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Abstract

In this paper we propose and analyse a novel numerical method for a Helmholtz transmission problem in a bounded domain, with non–homogeneous mixed conditions on the exterior boundary. The method relies on the superposition of a classical Lagrange finite element method on a triangulation of the domain without the interior obstacles and a general stable boundary element method for an exterior Helmholtz transmission problem. The analysis of the scheme is carried out by transforming the exact and the approximated equations into an abstract operator equation of the second kind with a non–standard approximation of it. A numerical example for a two dimensional case shows the good behaviour of the method and its advantages for some particular geometries.

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1 Introduction

In this work we propose and analyse a novel formulation and its numerical approximation for a class of Helmholtz transmission problems in bounded domains. The basic principle underlying this formulation is the decomposition of the solution in two parts. One of them takes care of boundary conditions in the outer boundaries and ignores interior obstacles (acting as a sort of incident wave that, unlike in the usual exterior scattering problems, is unknown). The second part solves a Helmholtz transmission problem in the whole of the space by employing layer potentials.

With a geometry in two or three dimensions as that described in Figure 1 (we will specify conditions later on), we are basically trying to solve the following problem (γ_Γ and

∂_Γ denote traces and normal derivatives respectively):

$$\begin{aligned}
&\Delta\omega^+ + \lambda^2\omega^+ = 0, \qquad \text{in } \Omega^+,\\
&\Delta\omega^- + \mu^2\omega^- = 0, \qquad \text{in } \Omega^-,\\
&\gamma_\Gamma\omega^+ - \gamma_\Gamma\omega^- = g_\Gamma, \qquad \partial_\Gamma\omega^+ - \alpha\,\partial_\Gamma\omega^- = \chi_\Gamma,\\
&\gamma_D\omega^+ = g_D, \qquad \partial_N\omega^+ = \chi_N.
\end{aligned}$$

The solution is written formally as

$$\omega^+ = u + \int_\Gamma \phi_\lambda(\,\cdot - \mathbf{x})\psi^+(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}), \qquad \omega^- = \int_\Gamma \phi_\mu(\,\cdot - \mathbf{x})\psi^-(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}),$$

where ϕ_ρ is the outgoing fundamental solution of $\Delta+\rho^2$. The unknowns are two densities $\psi^\pm$, defined on the interface Γ and a function u, such that $\Delta u + \lambda^2 u = 0$ in the domain $Q := \Omega^+ \cup \overline{\Omega}^-$. If u were known, $\psi^\pm$ could be computed by solving a system of boundary integral equations, equivalent to the transmission conditions on Γ, with traces of u on this surface as part of the data. If, on the other hand, the exterior density would be known, then u could be computed by solving a variational problem in Q, with the potential generated by ψ^+ evaluated in points of Σ as part of the boundary data. Since the whole set is unknown, we have a coupled set of boundary integral equations and variational formulation for a BVP.

The elementary idea for numerical approximation of the problem is using the decomposition above and treating the components separately. For the boundary integral system we use a convergent BEM (we keep this part in full generality throughout) and for the variational problem we use a polynomial finite element method based on a simplicial triangulation of the domain. The potential generated by the BEM solution is seen in the outer boundaries after an interpolation process, that allows for a finite and reduced number of evaluation of the smooth kernels of the transition operators from the inner to the outer boundary. The finite element solution creates some problems, because taking the gradient in the interior of the domain gives suboptimal results (it clearly lacks the correct boundedness properties). We have devised an approach based upon boundary integrals, that, using an approximation of the representation formula from the boundary and an elementary postprocessing of the finite element solution, creates a smooth version of this one, superconvergent in any domain that does not touch the boundary. This is a minor difficulty at the implementation level and only requires some care from the analytical point of view.

The advantage of the method lies in the fact that the FEM routine can take care of difficulties due to changing boundary conditions or geometrical complications, whereas the BEM part deals with smooth obstacles in a very natural and simple way, allowing for simple grids (there is no condition connecting the BEM and FEM discretizations) and for high order methods. The method is especially well–suited for inverse problems related to the geometry of the obstacles or its position in the interior domain, since the FEM grid can be fixed and we would just have to move the interface and thus recalculate relative positions of grid points with respect to the moving BEM discretization.

An apparently simpler approach that could be easily suggested consists of using boundary elements also in the outer boundary. In some examples, where the exterior geometry is simple (or smooth) and only Dirichlet or Neumann conditions appear, this would be obviously more convenient. When the outer boundary is polyhedral and has complications or when we have mixed boundary conditions, boundary elements begin to cause some problems and the overlapping is, we believe, a good chance to have good approximations. For this we lack an *a posteriori* error estimator, that is able to prepare the grids to tackle with the difficulties of the particular solution. We also think that the method has its own interest in the realm of theoretical numerical analysis of elliptic transmission problems.

In the field of applications, Helmholtz transmission problems have been studied related to the scattering of acoustic waves. A field where they have only recently received more attention is that of thermal waves, i.e., time–harmonic solutions to the heat equation. These problems are relevant for inverse problems and non–destructive testing of materials, with physical techniques such as photothermal excitation of the boundaries of objects containing hidden materials or cracks. The use of Laplace transforms as in [11] has also extended the field of application of boundary integral formulations to time–dependent problems of parabolic type. The kind of ideas we expose in the present paper can be applied both to direct and inverse problems in these situations.

A previous paper related to the present work is [4]. In that work, only the Laplace equation with Dirichlet boundary conditions in the exterior and interior domains (there is, therefore, no transmission to Ω^-) are considered. The situation here is, hence, much more general. Although part of the structure of the underlying analysis appeared already in [4], there are some new more general approaches, that allow for further generalizations, for instance to other transmission problems for strongly elliptic systems for which BIE formulations are available.

Throughout the forthcoming work we will make extensive use of the Sobolev spaces $H^s(\Omega)$ with non–negative real index s on a bounded domain with Lipschitz or smooth boundary. For boundaries, we will also consider Sobolev spaces $H^s(E)$ for positive and negative indices, E being the smooth or Lipschitz boundary of a domain or part of it. For these we refer to any classical monograph on Sobolev space theory or analysis of elliptic boundary value problems (see for instance [13] for a clear exposition in the very general Lipschitz setting). The duality bracket of Sobolev spaces on boundaries will always be denoted $\langle \cdot\, , \cdot \rangle$.

2 Statement of the problem

Let $Q \subset \mathbb{R}^d$ ($d = 2$ or 3), be a bounded domain with polygonal/polyhedral boundary $\Sigma := \partial Q$ that is subdivided into two non overlapping parts D and N. In the three dimensional case we also assume that the interfaces between D and N are polygonal lines. Let Ω^- be another domain, strictly contained in Q and whose boundary is a Lyapunov curve/surface $\Gamma := \partial\Omega^-$. Let finally $\Omega^+ := Q \setminus \overline{\Omega}^-$.

We consider the trace operator γ_B with $B \in \{\Gamma, D, N, \Sigma\}$. In the case of Γ, it will be clear from the context whether we are considering interior or exterior traces. Also ∂_B denotes the exterior normal derivative on B, pointing always outwards of Ω^- in the case

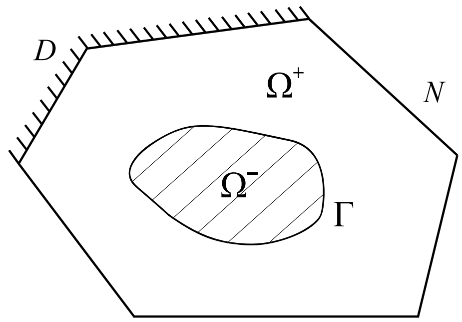

Figure 1: Geometry of the problem. Q is the whole polyhedron and $\Sigma = D \cup N$.

of Γ.

