PRE-PUBLICACIONES del seminario matematico "garcia de galdeano"

2004

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n. 23

seminario matemático
garcía de galdeano
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Groups with the Minimal Condition for Abnormal Subgroups*

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to Professor Otto H. Kegel on his 70th birthday

Abstract

We study the relationship between Carter subgroups and groups satisfying the minimal condition on their abnormal subgroups in some classes of generalized soluble groups.

Key Words: *Carter subgroup; Abnormal subgroup; Minimal condition*
A.M.S. Classification: 20 F 19; 20 F 22

1 Introduction

A subgroup H of a group G is said to be *abnormal* in G if $g \in \langle H, H^g \rangle$ for each element $g \in G$. This type of subgroups appeared in the paper [4] of P. Hall, though the specific term *abnormal subgroup* is due to R. Carter (see [1]). Since abnormal subgroups are clearly self-normalizing, abnormality is a strong form of nonnormality, which yields characterizations of finite nilpotent groups as it is known that a finite non-nilpotent group has a proper abnormal subgroup and conversely. In finite (especially soluble) groups the properties of abnormal subgroups have been studied in detail (see, for example, [2]).

In infinite groups, the situation is quite different, and we cannot even assure the existence of proper abnormal subgroups. Indeed, it is easy to prove that a locally nilpotent group has no proper abnormal subgroups but the converse (if true) is not established yet. On the other hand, in [5] and in [7] some classes of groups in which the absence of proper abnormal subgroups implies the locally nilpotency have been considered.

A subgroup H of a finite group is said to be a *Carter subgroup* if it is nilpotent and self-normalizing. It is well known that in finite soluble groups

*This research has been made during the visits of the first author at University of Zaragoza and at University of Lecce. The research was supported by Proyecto BFM2001-2452 of CIGYT (Spain) and Proyecto 100/2001 of Gobierno de Aragon (Spain) and also by University of Lecce (Italy)

Carter subgroups always exist and they are all conjugate, and so even abnormal (see, for example, [2, Theorem 4.6]). Therefore, it is natural to define *Carter subgroups of a generalized soluble group as the locally nilpotent abnormal subgroups*. The existence of a unique conjugacy class of Carter subgroups was proved by Stonehewer in [10] for periodic locally soluble groups with a locally nilpotent subgroup of finite index. We have already noticed that a locally nilpotent group has no proper abnormal subgroups, so that Carter subgroups of generalized soluble groups are minimal abnormal. This point of view suggests to develop the study of Carter subgroups in the class of groups satisfying the *minimal condition on abnormal subgroups* (*Min–abn*).

In this article the quoted Carter's results will be extended to some generalized soluble groups with *Min–abn* in the following two theorems. But first we must give here all the needed concepts.

Let G be a group, H a normal subgroup of G and A a $\mathbb{Z}G$–module. Put

$$\zeta_{\mathbb{Z}H}(A) = \{a \in A \mid ax = a \text{ for each } x \in H\}.$$

It is rather easy to see that $\zeta_{\mathbb{Z}H}(A)$ is a $\mathbb{Z}G$–submodule of A, which is called *the $\mathbb{Z}H$–center of A*. Starting from the $\mathbb{Z}H$–center, we construct *the upper $\mathbb{Z}H$–central series of A*

$$\{0\} = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma$$

defined by the following rules: $C_1 = \zeta_{\mathbb{Z}H}(A)$, $C_{\alpha+1}/C_\alpha = \zeta_{\mathbb{Z}H}(A/C_\alpha)$, if $\alpha < \gamma$, $C_\lambda = \bigcup_{\beta<\lambda} C_\beta$ if $\lambda < \gamma$ is a limit ordinal, and $\zeta_{\mathbb{Z}H}(A/C_\gamma) = \{0\}$. The last term C_γ of this series is called *the upper $\mathbb{Z}H$-hypercenter of A* and will be denoted by $\zeta^\infty_{\mathbb{Z}H}(A)$. A module A is called *$\mathbb{Z}H$–hypercentral* if $A = \zeta^\infty_{\mathbb{Z}H}(A)$. If A is $\mathbb{Z}H$–hypercentral and the upper $\mathbb{Z}H$–central series has finite length, then A is called *$\mathbb{Z}H$–nilpotent*.

