n^2$) to be any line passing through the node $L_0^1 \cap L_0^2 \cap X$ (resp., $L_0^0 \cap L_0^2 \cap X$, $L_0^0 \cap L_0^1 \cap X$) and not containing any other node. In this way three families of lines $L_{0,n}^r$, $r = 0, 1, 2$, have been defined.

Let us observe that $X_{110} = X \setminus (L_0^0 \cup L_0^2)$ is a generalized principal sublattice of $X_{100}$ and of $X_{010}$. So the families of lines of $L_{0,n-3}^r(X_{110})$ are determined by $X_{110}$ up to a permutation of the indices. A suitable permutation of indices allows us to choose

$$L_0^0(X_{110}) = L_1^0, \quad L_0^1(X_{110}) = L_1^1, \quad L_0^2(X_{110}) = L_1^2 \quad (3.4)$$

since $L_1^0$, $L_1^1$ and $L_1^2$ contain $n - 1$ nodes of the set $X_{110}$. Using Proposition 2.5 (c), we can identify the sets of lines

$$L_{0,n-3}^0(X_{100}) = L_{0,n-3}^0(X_{110}) = L_{1,n-2}^0(X_{100}),$$

$$L_{0,n-3}^1(X_{010}) = L_{0,n-3}^1(X_{110}) = L_{1,n-2}^1(X_{100}). \quad (3.5)$$

Combining (3.2) and (3.3) with (3.5), we deduce that $L_i^0 = L_i^0(X_{010})$ and $L_i^1 = L_i^1(X_{100})$ for all $i = 0, 1, \ldots, n - 2$. Considering $X_{101}$ and $X_{011}$ we can obtain the following identifications

$$L_i^0(X_{010}) = L_i^0(X_{001}) = L_i^0, \quad L_i^1(X_{100}) = L_i^1(X_{001}) = L_i^1, \quad L_i^2(X_{100}) = L_i^2(X_{010}) = L_i^2, \quad (3.6)$$

for all $i = 0, \ldots, n - 2$.

Now we can show that $X$ is a generalized principal lattice. Let $(i, j, k) \in S_n$ and let us show that the lines $L_i^0, L_j^1, L_k^2$ are concurrent. If one of the indices is $n$, the other ones
must be 0 and the choice of the lines $L_n^r$ allow us to ensure the concurrence of $L_i^0, L_j^1, L_k^2$. So, we can assume that $\max(i, j, k) \leq n - 1$. If $i = \max(i, j, k)$, since $i + j + k = n \geq 3$, we must have that

$$3i \geq i + j + k = n \geq 3, \quad 2j \leq i + j + k = n < 2n - 2, \quad 2k \leq i + j + k = n < 2n - 2,$$

and so we have that $1 \leq i \leq n - 1$, $j, k \leq n - 2$. Then, by (3.1), (3.2), (3.3), (3.6)

$$L_i^0 \cap L_j^1 \cap L_k^2 = L_{i-1}^0(X_{100}) \cap L_j^1(X_{100}) \cap L_k^2(X_{100}) \neq \emptyset.$$

Therefore the lines $L_i^0, L_j^1, L_k^2$ meet at a point in $X_{100}$. Analogously if $j = \max(i, j, k)$ (resp. $k = \max(i, j, k)$) it can be shown that the lines $L_i^0, L_j^1, L_k^2$ meet at a point in $X_{010}$ (resp., in $X_{001}$).

Let us see that the points $x_{ijk} = L_i^0 \cap L_j^1 \cap L_k^2$, $(i, j, k) \in S_n$ are distinct. The nodes $x_{ijk}$ with $i \geq 1$ are all distinct because they belong to points of the principal lattice $X_{100}$ obtained as intersection of different lines. In addition, the nodes $X_{100}$ are distinct from the nodes $x_{0jk} \in L_i^0 \cap X$ because $L_i^0 \cap X$ and $X_{100}$ are disjoint sets. The nodes $x_{0jk}$ with $j > 0$ are distinct because they are intersections of different lines corresponding to the generalized principal lattice $X_{010}$. Finally the nodes in $X_{100} \cap L_0^n$ are distinct from the node $x_{000} \in L_i^0 \cap L_j^1 \cap X$ because $X_{010}$ and $L_0^1 \cap X$ are disjoint sets.

Now let us see that condition

$$x \in L_i^0 \cap L_j^1 \cap L_k^2 \cap X \neq \emptyset, \quad i, j, k \in \mathbb{N}_n$$

implies that $i + j + k = n$.

Taking into account that $X = X_{100} \cup X_{010} \cup X_{001}$, we may assume without loss of generality that $x \in X_{100}$, that is,

$$x \in L_i^0 \cap L_j^1 \cap L_k^2 \cap X_{100} \neq \emptyset, \quad i, j, k \in \mathbb{N}_n. \quad (3.7)$$

Since $X_{100} = X \setminus L_0^n$, we must have that, if (3.7) holds, then $i > 0$. On the other hand, no node of $X_{100}$ belongs to $L_n^1$ or $L_n^2$ by the choice of these lines. So we can ensure that $j, k < n$.

If $i = n$, $j, k \leq n - 2$, and (3.7) holds then $x \in L_n^0 \cap X$ and, by the choice of $L_n^0$, $x = L_0^1(X_{100}) \cap L_0^2(X_{100})$. Since $X_{100}$ is a generalized principal lattice and

$$x = L_n^0 \cap L_j^1(X_{100}) \cap L_k^2(X_{100}) = L_{n-1}^0(X_{100}) \cap L_j^1(X_{100}) \cap L_k^2(X_{100}),$$

we deduce from Proposition 2.2 (a) that $j = k = 0$ and then we have $i + j + k = n$.