For fixed $\alpha, \lambda, \mu \neq 0$, we consider the following Helmholtz transmission problem: find $\omega^{\pm} \in H^1(\Omega^{\pm})$ such that

$$\begin{aligned} &\Delta\omega^+ + \lambda^2\omega^+ = 0, \qquad \text{in } \Omega^+,\\ &\Delta\omega^- + \mu^2\omega^- = 0, \qquad \text{in } \Omega^-,\\ &\gamma_\Gamma\omega^+ - \gamma_\Gamma\omega^- = g_\Gamma, \qquad \partial_\Gamma\omega^+ - \alpha\,\partial_\Gamma\omega^- = \chi_\Gamma,\\ &\gamma_D\omega^+ = g_D, \qquad \partial_N\omega^+ = \chi_N. \end{aligned} \tag{1}$$

The data functions belong to the following spaces:

$$\begin{aligned} (g_D, \chi_N) &\in \mathbb{H}(\Sigma) := H^{1/2}(D) \times H^{-1/2}(N),\\ (g_\Gamma, \chi_\Gamma) &\in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma). \end{aligned}$$

In $\mathbb{H}(\Sigma)$ we will use the product norm

$$\|(g_D, \chi_N)\|_\Sigma := \Big[\|g_D\|^2_{1/2,D} + \|\chi_N\|^2_{-1/2,N}\Big]^{1/2}.$$

Let ϕ_ρ be the fundamental solution to $\Delta + \rho^2$

$$\phi_\rho(\mathbf{x}) := \begin{cases} (\imath/4)H_0^{(1)}(\rho|\mathbf{x}|), & d = 2,\\ \exp(\imath\,\rho|\mathbf{x}|)\,/(4\pi|\mathbf{x}|), & d = 3, \end{cases}$$

and let $\mathcal{S}_\rho$ be the associated single layer potential

$$\mathcal{S}_\rho\psi := \int_\Gamma \phi_\rho(\,\cdot\, - \mathbf{x})\psi(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) \,:\, \mathbb{R}^d \to \mathbb{C},$$

which can be defined weakly to admit densities $\psi \in H^{-1/2}(\Gamma)$. Associated to the traces of this potential, there are two boundary integral operators,

$$\begin{aligned} V_\rho\psi &:= \int_\Gamma \phi_\rho(\,\cdot\, - \mathbf{x})\psi(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) \,:\, \Gamma \to \mathbb{C},\\ K_\rho\psi &:= \int_\Gamma \mathbf{n}(\,\cdot\,)\cdot\nabla\phi_\rho(\,\cdot\, - \mathbf{x})\psi(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) \,:\, \Gamma \to \mathbb{C}. \end{aligned}$$

As a solution to (1) we propose

$$\omega^+ := u + \mathcal{S}_\lambda \psi^+, \qquad \omega^- := \mathcal{S}_\mu \psi^-, \tag{2}$$

where $\psi^\pm \in H^{-1/2}(\Gamma)$, and

$$u \in H^1(Q), \qquad \Delta u + \lambda^2 u = 0. \tag{3}$$

Therefore, in order for the transmission and boundary value conditions to hold, we have to impose some restrictions to the unknowns. We are thus led to solving the following problem: find $u \in H^1(Q)$ and $\psi^\pm \in H^{1/2}(\Gamma)$ satisfying

$$\begin{array}{lcccl} \Delta u + \lambda^2 u = 0, & & \text{in } Q, & & \\ \gamma_D u & + & \gamma_D \mathcal{S}_\lambda \psi^+ & = & g_D \\ \partial_N u & + & \partial_N \mathcal{S}_\lambda \psi^+ & = & \chi_N \\ \begin{bmatrix} \gamma_\Gamma u \\ \partial_\Gamma u \end{bmatrix} & + & \mathcal{W} \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} & = & \begin{bmatrix} g_\Gamma \\ \chi_\Gamma \end{bmatrix} \end{array} \tag{4}$$

where $\mathcal{W}$ is the following matrix of boundary integral operators on Γ:

$$\mathcal{W} := \begin{bmatrix} V_\lambda & -V_\mu \\ -\frac{1}{2} I + K_\lambda & -\alpha\left(\frac{1}{2} I + K_\mu\right) \end{bmatrix}.$$

The transmission conditions in terms of the boundary integral operators are obtained by applying the jump relations for potentials [13].

To simplify some forthcoming notations we define the space

$$\mathbb{H}^{r,s}(\Gamma) := H^r(\Gamma) \times H^s(\Gamma)$$

endowed with the product norm

$$\|\boldsymbol{\psi}\|_{r,s,\Gamma} = \left[\|\psi^+\|^2_{r,\Gamma} + \|\psi^-\|^2_{s,\Gamma}\right]^{1/2}, \qquad \boldsymbol{\psi} = (\psi^+, \psi^-)^\top.$$

Notice that, as a consequence of classical results of boundary integral operators $\mathcal{W} : \mathbb{H}^{s,s}(\Gamma) \to \mathbb{H}^{s+1,s}(\Gamma)$ is bounded for all $s \in [-1, 0]$ (see [13, 5, 7]).

Hypotheses We further assume the following set of hypotheses:

(a) Uniqueness of solution for (1).

(b) $-\lambda^2$ and $-\mu^2$ are not Dirichlet eigenvalues of Δ in Ω^-. This hypothesis implies the invertibility of V_λ and V_μ and is imposed in order to be able to deal with single layer potentials. A way of circumventing it would be using Brakhage–Werner potentials (see [6]) in (2). This option introduces some non–essential difficulties in our forthcoming analysis. We restrict ourselves to this case for the sake of simplicity.

(c) Uniqueness of solution for the problem in Q, without the inclusion Ω^-:

$$\left.\begin{array}{ll} \Delta u + \lambda^2 u = 0, & \text{in } Q, \\ \gamma_D u = 0, & \partial_N u = 0 \end{array}\right| \qquad \Longrightarrow \qquad u = 0$$

and of the tranmission problem in the whole of $\mathbb{R}^d$:

$$\left.\begin{array}{ll} \Delta\omega^+ + \lambda^2\omega^+ = 0, & \text{in } \mathbb{R}^d \setminus \overline{\Omega^-}, \\ \Delta\omega^- + \mu^2\omega^- = 0, & \text{in } \Omega^-, \\ \gamma_\Gamma\omega^+ - \gamma_\Gamma\omega^- = 0, & \partial_\Gamma\omega^+ - \alpha\,\partial_\Gamma\omega^- = 0, \\ \lim\limits_{r\to\infty} r^{\frac{d-1}{2}}(\partial_r u - \imath\lambda u) = 0 & \end{array}\right| \qquad \Longrightarrow \qquad \omega^+ = 0, \quad \omega^- = 0.$$

The behaviour at infinity demanded in the transmission problem above is the so–called Sommerfeld radiation condition [5, 13]. With (b) and the last part of this hypothesis we can prove (see [15]) that $\mathcal{W}$ is invertible.

3 Theoretical results

We begin by reformulating (4) as an operator equation of the second kind. This will facilitate the forthcoming analysis. In order to do that, consider first the operator $L : \mathbb{H}(\Sigma) \to H^1(Q)$ given by

$$L(f_D, \varphi_N) := u, \qquad \left|\begin{array}{ll} \Delta u + \lambda^2 u = 0, & \text{in } Q, \\ \gamma_D u = f_D, & \partial_N u = \varphi_N. \end{array}\right. \tag{5}$$

The equivalent variational form for the problem above is

$$\left|\begin{array}{ll} u \in H^1(Q), & \gamma_D u = f_D, \\ a(u,v) = \langle\varphi_N, \gamma_N v\rangle, & \forall v \in H^1_D(Q), \end{array}\right. \tag{6}$$

where $H^1_D(Q) = \{v \in H^1(Q) \,|\, \gamma_D v = 0\}$ and

$$a(u,v) = \int_Q \Big(\nabla u \cdot \nabla v - \lambda^2\, u\, v\Big).$$

Let

$$T_{\Gamma\Sigma} := \begin{bmatrix} \gamma_\Gamma \\ \partial_\Gamma \end{bmatrix} L \,:\, \mathbb{H}(\Sigma) \to \mathbb{H}^{1/2,-1/2}(\Gamma)$$

and

$$K_{\Sigma\Gamma} := \begin{bmatrix} \gamma_D \\ \partial_N \end{bmatrix} \begin{bmatrix} \mathcal{S}_\lambda & 0 \end{bmatrix} \,:\, \mathbb{H}^{-1/2,-1/2}(\Gamma) \to \mathbb{H}(\Sigma).$$

Consider also

$$\mathbf{g}_\Sigma := (g_D, \chi_N)^\top \in \mathbb{H}(\Sigma), \qquad \mathbf{g}_\Gamma := (g_\Gamma, \chi_\Gamma)^\top \in \mathbb{H}^{1/2,-1/2}(\Gamma).$$

If $(u, \boldsymbol{\psi}) \in H^1(Q) \times \mathbb{H}^{-1/2,-1/2}(\Gamma)$ solves (4), then $\mathbf{f}_\Sigma := (\gamma_D u, \partial_N u) \in \mathbb{H}(\Sigma)$ and $\boldsymbol{\psi}$ are a solution to

$$\left| \begin{array}{lclcl} \mathbf{f}_\Sigma & + & K_{\Sigma\Gamma}\boldsymbol{\psi} & = & \mathbf{g}_\Sigma, \\ T_{\Gamma\Sigma}\mathbf{f}_\Sigma & + & \mathcal{W}\boldsymbol{\psi} & = & \mathbf{g}_\Gamma. \end{array} \right. \tag{7}$$