Let B and C two $\mathbb{Z}G$–submodules of A such that $B \leq C$. The factor C/B is called *H–central* (respectively, *H–eccentric*) if $C_H(C/B) = H$ (respectively, $C_H(C/B) \neq H$). A $\mathbb{Z}G$–submodule C is called *H–hypereccentric* if C has an ascending series of $\mathbb{Z}G$–submodules of A

$$\{0\} = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = C$$

such that $C_{\alpha+1}/C_\alpha$ is a simple $\mathbb{Z}G$–module, which is an H–eccentric factor for every $\alpha < \gamma$.

Following D.I. Zaitsev [12], we say that *the $\mathbb{Z}G$–module A has the Z–decomposition* if

$$A = \zeta^\infty_{\mathbb{Z}G}(A) \bigoplus E^\infty_{\mathbb{Z}G}(A),$$

where $E^\infty_{\mathbb{Z}G}(A)$ is the maximal G–hypereccentric $\mathbb{Z}G$–submodule of A. It is worth to mentioning that if A has the Z–decomposition the submodule

$E = E^{\infty}_{\mathbb{Z}G}(A)$ includes every G–hypereccentric $\mathbb{Z}G$–submodule of A; in particular, this decomposition is unique. In fact, let B be a G–hypereccentric $\mathbb{Z}G$–submodule of A. If $(B+E)/E$ is not zero, it includes a non-zero simple $\mathbb{Z}G$–submodule D/E. Since $(B+E)/E \cong B/(B \cap E)$, D/E is $\mathbb{Z}G$–isomorphic to some simple $\mathbb{Z}G$–factor of B and it follows that $C_G(D/E) \neq G$. On the other hand, $(B+E)/E \leq A/E \cong C$ and so $C_G(D/E) = G$. This contradiction proves that $B \leq E$, as we claimed.

Let now G be a locally nilpotent group. A $\mathbb{Z}G$–module A is said to be *locally nilpotent* if for every finite subset M of A and every finitely generated subgroup F of G the $\mathbb{Z}F$–submodule $M\mathbb{Z}F$ generated by M is $\mathbb{Z}F$–nilpotent. Clearly, if A is a locally nilpotent $\mathbb{Z}G$–module, then the natural semidirect product $G \ltimes A$ is a locally nilpotent group.

Put

$$LN_{\mathbb{Z}G}(A) = \{a \in A \mid a\mathbb{Z}G \text{ is locally nilpotent}\}.$$

Note that $L = LN_{\mathbb{Z}G}(A)$ is a $\mathbb{Z}G$–submodule of A. Indeed, let $a, b \in L$ and H be an arbitrary finitely generated subgroup of G. Then the submodules $a\mathbb{Z}H$ and $b\mathbb{Z}H$ are nilpotent. It follows that there exists some $m, k \geq 1$ such that

$$ax(h-1)^m = 0 \text{ and } by(h-1)^k = 0$$

for all $h \in H$ and $x, y \in \mathbb{Z}H$. Put $t = \max\{m, k\}$. Then $(ax+by)(h-1)^t = 0$. It follows that $a\mathbb{Z}H + b\mathbb{Z}H$ is nilpotent, and, since $(a-b)\mathbb{Z}H \leq a\mathbb{Z}H + b\mathbb{Z}H$, $(a-b)\mathbb{Z}H$ is too nilpotent. Thus $(a-b)\mathbb{Z}G$ is locally nilpotent and so $(a-b) \in L$. Similarly $ag \in L$ for each $g \in G$. The submodule $LN_{\mathbb{Z}G}(A)$ is called *the locally nilpotent radical of the module A.*

We say that the $\mathbb{Z}G$–module A has *the LN–decomposition* if

$$A = LN_{\mathbb{Z}G}(A) \bigoplus E_{\mathbb{Z}G}(A).$$

Remark that if A has an LN–decomposition, then this decomposition is unique.