If we assume that $1 \leq i \leq n - 1$, $j, k \leq n - 2$ and that (3.7) holds, we can write

$$x \in L_i^0 \cap L_j^1 \cap L_k^2 \cap X_{100} = L_{i-1}^0(X_{100}) \cap L_j^1(X_{100}) \cap L_k^2(X_{100}) \cap X_{100}.$$ 

Since $X_{100}$ is a generalized principal lattice, we deduce from (2.5) that $i - 1 + j + k = n - 1$, that is, $i + j + k = n$.

Therefore, the only remaining cases to study are $1 \leq i \leq n$, $\max(j, k) = n - 1$. 

10
If \(1 \leq i \leq n - 1, j = n - 1, k \leq n - 2\) and (3.7) holds we can write \(x \in L^0_{i-1}(X_{100}) \cap L^2_k(X_{100}) \cap X_{100}\). By Proposition 2.2 (e), we have that \(i - 1 + k \leq n - 1\), that is \(i + k \leq n\). If \(i = n, j = n - 1, k \leq n - 2\) and (3.7) holds, then

\[
x = L^0_n \cap L^2_k(X_{100}) = L^0_{n-1}(X_{100}) \cap L^1_j(X_{100}) \cap L^2_k(X_{100})
\]

and by Proposition 2.2 (a), \(k = 0\), so we have again \(i + k = n + 0 \leq n\). Analogously, if \(1 \leq i \leq n, j \leq n - 2, k = n - 1\) and (3.7) holds, we have \(i + j \leq n\).

Therefore the cases to check are reduced to \(1 \leq i \leq n, \min(j,k) \leq n - i, \max(j,k) = n - 1\).

If \(1 \leq i \leq n - 1, j = n - 1, i + k \leq n - 1\) and (3.7) holds we can write

\[
x \in L^0_i \cap L^1_j \cap L^2_k = L^0_i(X_{010}) \cap L^1_j(X_{010}) \cap L^2_k(X_{010})
\]

Taking into account that \(\min(i + j - 1, i + k, j - 1 + k) \leq n - 1\), Proposition 2.2 (d) implies that, \(i + j - 1 + k = n - 1\), that is, \(i + j + k = n\). Analogously, if \(1 \leq i \leq n - 1, k = n - 1, i + j \leq n - 1\) and (3.7) holds, then \(i + j + k = n\).

So, it only remains to check the case where \(1 \leq i \leq n, \min(j,k) = n - i, \max(j,k) = n - 1\).

If \(1 \leq i \leq n, j = n - 1, i + k = n\), and (3.7) holds, then we have that

\[
x \in L^0_i \cap L^1_{n-1} \cap L^2_k \cap X_{100} = L^0_i \cap L^2_k \cap X_{100} = L^0_i(X_{100}) \cap L^1_k(X_{100}) \cap X_{100}.
\]

Therefore

\[
x = x_{i0k} = L^0_i \cap L^1_0 \cap L^2_k \in L^1_{n-1}.
\]

Since \(L^1_{n-1}\) is the line passing through the points

\[
x_{1,n-1,0} = L^1_1 \cap L^1_{n-1} \cap L^2_0, \quad x_{0,n-1,1} = L^0_0 \cap L^1_{n-1} \cap L^2_1
\]

we deduce that \(x_{i0k}, x_{1,n-1,0}, x_{0,n-1,1}\) are collinear. Let \(\Gamma_1\) be the curve of degree \(i + 1\) formed by the union of the lines of the family \(L^0_0,\ldots, L^0_{i-1}\) and let \(\Gamma_2\) be the curve of degree \(k + 1\) formed by the union of the lines of the family \(L^2_{i+1}, \ldots, L^2_{n-k}\) and let \(\Gamma\) be the curve formed by the union of the lines \(L^1_{i,n-1}\). Since \(i + k < n - 1\), we see that \(Y = \Gamma_1 \cap \Gamma_2\), consists of the \((i + 1)(k + 1)\) distinct points

\[
x'_{i',n-i'-k',k'} = 0, \ldots, i, \quad k' = 0, \ldots, k.
\]

The curve \(\Gamma\) contains all points of \(Y\) except \(x_{0n0} = L^0_0 \cap L^1_n \cap L^2_0\). Theorem 3.5, implies that \(x_{0n0} \in L^1_{j'}\) for some \(j' \in \{1, \ldots, n - 1\}\), that is

\[
L^0_0(X_{010}) \cap L^1_{j'-1}(X_{010}) \cap L^2_0(X_{010}) \cap X_{010} \neq \emptyset.
\]

Since \(X_{010}\) is a generalized principal lattice and \(x_{0n0} \in L_{n-1}(X_{010})\), we deduce from Proposition 2.2 (a) that \(j' - 1 = n\), which is a contradiction. Analogously, we can show that, if \(1 \leq i \leq n, k = n - 1, i + j = n\), then (3.7) cannot hold.

So we have shown that (3.7) implies that \(i + j + k = n\). Analogously we deduce that

\[
x \in L^0_i \cap L^1_j \cap L^2_k \cap X_{100} \neq \emptyset, \quad i,j,k \in \mathbb{N}_n
\]

imply that \(i + j + k = n\).

In view of the previous result, generalized principal lattices represent the GC\(_n\) set with maximal defect, assuming that Conjecture 3.1 holds for all degrees less than or equal to \(n - 3\).
**Remark 3.6.** We observe that the conjecture holds for all degrees up to 4 (see \([1,3]\)). Then Theorem 3.5, implies that all \(GC_{n,n-1}\) sets are generalized principal lattices at least for degrees \(n \leq 7\).

**References.**