Reciprocally, given a solution to (7), $u := L\mathbf{f}_\Sigma$ and $\boldsymbol{\psi}$ define a solution to (4). Finally we define $K_{\Gamma\Sigma} := \mathcal{W}^{-1} T_{\Gamma\Sigma}$ and

$$\mathcal{K} := \begin{bmatrix} 0 & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & 0 \end{bmatrix} : \mathbb{H}(\Sigma) \times \mathbb{H}^{-1/2,-1/2}(\Gamma) \to \mathbb{H}(\Sigma) \times \mathbb{H}^{-1/2,-1/2}(\Gamma).$$

It is now clear that (7) is equivalent to the operator equation

$$(\mathcal{I} + \mathcal{K}) \begin{bmatrix} \mathbf{f}_\Sigma \\ \boldsymbol{\psi} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Sigma \\ \mathcal{W}^{-1}\mathbf{g}_\Gamma \end{bmatrix}. \tag{8}$$

Theorem 1 *The operator $\mathcal{I} + \mathcal{K}$ is invertible.*

Proof. Notice first that by elliptic regularity both $T_{\Gamma\Sigma}$ and $K_{\Sigma\Gamma}$ are compact and, therefore, so is $\mathcal{K}$. Then, by the Fredholm alternative, we only have to show injectivity of $\mathcal{I} + \mathcal{K}$. Let then

$$(\mathcal{I} + \mathcal{K})(\mathbf{f}_\Sigma, \boldsymbol{\psi})^\top = (0, 0)^\top$$

and define

$$\omega^+ := L\mathbf{f}_\Sigma + \mathcal{S}_\lambda \psi^+, \qquad \omega^- := \mathcal{S}_\mu \psi^-.$$

Since this pair of functions satisfies (1) with homogeneous data, then by Hypothesis (a), $\omega^+ = 0$ and $\omega^- = 0$. Since $V_\mu \psi^- = \gamma_\Gamma \omega^- = 0$, it follows by Hypothesis (b) that $\psi^- = 0$.

Consider now the function $\omega := L\mathbf{f}_\Sigma + \mathcal{S}_\lambda \psi^+ : \Omega^- \to \mathbb{C}$. It is clear that $\gamma_\Gamma \omega = \gamma_\Gamma \omega^+ = 0$ and that $\Delta\omega + \lambda^2\omega = 0$ in Ω^-. Therefore $\omega = 0$. By the jump relations of potentials $\psi^+ = \partial_N \omega - \partial_N \omega^+ = 0$, which also implies that $L\mathbf{f}_\Sigma = 0$ and therefore $\mathbf{f}_\Sigma = 0$. □

So far we have proved that if $(\mathbf{f}_\Sigma, \boldsymbol{\psi})$ is the unique solution of (8), then $(u, \boldsymbol{\psi}) := (L\mathbf{f}_\Sigma, \boldsymbol{\psi})$ is the unique solution of (4) and $(\omega^+, \omega^+) := (u + \mathcal{S}_\lambda \psi^+, \mathcal{S}_\mu \psi^-)$ is the unique solution of (1). Hence, all the problems we are considering are well posed and equivalent.

4 Numerical routines

We propose to solve numerically the problem by discretising (4) using finite elements to approximate u and Petrov–Galerkin boundary elements to approximate $\boldsymbol{\psi}$. In order to define the method we need three routines: a BEM module for exterior transmission problems, a FEM module for Helmholtz problems in Q with mixed boundary conditions and, finally, a smoother for finite element solution of homogenous elliptic problems.

As discretization parameter we will be using the symbol h. There will be however two different parameters, one controlling the level of discretization for boundary elements (this could have no geometrical meaning) and another showing the size of the mesh used for the finite element space. Both will be decaying to zero when we talk of convergence. We remark here, however, that the behaviour of both parameters will be independent: there will not be conditions lying one to another in order to obtain convergence. In general we will use h as a subindex denoting discretization. However, to avoid too many subindices in the same expression, we will write h as a superscript for all operators and data functions already labelled by one or more subscripts.

In the sequel we will write for two quantities depending on h that $a \lesssim b$ when there exists a constant C independent of h such that $a \leq Cb$.

4.1 Petrov–Galerkin boundary elements

Let $X_\Gamma^h \subset H^{-1/2}(\Gamma)$ and $Y_\Gamma^h \subset H^{1/2}(\Gamma)$ be two families of finite dimensional spaces such that: $\mathrm{dim} X_\Gamma^h = \mathrm{dim} Y_\Gamma^h$, there exists $\beta > 0$ such that

$$\sup_{0 \neq \eta_h \in Y_\Gamma^h} \frac{|\langle \varphi_h, \eta_h \rangle|}{\|\eta_h\|_{1/2,\Gamma}} \geq \beta \|\varphi_h\|_{-1/2,\Gamma}, \qquad \forall \varphi_h \in X_\Gamma^h$$

and for all $\varphi \in H^{-1/2}(\Gamma)$

$$\inf_{\varphi_h \in X_\Gamma^h} \|\varphi - \varphi_h\|_{-1/2,\Gamma} \to 0, \qquad \text{as } h \to 0.$$

These hypotheses imply that the discrete equations

$$\left| \begin{array}{l} \boldsymbol{\psi}_h \in X_\Gamma^h \times X_\Gamma^h, \\ \langle \mathcal{W} \boldsymbol{\psi}_h, \boldsymbol{\xi}_h \rangle = \langle \mathbf{g}, \boldsymbol{\xi}_h \rangle, \qquad \forall \boldsymbol{\xi}_h \in X_\Gamma^h \times Y_\Gamma^h, \end{array} \right. \tag{9}$$

are uniquely solvable for h small enough (see [15]). Then we can define the operator $G_h : \mathbb{H}^{1/2,-1/2}(\Gamma) \to X_\Gamma^h \times X_\Gamma^h$, mapping $\mathbf{g}$ to the solution of (9). By [15], we also have uniform boundedness of G_h

$$\|G_h \mathbf{g}\|_{-1/2,-1/2,\Gamma} \lesssim \|\mathbf{g}\|_{1/2,-1/2,\Gamma}, \qquad \forall \mathbf{g} \in \mathbb{H}^{1/2,-1/2}(\Gamma)$$

and the usual Céa estimate (here we write $\boldsymbol{\psi} = \mathcal{W}^{-1}\mathbf{g}$)

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-1/2,-1/2,\Gamma} \lesssim \inf_{\boldsymbol{\eta}_h \in X_\Gamma^h \times X_\Gamma^h} \|\boldsymbol{\psi} - \boldsymbol{\eta}_h\|_{-1/2,-1/2,\Gamma}.$$

4.2 Finite element discretization

Consider a class of triangulations of Q depending on the parameter h, formed by $d-$simpli--ces in the usual conditions for conforming continuous finite elements. We assume that the triangulations respect the interfaces of the Dirichlet and Neumann boundaries. Let V_h be the space of continuous finite elements of degree $k \geq 1$ on the triangulation and let

$$V_D^h := \{v_h \in V_h \,|\, \gamma_D v_h = 0\}.$$

Consider the spaces

$$X_D^h := \gamma_D V_h, \qquad \widetilde{X}_N^h := \gamma_N V_D^h.$$

Notice that elements of X_D^h and $\widetilde{X}_N^h$ are finite elements in $d-1$ dimension, on the inherited triangulation, and that elements of $\widetilde{X}_N^h$ satisfy homogeneous boundary conditions on the interfaces between D and N. Therefore, if we take an element of $\widetilde{X}_N^h$ and extend it by zero to D, it still belongs to $\gamma_\Sigma V_h$. Given $\varphi_N \in H^{-1/2}(N)$ we define

$$\rho_N^h \varphi_N \in Y_N^h := (\widetilde{X}_N^h)^* \tag{10}$$

by restricting the action of φ_N to the elements of $\widetilde{X}_N^h$.

To approximate the solution of (6) we consider the scheme

$$\left| \begin{array}{l} u_h \in V_h, \qquad \gamma_D u_h = f_D^h, \\ a(u_h, v_h) = \langle \varphi_N^h, \gamma_N v_h \rangle, \qquad \forall v_h \in V_D^h, \end{array} \right. \tag{11}$$

where $f_D \approx f_D^h \in X_D^h$ is 'user–chosen' and $\varphi_N^h := \rho_N^h \varphi_N$. Notice that we can write φ_N in (11) instead of φ_N^h. However, what the scheme takes into account is how φ_N acts on $\widetilde{X}_N^h$. Therefore the data for (11) are

$$(f_D^h, \varphi_N^h) \in X_D^h \times Y_N^h.$$

The discrete problem (11) is uniquely solvable for h small enough, since we are dealing with a compact perturbation of an elliptic operator and we have assumed that the continuous problem is well posed (see [12, Chapter 10]).