Theorem 1.1 *Let G be a periodic group, and suppose that L is a nilpotent normal subgroup of G such that G/L is hypercentral. If G satisfies Min–abn, then either G is locally nilpotent or G satisfies the following conditions:*

(1) *$G = HL$, where H is a minimal abnormal subgroup;*

(2) *H is locally nilpotent;*

(3) *If U/V is a factor of the upper central series of L then U/V has the LN–decomposition $U/V = Z/V \times E/V$, where $Z/V = LN_{\mathbb{Z}G}(U/V)$ and $E/V = E^{\infty}_{\mathbb{Z}G}(U/V)$;*

(4) *$Z \leq HV$, and hence $(U/V)/((U/V) \cap (HV/V))$ is G–hypereccentric and satisfies Min–G;*

(5) *if H_1 is another minimal abnormal subgroup then H_1 and H are conjugate.*

Conversely, if G satisfies the conditions (1)–(5), then G has Min–abn.

Corollary 1.2 *Let G be a periodic nilpotent–by–hypercentral group. If G satisfies Min–abn, then G has a unique conjugacy class of Carter subgroups.*

Theorem 1.3 *Let G be an FC–nilpotent group, and suppose that L is a nilpotent normal subgroup of G such that G/L is hypercentral. If G satisfies Min–abn, then either G is hypercentral or G satisfies the following conditions:*

(1) *$G = HL$, where H is a minimal abnormal subgroup;*

(2) *H is hypercentral;*

(3) *If U/V is a factor of the upper central series of L, then U/V has the Z–decomposition $U/V = Z/V \times E/V$, where $Z/V = \zeta^{\infty}_{\mathbb{Z}G}(U/V)$ and $E/V = E^{\infty}_{\mathbb{Z}G}(U/V)$;*

(4) *$Z \leq HV$, and hence $(U/V)/((U/V) \cap (HV/V))$ is G–hypereccentric and satisfies Min–G;*

(5) *if H_1 is another minimal abnormal subgroup then H_1 and H are conjugate.*

Conversely, if G satisfies the conditions (1)–(5), then G has Min–abn.

Corollary 1.4 *Let G be an FC–nilpotent group which is nilpotent–by–hypercentral. If G satisfies Min–abn, then G has a unique conjugacy class of Carter subgroups.*

It is interesting to note that in the classes of groups considered, the property of being abnormal is a transitive relation (see Corollary 3.5), and moreover, such groups are locally nilpotent when they have no proper abnormal subgroups (see Theorem 3.6 and [7]). Therefore, in these classes the Carter subgroups are all the minimal abnormal subgroups.

Most of our notation is standard and can be found in [9].

2 Module Theoretic Results

Lemma 2.1 *Let G be an hypercentral group, A a $\mathbb{Z}G$–module and C a $\mathbb{Z}G$–submodule of A. Suppose that $C_G(C) = G$, A/C is a simple $\mathbb{Z}G$–module and $C_G(A/C) \neq G$. Then A has a simple $\mathbb{Z}G$–submodule E such that $A = C \oplus E$.*

Proof – There is no loss if we assume that $C_G(A) = \{1\}$. Let $1 \neq z \in \zeta(G)$. Then the mapping $\phi_z : a \mapsto a(z-1),\ a \in A$, is a $\mathbb{Z}G$–endomorphism of A. In particular, Im ϕ_z and Ker ϕ_z are $\mathbb{Z}G$–submodules of A. Since $C_G(C) = G$, Ker $\phi_z \geq C$. If we assume that Ker $\phi_z \neq C$, then Ker $\phi_z = A$, because A/C is simple. But in this case $z \in C_G(A) = \{1\}$. This contradiction shows that Ker $\phi_z = C$. The isomorphism $A/\text{Ker } \phi_z \cong_{\mathbb{Z}G} \text{Im } \phi_z = E$ shows that E is a $\mathbb{Z}G$–submodule of A such that $C_G(E) \neq G$. Moreover, E is a simple $\mathbb{Z}G$–submodule, and so $C \cap E = \{0\}$, because $C \neq E$. Finally, since A/B is a simple $\mathbb{Z}G$–module, $C + B = A$, and thus $A = C \oplus E$. □