The operator $L_h : X_D^h \times Y_N^h \to V_h$, that maps data to solution in (11), is an approximation of L given in (5). In the following result the norm of the space $\widetilde{H}^{1/2}(N)$ will appear: this is the trace space of $H_D^1(\Omega)$ onto N, endowed, for instance, with the image norm. More characterizations on this space can be found in [13].

Proposition 2 *Let $u := L(f_D, \varphi_N)$ and $u_h := L_h(f_D^h, \varphi_N^h)$. Then*

$$\|u - u_h\|_{1,Q} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{1,Q} + \|f_D - f_D^h\|_{1/2,D} + \sup_{0 \neq g_h \in \widetilde{X}_N^h} \frac{|\langle \varphi_N - \varphi_N^h, g_h \rangle|}{\|g_h\|_{\widetilde{H}^{1/2}(N)}}.$$

The last term disappears when $\varphi_N^h = \rho_N^h \varphi_N$.

Proof. The proof follows from an argument similar to the First Strang Lemma (see [3]) modified to handle non–homogeneous Dirichlet conditions. These modifications follow readily from arguments given in an abstract setting in [10] and are lied to the existence of a stable lifting operator from X_D^h into V_h, which can be derived from the Scott–Zhang operator [16] or from a different lifting in [2] (see also the proof of Lemma 3 below). □

4.3 Smoothed finite element solution

The single and double layer potentials from Σ, related to the operator $\Delta + \lambda^2$, are

$$\begin{aligned} \mathcal{S}_\lambda^\Sigma \psi &:= \int_\Sigma \phi_\lambda(\,\cdot\, - \mathbf{x})\psi(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) \,:\, Q \to \mathbb{C}, \\ \mathcal{D}_\lambda^\Sigma \psi &:= -\int_\Sigma \mathbf{n}(\mathbf{x}) \cdot \nabla\phi_\lambda(\,\cdot\, - \mathbf{x})\psi(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) \,:\, Q \to \mathbb{C}. \end{aligned}$$

Then, the Third Green Identity [13] states that if $u \in H^1(Q)$ satisfies $\Delta u + \lambda^2 u = 0$, then

$$u = \mathcal{S}_\lambda^\Sigma \partial_\Sigma u - \mathcal{D}_\lambda^\Sigma \gamma_\Sigma u.$$

We will use a discretised version of this formula to obtain a smoothed version of $u_h \in V_h$ from which we can easily take traces and normal derivatives in Γ. Notice that, while there is no problem in taking a trace of u_h in Γ, the *naïve* normal derivative $\partial_\Gamma u_h = \mathbf{n}\cdot\nabla u_h$ fails to define a bounded operator from V_h to $H^{-1/2}(\Gamma)$. Also, $\gamma_\Gamma u_h$ is piecewise smooth on a grid that has no relationship whatsoever with the discretization imposed by the BEM spaces.

Let $\{\mathbf{x}_j\}$ be an ordering of the nodes of the triangulation lying on Σ and let $\psi_j \in V_h$ be the nodal function associated to $\mathbf{x}_j$. Then, we define

$$S_h u_h := \sum_j a(u_h, \psi_j)\phi_\lambda(\,\cdot\, - \mathbf{x}_j) - \mathcal{D}_\lambda^\Sigma \gamma_\Sigma u_h. \tag{12}$$

Notice that since $\Delta u + \lambda^2 u = 0$, then $\langle \partial_\Sigma u, \gamma_\Sigma v\rangle = a(u, v)$ for all $v \in H^1(Q)$. Given that $a(u_h, v_h) = 0$ for all $v_h \in V_h \cap H^1_0(Q)$, we are essentially taking a sum of Dirac delta distributions on the nodes $\partial_\Sigma u \approx \sum_j a(u_h, \psi_j)\delta_{\mathbf{x}_j} =: \partial^h_\Sigma u_h$, so that $\langle \partial^h_\Sigma u_h, \gamma_\Sigma v_h\rangle = a(u_h, v_h)$ for all $v_h \in V_h$.

Evaluation of $S_h u_h$ requires that of the functions

$$\mathcal{D}_\lambda^\Sigma \gamma_\Sigma \psi_j = \int_\Sigma \mathbf{n}(\mathbf{x}) \cdot \nabla\phi_\lambda(\,\cdot\, - \mathbf{x})\gamma_\Sigma\psi_j(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}).$$

The integral above is restricted to the small support of $\gamma_\Sigma\psi_j$. There, the function is piecewise smooth as long as we do not get near the boundary Σ.

5 Numerical approximation

The variational formulation corresponding to (4) is:

$$\left|\begin{array}{llll} u \in H^1(Q), & \boldsymbol{\psi} = (\psi^+, \psi^-) \in \mathbb{H}^{-1/2,-1/2}(\Gamma), & & \\ \gamma_D u & + \ \gamma_D \mathcal{S}_\lambda \psi^+ & = \ g_D, & \\ a(u, v) & + \ \langle \partial_N \mathcal{S}_\lambda \psi^+, \gamma_N v\rangle & = \ \langle \chi_N, \gamma_N v\rangle, & \forall v \in H^1_D(Q), \\ \langle \boldsymbol{\vartheta}_\Gamma u, \boldsymbol{\xi}\rangle & + \ \langle \mathcal{W}\boldsymbol{\psi}, \boldsymbol{\xi}\rangle & = \ \langle \mathbf{g}_\Gamma, \boldsymbol{\xi}\rangle, & \forall \boldsymbol{\xi} \in \mathbb{H}^{-1/2,1/2}(\Gamma), \end{array}\right. \tag{13}$$

where we have denoted $\boldsymbol{\vartheta}_\Gamma := (\gamma_\Gamma, \partial_\Gamma)^\top$.

The final elements to approximate the solution to (4) are the two nodal interpolation operators on the Dirichlet and Neumann boundaries

$$\begin{aligned} I_D^h &: \mathcal{C}(D) \to \gamma_D V_h = X_D^h, \\ I_N^h &: \mathcal{C}(N) \to \gamma_N V_h, \end{aligned}$$

as well as an approximation $g_D^h \approx g_D$. The forthcoming analysis does not depend on the way we choose g_D^h. In practice, if g_D is a continuous function, we could simply use an interpolate of g_D on the Dirichlet nodes.

Then we can consider the method

$$\left|\begin{array}{lcll} u_h \in V_h, \qquad \boldsymbol{\psi}_h = (\psi_h^+, \psi_h^-) \in X_\Gamma^h \times X_\Gamma^h, & & & \\ \gamma_D u_h \quad + \quad I_D^h \gamma_D \mathcal{S}_\lambda \psi_h^+ & = & g_D^h, & \\ a(u_h, v_h) \quad + \quad \langle I_N^h \partial_N \mathcal{S}_\lambda \psi_h^+, \gamma_N v_h \rangle & = & \langle \chi_N, \gamma_N v_h \rangle, & \forall v_h \in V_D^h, \\ \langle \boldsymbol{\vartheta}_\Gamma S_h u_h, \boldsymbol{\xi}_h \rangle \quad + \quad \langle \mathcal{W} \boldsymbol{\psi}_h, \boldsymbol{\xi}_h \rangle & = & \langle \mathbf{g}_\Gamma, \boldsymbol{\xi}_h \rangle, & \forall \boldsymbol{\xi}_h \in X_\Gamma^h \times Y_\Gamma^h. \end{array}\right. \tag{14}$$

As we did with (4) to get to (7) and eventually to the operator equation (8), we are going to introduce some notations to write (14) in a form more appropriate for the sake of analysis. First we notice that (14) is equivalent to the following problem, written in a very compact form using the FEM and BEM solvers L_h and G_h (recall the definition of ρ_N^h in (10)):

$$\left|\begin{array}{l} u_h \in V_h, \qquad \boldsymbol{\psi}_h = (\psi_h^+, \psi_h^-) \in X_\Gamma^h \times X_\Gamma^h, \\ u_h = L_h(g_D^h - I_D^h \gamma_D \mathcal{S}_\lambda \psi_h^+, \rho_N^h(\chi_N - I_N^h \partial_N \mathcal{S}_\lambda \psi_h^+)), \\ G_h \boldsymbol{\vartheta}_\Gamma S_h u_h + \boldsymbol{\psi}_h = G_h \mathbf{g}_\Gamma. \end{array}\right. \tag{15}$$

The choice of the operator I_N^h as the interpolation operator onto $\gamma_N V_h$ admits many other variants (for instance, interpolating with discontinuous functions), as long as they respect the global order of approximation. The main idea behind using this operator is demanding a finite number of evaluations of the potential generated from Γ in the outer boundary N.

Lemma 3 *There exists a class of operators $\pi_D^h : H^{1/2}(D) \to X_D^h$ satisfying:*

(a) $\pi_D^h g_h = g_h$, *for all* $g_h \in X_D^h$,

(b) $\|\pi_D^h g\|_{1/2,D} \lesssim \|g\|_{1/2,D}$, *for all* $g \in H^{1/2}(D)$,

(c) $\|\pi_D^h g - g\|_{1/2-\eta,D} \lesssim h^\eta \|g\|_{1/2,D}$, *for all* $g \in H^{1/2}(D)$ *and* $\eta \in [0, 1/2]$.

Proof. By [16] there exists an operator $\Pi_h : H^1(Q) \to V_h$ such that:

$$\begin{aligned} \Pi_h u_h = u_h, &\qquad \forall u_h \in V_h, \\ \|\Pi_h u\|_{1,Q} \lesssim \|u\|_{1,Q}, &\qquad \forall u \in H^1(Q), \\ \gamma_D \Pi_h u = 0, &\qquad \forall u \in H_D^1(Q), \\ \|\Pi_h u - u\|_{0,Q} \lesssim h\, \|u\|_{1,Q}, &\qquad \forall u \in H^1(Q). \end{aligned}$$

Let γ_D^+ be the pseudoinverse of $\gamma_D : H^1(Q) \to H^{1/2}(D)$ (or any other continuous lifting) and define $\pi_D^h := \gamma_D \Pi_h \gamma_D^+$. It is simple to see that this operator satisfies all the requirements of the statement of the lemma.

□

To reach an operator form of (14) that matches (8) we have to introduce some new elements. We remark that our aim is to write an operator equation in $\mathbb{H}(\Sigma)\times\mathbb{H}^{-1/2,-1/2}(\Gamma)$ whose solution is univocally tied to that of (14), instead of writing an operator equation in the discrete space $(X_D^h \times Y_N^h) \times (X_\Gamma^h \times X_\Gamma^h)$.

Let now $T_{\Gamma\Sigma}^h$ be defined by

$$T_{\Gamma\Sigma}^h := \boldsymbol{\vartheta}_\Gamma S_h L_h \begin{bmatrix} \pi_D^h & 0 \\ 0 & \rho_N^h \end{bmatrix}$$

and $K_{\Gamma\Sigma}^h := G_h T_{\Gamma\Sigma}^h : \mathbb{H}(\Sigma) \to X_\Gamma^h \times X_\Gamma^h \subset \mathbb{H}^{-1/2,-1/2}(\Gamma)$. Also, let $K_{\Sigma\Gamma}^h : \mathbb{H}^{-1/2,-1/2}(\Gamma) \to \mathbb{H}(\Sigma)$ be given by

$$K_{\Sigma\Gamma}^h := \begin{bmatrix} I_D^h & 0 \\ 0 & I_N^h \end{bmatrix} K_{\Sigma\Gamma} = \begin{bmatrix} I_D^h \gamma_D \\ I_N^h \partial_N \end{bmatrix} \begin{bmatrix} \mathcal{S}_\lambda & 0 \end{bmatrix}.$$

Notice that the image of I_N^h is included in $L^2(N) \subset H^{-1/2}(N)$. Finally, we consider the operator

$$\mathcal{K}_h := \begin{bmatrix} 0 & K_{\Sigma\Gamma}^h \\ K_{\Gamma\Sigma}^h & 0 \end{bmatrix} : \mathbb{H}(\Sigma) \times \mathbb{H}^{-1/2,-1/2}(\Gamma) \to \mathbb{H}(\Sigma) \times \mathbb{H}^{-1/2,-1/2}(\Gamma),$$

the semidiscrete right–hand side $\mathbf{g}_\Sigma^h := (g_D^h, \chi_N)^\top$ and the operator equation

$$(\mathcal{I} + \mathcal{K}_h) \begin{bmatrix} \mathbf{f}_\Sigma^h \\ \boldsymbol{\psi}_h \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Sigma^h \\ G_h \mathbf{g}_\Gamma \end{bmatrix}. \tag{16}$$

This operator equation can also be seen as the following group of equations:

$$\left|\begin{array}{lcrcl} \mathbf{f}_\Sigma^h = (f_D^h, \varphi_N^h) \in \mathbb{H}(\Sigma), & & \boldsymbol{\psi}_h = (\psi_h^+, \psi_h^-) \in \mathbb{H}^{-1/2,-1/2}(\Gamma), & & \\ f_D^h & + & I_D^h \gamma_D \mathcal{S}_\lambda \psi_h^+ & = & g_D^h, \\ \varphi_N^h & + & I_N^h \partial_N \mathcal{S}_\lambda \psi_h^+ & = & \chi_N, \\ G_h \boldsymbol{\vartheta}_\Gamma S_h L_h (\pi_D^h f_D^h, \rho_N^h \varphi_N^h)^\top & + & \boldsymbol{\psi}_h & = & G_h \mathbf{g}_\Gamma. \end{array}\right. \tag{17}$$

Equivalence of (16) and (14) is clarified in the following result.

Proposition 4 *If $((f_D^h, \varphi_N^h), \boldsymbol{\psi}_h)$ is a solution of* (16)*, then $(L_h(f_D^h, \rho_N^h \varphi_N^h), \boldsymbol{\psi}_h)$ is a solution of* (14)*. Reciprocally, if $(u_h, \boldsymbol{\psi}_h)$ solves* (14)*, then $((\gamma_D u_h, \chi_N - I_N^h \partial_N \mathcal{S}_\lambda \psi_h^+), \boldsymbol{\psi}_h)$ solves* (16)*. Finally, if $\mathcal{I} + \mathcal{K}_h$ is an isomorphism, then* (14) *is uniquely solvable.*

Proof. For the first result, we simply have to notice that any solution to (17) satisfies that $f_D^h \in X_D^h$ (and therefore $\pi_D^h f_D^h = f_D^h$) and $\boldsymbol{\psi}_h \in X_\Gamma^h \times X_\Gamma^h$. It is trivial then to recognize $u_h := L_h(f_D^h, \rho_N^h \varphi_N^h) = L_h(\pi_D^h f_D^h, \rho_N^h \varphi_N^h)$ and $\boldsymbol{\psi}_h$ as the components of a solution of (15). The reciprocal statement is a simple verification.

If $\mathcal{I} + \mathcal{K}_h$ is an isomorphism, then it is clear that we can always build a solution of (14) as explained before. Then we only have to deal with uniqueness. Let $(u_h, \boldsymbol{\psi}_h)$ satisfy

$$\left|\begin{array}{lcll} u_h \in V_h, \qquad \boldsymbol{\psi}_h = (\psi_h^+, \psi_h^-) \in X_\Gamma^h \times X_\Gamma^h, & & & \\ \gamma_D u_h \quad + \quad I_D^h \gamma_D \mathcal{S}_\lambda \psi_h^+ & = & 0, & \\ a(u_h, v_h) \quad + \quad \langle I_N^h \partial_N \mathcal{S}_\lambda \psi_h^+, \gamma_N v_h \rangle & = & 0, & \forall v_h \in V_D^h, \\ \langle \boldsymbol{\vartheta}_\Gamma S_h u_h, \boldsymbol{\xi}_h \rangle \quad + \quad \langle \mathcal{W} \boldsymbol{\psi}_h, \boldsymbol{\xi}_h \rangle & = & 0, & \forall \boldsymbol{\xi}_h \in X_\Gamma^h \times Y_\Gamma^h. \end{array}\right.$$

Defining $\mathbf{f}_\Sigma^h := (\gamma_D u_h, -I_N^h \partial_N \mathcal{S}_\lambda \psi_h^+)$, we have that $(\mathcal{I} + \mathcal{K}_h)(\mathbf{f}_\Sigma^h, \boldsymbol{\psi}_h)^\top = (0,0)^\top$. Therefore $\boldsymbol{\psi}_h = 0$ and looking at the first two equations above, it follows that $u_h = 0$. $\square$

6 Analysis of the smoothed finite element

Throughout this section we take u, u_h to be the respective solutions of

$$\left|\begin{array}{l} u \in H^1(Q), \qquad \gamma_D u = f_D, \\ a(u, v) = \langle \varphi_N, \gamma_N v \rangle, \qquad \forall v \in H_D^1(Q), \end{array}\right. \qquad \left|\begin{array}{l} u \in V_h, \qquad \gamma_D u_h = f_D^h, \\ a(u_h, v_h) = \langle \varphi_N, \gamma_N v_h \rangle, \qquad \forall v_h \in V_D^h, \end{array}\right.$$

i.e., $u = L(f_D, \varphi_N)$, $u_h = L_h(f_D^h, \rho_N^h \varphi_N)$. Notice that at this point f_D^h can be completely unrelated to f_D.

Let $\eta \in (0, 1/2)$ be such that the solutions of the mixed problem, with $g \in L^2(Q)$,

$$\left|\begin{array}{l} v \in H^1(Q), \\ \Delta v + \lambda^2 v = g, \qquad \text{in } Q \\ \gamma_D v = 0, \qquad \partial_N v = 0, \end{array}\right. \tag{18}$$

belong to $H^{1+\eta}(Q)$ and satisfy a bound, independent of g,

$$\|v\|_{1+\eta,Q} \le C\, \|g\|_{0,Q}. \tag{19}$$

In general we can always have $\eta = 1/4$. This result follows from results in [9] (cf. [8]). In some particular situations we can make η greater than $1/2$. This additional regularity is however lost in the forthcoming bounds.

Proposition 5 *If $u = L(f_D, \varphi_N)$ and $u_h = L_h(f_D^h, \rho_N^h \varphi_N)$, then*

$$\|u - u_h\|_{0,Q} \lesssim h^\eta \|u - u_h\|_{1,Q} + \|f_D - f_D^h\|_{1/2-\eta,D}.$$

Proof. The proof follows the lines of the classical Aubin–Nitsche trick. Given $g \in L^2(Q)$, we take v satisfying (18) and $v_h \in V_D^h$ the corresponding finite element approximation. By the First Green Identity

$$\begin{aligned}\int_Q (u - u_h)\, g &= -a(u - u_h, v) + \langle \gamma_D(u - u_h), \partial_D v\rangle \\ &= -a(u - u_h, v - v_h) + \langle f_D - f_D^h, \partial_D v\rangle.\end{aligned}$$

By Céa's estimate, approximation results in Sobolev spaces and (19), we have that $\|v - v_h\|_{1,Q} \lesssim h^\eta \|g\|_{0,Q}$. Also

$$|\langle f_D - f_D^h, \partial_D v\rangle| \le \|f_D - f_D^h\|_{1/2-\eta,D} \|\partial_D v\|_{-1/2+\eta,D} \lesssim \|f_D - f_D^h\|_{1/2-\eta,D}\, \|g\|_{0,Q}.$$

The previous estimates prove the result. □

Proposition 6 *If* $u = L(f_D, \varphi_N)$ *and* $u_h = L_h(\pi_D^h f_D, \rho_N^h \varphi_N)$, *then*

$$\begin{aligned}\|u - u_h\|_{0,Q} &\lesssim h^\eta\, \|u - u_h\|_{1,Q}, &(20)\\ \|\gamma_\Sigma u - \gamma_\Sigma u_h\|_{0,\Sigma} &\lesssim h^{\eta/2} \|u - u_h\|_{1,Q}, &(21)\\ \|u - u_h\|_{1,Q} &\lesssim \inf_{v_h \in V_h} \|u - v_h\|_{1,Q}. &(22)\end{aligned}$$

Proof. By Lemma 3, it follows that

$$\|f_D - \pi_D^h f_D\|_{1/2-\eta,D} \lesssim h^\eta \|f_D - \pi_D^h f_D\|_{1/2,D} \lesssim h^\eta \|u - u_h\|_{1,Q}.$$

Then, by Proposition 5, the first result follows readily from this inequality. By the trace inequality [3, Theorem 1.6.6],

$$\|\gamma_\Sigma v\|_{0,\Sigma} \le C\Big[(1 + \tau^{-1})\|v\|_{0,Q} + \tau\, \|v\|_{1,Q}\Big],$$

that holds for arbitrarily small τ, and taking $\tau = h^{\eta/2}$, (21) is a consequence of (20). From Lemma 3 it also follows that

$$\|f_D - \pi_D^h f_D\|_{1/2,D} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{1,Q}.$$

Then (22) is a consequence of Proposition 2. □

Let

$$e(u, u_h) := h^{k+1/2}\|u\|_{1,Q} + h^{\eta/2}\|u - u_h\|_{1,Q}.$$

Proposition 7 *If* $u = L(f_D, \varphi_N)$ *and* $u_h = L_h(\pi_D^h f_D, \rho_N^h \varphi_N)$, *then for all* $\phi \in H^{k+2}(Q)$

$$|\langle \partial_\Sigma u, \gamma_\Sigma \phi\rangle - \textstyle\sum_j a(u_h, \psi_j)\phi(\mathbf{x}_j)| \lesssim e(u, u_h)\|\phi\|_{k+2,Q}. \qquad (23)$$

Proof. Let $\phi_h \in V_h$ be the nodal interpolant of $\phi \in H^{k+2}(Q) \subset \mathcal{C}(\overline{Q})$ (notice that $k+2 \geq 2 > d/2$). Then

$$\begin{aligned} \langle \partial_\Sigma u, \gamma_\Sigma \phi\rangle - \textstyle\sum_j a(u_h,\psi_j)\phi(\mathbf{x}_j) &= a(u,\phi) - a(u_h,\phi_h) \\ &= a(u,\phi-\phi_h) + a(u_h-u,\phi-\phi_h) + a(u-u_h,\phi). \end{aligned} \tag{24}$$

Notice that $\|\gamma_\Sigma(\phi-\phi_h)\|_{1/2,\Sigma} \lesssim h^{k+1/2}\|\phi\|_{k+2,Q}$, since the interpolation operator commutes with the trace operator and we can benefit from the lower Sobolev index in the boundary (there is an overestimate in the Sobolev index of the requirement $\phi \in H^{k+2}(Q)$ which is immaterial for our purposes). Thus

$$|a(u,\phi-\phi_h)| = |\langle \partial_\Sigma u, \gamma_\Sigma(\phi-\phi_h)\rangle| \lesssim h^{k+1/2}\|u\|_{1,Q}\|\phi\|_{k+2,Q}. \tag{25}$$

Also

$$|a(u-u_h,\phi-\phi_h)| \lesssim h^k\|u-u_h\|_{1,Q}|\phi|_{k+1,Q}. \tag{26}$$

From the Second Green Identity we obtain

$$a(u-u_h,\phi) = -\int_Q (u-u_h)\,(\Delta\phi + \lambda^2\phi) + \langle \gamma_\Sigma(u-u_h), \partial_\Sigma\phi\rangle,$$

and therefore, by Proposition 6

$$|a(u-u_h,\phi)| \lesssim h^{\eta/2}\|u-u_h\|_{1,Q}\|\phi\|_{2,Q}. \tag{27}$$

Hence, (24,25,26,27) prove (23), considering that $h^k \lesssim h^{\eta/2}$. □

Let now Θ be any open set strictly contained in Q. We will use the classical multiindex notation for partial derivatives: ∂^α with $\alpha \in \mathbb{N}^d$.

Proposition 8 *For all* α

$$\|\partial^\alpha \mathcal{S}_\lambda^\Sigma \partial_\Sigma u - \partial^\alpha(\textstyle\sum_j a(u_h,\psi_j)\phi_\lambda(\,\cdot\, - \mathbf{x}_j))\|_{0,\Theta} \lesssim e(u,u_h).$$

Proof. Let M be a neighbourhood of Θ, strictly contained in Q and with smooth boundary (see figure 2). We consider a cut–off function $\chi \in \mathcal{C}^\infty(\overline{Q})$ satisfying $\chi|_\Theta \equiv 0$ and $\chi_{Q\setminus M} \equiv 1$. For each $\mathbf{x} \in \Theta$ we define the function

$$\phi_\mathbf{x}(\mathbf{y}) := \chi(\mathbf{y})\,\partial^\alpha\phi_\lambda(\mathbf{x}-\mathbf{y}).$$

Notice that $\phi_\mathbf{x} \in \mathcal{C}^\infty(\overline{Q})$ and that

$$\sup_{\mathbf{x}\in\Theta}\|\phi_\mathbf{x}\|_{k+2,Q} \leq C.$$

Since for all $\mathbf{x} \in \Theta$

$$(\partial^\alpha \mathcal{S}_\lambda^\Sigma \partial_\Sigma u)(\mathbf{x}) - \partial^\alpha(\textstyle\sum_j a(u_h,\psi_j)\phi_\lambda(\mathbf{x}-\mathbf{x}_j)) = \langle \partial_\Sigma u, \gamma_\Sigma\phi_\mathbf{x}\rangle - \textstyle\sum_j a(u_h,\psi_j)\phi_\mathbf{x}(\mathbf{x}_j),$$

the result is a simple consequence of Proposition 7. □

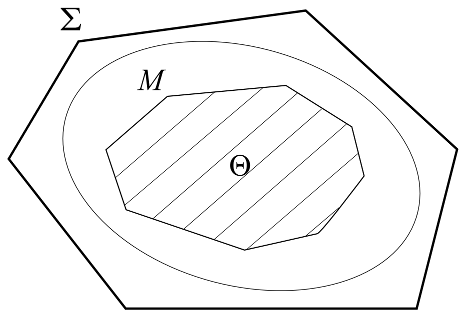

Figure 2: The domains in the proof of Proposition 8.

Theorem 9 *For all m,*

$$\|u - S_h u_h\|_{m,\Theta} \lesssim e(u, u_h).$$

Proof. Recall that $u = \mathcal{S}^\Sigma_\lambda \partial_\Sigma u - \mathcal{D}^\Sigma_\lambda \gamma_\Sigma u$ and the definition of $S_h u_h$ given in (12). Then, by Proposition 6

$$\|\mathcal{D}^\Sigma_\lambda \gamma_\Sigma u - \mathcal{D}^\Sigma_\lambda \gamma_\Sigma u_h\|_{m,\Theta} \lesssim \|\gamma_\Sigma u - \gamma_\Sigma u_h\|_{0,\Sigma} \lesssim h^{\eta/2} \|u - u_h\|_{1,Q} \lesssim e(u, u_h).$$

The remaining term is bounded by Proposition 8. □

7 Convergence analysis of the method

Proposition 10 *In the operator norm from $\mathbb{H}(\Sigma)$ to $\mathbb{H}^{1/2,-1/2}(\Gamma)$ we have that*

$$\|T^h_{\Gamma\Sigma} - T_{\Gamma\Sigma}\| \to 0.$$

Moreover, the family of operators $T^h_{\Gamma\Sigma}$ is collectively compact.

Proof. Let $\mathbf{f}_\Sigma = (f_D, \varphi_N)$, $u = L(f_D, \varphi_N)$ and $u_h = L_h(\pi^h_D f_D, \rho^h_N \varphi_N)$. Then

$$T_{\Gamma\Sigma} \mathbf{f}_\Sigma - T^h_{\Gamma\Sigma} \mathbf{f}_\Sigma = \boldsymbol{\vartheta}_\Gamma (u - S_h u_h).$$

Notice that, as a simple consequence of (22), we can bound $\|u - u_h\|_{1,Q} \lesssim \|u\|_{1,Q} \lesssim \|\mathbf{f}_\Sigma\|_\Sigma$. Taking Θ to be a neighbourhood of Γ included in Q, it follows (with an overestimate on the norms needed for the bounds) that

$$\|T_{\Gamma\Sigma} \mathbf{f}_\Sigma - T^h_{\Gamma\Sigma} \mathbf{f}_\Sigma\|_{0,1,\Gamma} \lesssim \|u - S_h u_h\|_{2,\Theta} \lesssim e(u, u_h) \lesssim h^{\eta/2} \|\mathbf{f}_\Sigma\|_\Sigma, \tag{28}$$

where we have applied Theorem 9. This proves the convergence in a more restrictive norm than stated in the proposition. Inequality (28) proves that

$$\|T^h_{\Gamma\Sigma} \mathbf{f}_\Sigma\|_{1,0,\Gamma} \lesssim h^{\eta/2} \|\mathbf{f}_\Sigma\|_\Sigma + \|T_{\Gamma\Sigma} \mathbf{f}_\Sigma\|_{1,0,\Gamma}.$$

Since $T_{\Gamma\Sigma}$ is bounded from $\mathbb{H}(\Sigma)$ to $\mathbb{H}^{1,0}(\Gamma)$ and the embedding $\mathbb{H}^{1,0}(\Gamma) \subset \mathbb{H}^{1/2,-1/2}(\Gamma)$ is compact, collective compactness of the family $\{T^h_{\Gamma\Sigma}\}$ follows readily. □

Theorem 11 *In the operator norm of* $\mathbb{H}(\Sigma) \times \mathbb{H}^{1/2,-1/2}(\Gamma)$

$$\|\mathcal{K} - \mathcal{K}_h\| \to 0.$$

Therefore, for h small enough $\mathcal{I} + \mathcal{K}_h$ is invertible and has a uniformly bounded inverse.

Proof. We decompose

$$K^h_{\Gamma\Sigma} - K_{\Gamma\Sigma} = G_h T^h_{\Gamma\Sigma} - \mathcal{W}^{-1} T_{\Gamma\Sigma} = (G_h - \mathcal{W}^{-1}) T^h_{\Gamma\Sigma} + \mathcal{W}^{-1}(T^h_{\Gamma\Sigma} - T_{\Gamma\Sigma})$$

and notice that $(G_h - \mathcal{W}^{-1})\mathbf{g}_\Gamma \to 0$ for all $\mathbf{g}_\Gamma \in \mathbb{H}^{1/2,-1/2}(\Gamma)$. Then, by a classical result on collectively compact operators (see [12, Chapter 10] or [1, Chapter 1]) and Proposition 10, it follows that $\|K^h_{\Gamma\Sigma} - K_{\Gamma\Sigma}\| \to 0$.

Because of the definition of $K^h_{\Sigma\Gamma}$ by interpolation of both components of $K_{\Sigma\Gamma}$ with finite elements of degree k on the boundary, and using the mapping properties of the regularising operator $K_{\Sigma\Gamma}$, it follows that

$$\|K^h_{\Sigma\Gamma}\boldsymbol{\psi} - K_{\Sigma\Gamma}\boldsymbol{\psi}\|_\Sigma \lesssim h^{k+1/2}\|\boldsymbol{\psi}\|_{-1/2,-1/2,\Gamma} \tag{29}$$

for all $\boldsymbol{\psi} \in \mathbb{H}^{-1/2,-1/2}(\Gamma)$. Hence $\|K^h_{\Sigma\Gamma} - K_{\Sigma\Gamma}\| \to 0$ and the result is proven. $\square$

Theorem 12 *Let* $(\mathbf{f}_\Sigma, \boldsymbol{\psi})$ *and* $(\mathbf{f}^h_\Sigma, \boldsymbol{\psi}_h)$ *be the respective solutions of* (8) *and* (16). *Then we have the bound*

$$\begin{aligned}\|\mathbf{f}_\Sigma - \mathbf{f}^h_\Sigma\|_\Sigma + \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-1/2,-1/2,\Gamma} \quad \lesssim \quad & \|g_D - g^h_D\|_{1/2,D} + h^{k+1/2}\|\boldsymbol{\psi}\|_{-1/2,-1/2,\Gamma} \\ & + \inf_{\boldsymbol{\eta}_h \in X^h_\Gamma \times X^h_\Gamma} \|\boldsymbol{\psi} - \boldsymbol{\eta}_h\|_{-1/2,-1/2,\Gamma} \\ & + h^{k+1/2}\|u\|_{1,Q} + h^{\eta/2} \inf_{v_h \in V_h} \|u - v_h\|_{1,Q}.\end{aligned}$$

Proof. Because of the uniform boundedness of the inverse of $\mathcal{I} + \mathcal{K}_h$, if suffices to bound

$$\begin{aligned}(\mathcal{I} + \mathcal{K}_h)\begin{bmatrix} \mathbf{f}_\Sigma - \mathbf{f}^h_\Sigma \\ \boldsymbol{\psi} - \boldsymbol{\psi}_h \end{bmatrix} &= (\mathcal{K}_h - \mathcal{K})\begin{bmatrix} \mathbf{f}_\Sigma \\ \boldsymbol{\psi} \end{bmatrix} + \begin{bmatrix} \mathbf{g}_\Sigma - \mathbf{g}^h_\Sigma \\ (\mathcal{W}^{-1} - G_h)\mathbf{g}_\Gamma \end{bmatrix} \\ &= \begin{bmatrix} (K^h_{\Sigma\Gamma} - K_{\Sigma\Gamma})\boldsymbol{\psi} + (\mathbf{g}_\Sigma - \mathbf{g}^h_\Sigma) \\ (\mathcal{W}^{-1} - G_h)(\mathbf{g}_\Gamma - T_{\Gamma\Sigma}\mathbf{f}_\Sigma) + G_h(T^h_{\Gamma\Sigma} - T_{\Gamma\Sigma})\mathbf{f}_\Sigma \end{bmatrix},\end{aligned}$$

which reduces the problem to bound the four different terms of the last decomposition above in the corresponding norms. The definition of $\mathbf{g}^h_\Sigma$ and (29) serve to bound the first component above.

Also, since $\mathbf{g}_\Gamma - T_{\Gamma\Sigma}\mathbf{f}_\Sigma = \mathcal{W}\boldsymbol{\psi}$, then the Céa estimate for the BEM

$$\|\boldsymbol{\psi} - G_h\mathcal{W}\boldsymbol{\psi}\|_{-1/2,-1/2,\Gamma} \lesssim \inf_{\boldsymbol{\eta}_h \in X^h_\Gamma \times X^h_\Gamma} \|\boldsymbol{\psi} - \boldsymbol{\eta}_h\|_{-1/2,-1/2,\Gamma}$$

gives the bound for the third term. Finally, by (28) and (22),

$$\begin{aligned}\|G_h(T^h_{\Gamma\Sigma} - T_{\Gamma\Sigma})\mathbf{f}_\Sigma\|_{-1/2,-1/2,\Gamma} \quad &\lesssim \quad \|(T^h_{\Gamma\Sigma} - T_{\Gamma\Sigma})\mathbf{f}_\Sigma\|_{-1/2,1/2,\Gamma} \\ &\lesssim \quad e(u, L_h(\pi^h_D f_D, \rho^h_N \varphi_N)) \\ &\lesssim \quad h^{k+1/2}\|u\|_{1,Q} + h^{\eta/2} \inf_{v_h \in V_h} \|u - v_h\|_{1,Q}\end{aligned}$$

and the proof is thus finished. $\square$

Corollary 13 *Let $(u, \boldsymbol{\psi})$ and $(u_h, \boldsymbol{\psi}_h)$ be the respective solutions of* (4) *and* (14)

$$\begin{aligned}\|u - u_h\|_{1,Q} + \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-1/2,-1/2,\Gamma} \lesssim\ & h^{k+1/2}\Big(\|\boldsymbol{\psi}\|_{-1/2,-1/2,\Gamma} + \|u\|_{1,Q}\Big) \\ & + \|g_D - g_D^h\|_{1/2,D} \\ & + \inf_{\boldsymbol{\eta}_h \in X_\Gamma^h \times X_\Gamma^h} \|\boldsymbol{\psi} - \boldsymbol{\eta}_h\|_{-1/2,-1/2,\Gamma} + \inf_{v_h \in V_h} \|u - v_h\|_{1,Q}.\end{aligned}$$

Proof. By Proposition 2, it follows readily that

$$\|u - u_h\|_{1,Q} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{1,Q} + \|\mathbf{f}_\Sigma - \mathbf{f}_\Sigma^h\|_\Sigma.$$

We then apply Theorem 12. □

Notice that the right–hand side of the error bound of Corollary 13 comprises three types of terms: a super–optimal order term independent of the regularity of the components of the solution (and arising from the regularity effect of transferring information from Γ into Σ and backwards), a term related to the choice of the approximation to the Dirichlet datum g_D and the usual approximation terms in natural norms.

If $u \in H^{k+1}(Q)$, which is equivalent to ω^+ having the same Sobolev regularity near the boundary Σ, and we use interpolation to compute g_D^h, we can obtain a bound

$$\|u - u_h\|_{1,Q} + \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{-1/2,-1/2,\Gamma} = \mathcal{O}(h^k) + \inf_{\boldsymbol{\eta}_h \in X_\Gamma^h \times X_\Gamma^h} \|\boldsymbol{\psi} - \boldsymbol{\eta}_h\|_{-1/2,-1/2,\Gamma}.$$

Regularity of $\boldsymbol{\psi}$ depends on that of the interface Γ and on the corresponding data g_Γ and χ_Γ. As mentioned at the beginning of Section 4, the spaces X_Γ^h and Y_Γ^h are completely independent of V_h and the parameter h, when appearing accompanying variables in Γ refers to a different grid or discretization level as that of the finite element grid.

8 A numerical experiment

Consider the geometry described in Figure 3, with twenty circular–shaped interior obstacles. The parameters in the definition of the transmission problems are

$$\lambda = 1 + \imath, \qquad \mu = 0.5 + 0.5\imath, \quad \alpha = 0.5$$

and the transmission data are taken so that the exact solution is

$$\begin{aligned}\omega^+ \quad (\mathbf{x}) &= \ 1.5\phi_\lambda(\mathbf{x} - \mathbf{p}_0) + \phi_\lambda(\mathbf{x} - \mathbf{p}_1) + (1 + \imath)\phi_\lambda(\mathbf{x} - \mathbf{p}_2) - 2\exp(\imath\lambda x_2) \\ \omega^- \quad (\mathbf{x}) &= \ \exp(\mu\imath x_1) - \exp(\mu\imath x_2)\end{aligned}$$

where

$$\mathbf{p}_0 := (-2.6650, 0.6967), \quad \mathbf{p}_1 := (-1.8950, 0.3033), \quad \mathbf{p}_2 := (-1.53, -0.04).$$

We use quasi–uniform grids of the rectangle Q with N_Q elements. The basic one, with 112 triangles is the one given by the Matlab PDE Toolbox, and the refinements are

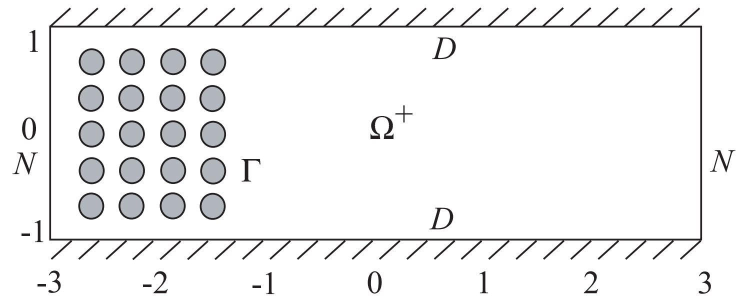

Figure 3: The domain for the numerical experiment

obtained by dividing each element into four similar triangles. We have implemented the elementary Courant $\mathbb{P}_1$ finite elements. On each component of the interface Γ we take N_Γ equally spaced elements and we use a $\mathbb{P}_0 - \mathbb{P}_1$ Petrov–Galerkin scheme with numerical integration like the one developed in [14, Chapter 5].

We measure an estimate of the $L^2(\Omega^+ \cup \Omega^-)$ error given by the following expression:

$$E_2 := \frac{1}{N_Q}\Big(\sum_{\mathbf{x}_j} |\omega(\mathbf{x}_j) - \omega_h(\mathbf{x}_j)|^2\Big)^{1/2}.$$

The errors, for different values of the parameters N_Γ and N_Q are listed in the following table.

$N_\Gamma \backslash N_Q$	112	448	1792	7168	28672
6	3.89E-3	5.72E-4	2.46E-4	1.33E-4	7.78E-5
12	3.82E-3	5.02E-4	1.18E-4	4.25E-5	2.78E-5
24	3.78E-3	3.79E-4	5.58E-5	1.51E-5	8.11E-6
48	3.78E-3	3.77E-4	4.96E-5	9.55E-6	3.08E-6

In Figure 4 we draw some of these numbers. The first graph corresponds to errors beginning with $(N_\Gamma, N_Q) = (6, 112)$ and then halving three progressive times both grids. These errors correspond to the diagonal in Table 8 beginning in the top left corner. The right–most graph corresponds to the same kind of refinement beginning with $(N_\Gamma, N_Q) = (6, 448)$. Both the graphs show convergence with order two. This superconvergence phenomenon can be due to the fact that we are observing in a weaker norm than the one given for analysis and to the very particular disposition of the boundary conditions in sides of the rectangle. A proper *a priori* analysis in weaker norms is lacking with our current theory, since it does not trivially follow from a classical stability plus Aubin–Nitsche analysis.

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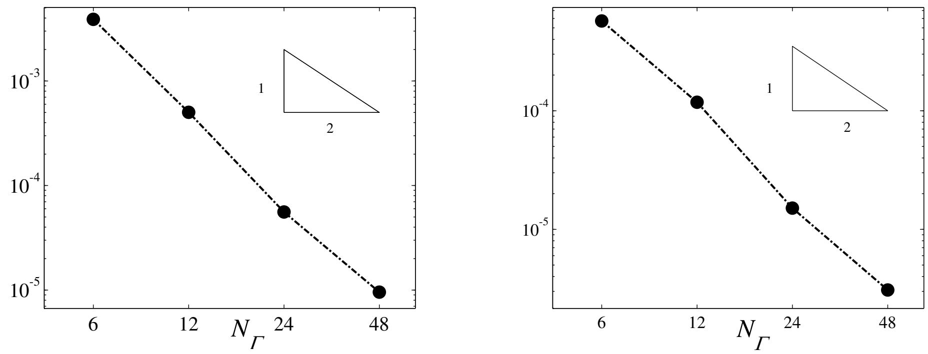

Figure 4: Some graphs of the errors

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