Corollary 2.2 *Let G be an hypercentral group and let A be a $\mathbb{Z}G$–module. Suppose that A has a finite series of $\mathbb{Z}G$–submodules*

$$\{0\} = A_0 \leq A_1 \leq \cdots \leq A_t \leq A_{t+1} = A$$

such that $C_G(A_{j+1}/A_j) = G$ $(0 \leq j \leq t-1)$, A/A_t is simple and $C_G(A/A_t) \neq G$. Then A has a simple $\mathbb{Z}G$–submodule E such that $A = \zeta^{\infty}_{\mathbb{Z}G}(A) \oplus E$.

The next result is due to D.I. Zaitsev and we omit its proof.

Lemma 2.3 *(see [11]) Let R be a commutative ring and suppose we have and R–module A such that*

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda,$$

where every A_λ is R–isomorphic to a simple module B. Then for each element $0 \neq a \in A$ the submodule aR is R–isomorphic to B.

Lemma 2.4 *Let G be a group and let A be a simple FG–module, where F is a field. If H is a periodic subgroup of the center $\zeta(G)$, then A is a semisimple FH–module.*

Proof – Let $\mathfrak{K}$ be a local system consisting of finite subgroups of H. Pick $K \in \mathfrak{K}$ arbitrary. If $0 \neq a \in A$ then the submodule a aFK has finite dimension over F and it ready follows that aFK has a simple FK–submodule B. Then there exists some subset $S \subseteq G$ such that $A = \bigoplus_{x \in S} Bx$. Since $K \leq \zeta(G)$, the mapping $b \mapsto bx,\ b \in B$, is an FK–isomorphism for every $x \in G$. By

Lemma 2.3, $aFK \cong_{FK} B$. In particular, aFK is simple FK–submodule. The equations

$$H = \bigcup_{K \in \mathfrak{K}} K \text{ and } aFH = \bigcup_{K \in \mathfrak{K}} aFK$$

yield that $C = aFH$ is simple FH–submodule of A. Since H is a normal subgroup of G, there exists some subset $M \subseteq G$ such that $A = \bigoplus_{y \in M} Cy$. Finally, as every FH–submodule Cy is isomorphic over FH to C, then Cy is a simple FH–submodule. Thus A is a semisimple FH–module as required. □

Proposition 2.5 *Let G be a periodic hypercentral group and suppose that A is a $\mathbb{Z}G$–module. If $A/LN_{\mathbb{Z}G}(A)$ is a simple $\mathbb{Z}G$–module then A has the LN–decomposition.*

Proof – There is no loss if we assume that $C_G(A) = \{1\}$. Put $L = LN_{\mathbb{Z}G}(A)$. If $C_G(A/L) = G$ then it is clear that A is a locally nilpotent $\mathbb{Z}G$–module. In particular, A has the LN–decomposition. Therefore, we assume that A/L is G–eccentric. Since A/L is a simple module over a periodic hypercentral group, the additive group of A/L is an elementary abelian p–group for some prime p (see, for example, [6, Corollary 1.18]). Moreover, $\zeta(G/C_G(A/L))$ is a locally cyclic p'–group (see, for example, [6, Theorem 3.1]). Since G is hypercentral, $G/C_G(A/L))$ is a p'–group.

Suppose first that the additive group of A is an elementary abelian p–group. Put $P = C_G(A/L)$. If $x \in P$ then $x = x_1x_2$, where $[x_1, x_2] = 1$, x_1 is a p–element and x_2 is a p'–element. By the choice of P, we have $A = LN_{\mathbb{Z}P}(A)$. In particular, the natural semidirect product $P \ltimes A$ is a locally nilpotent group. Since $\Pi(A) = \{p\}$, $x_2 \in C_G(A) = \{1\}$. In other words, P is a p–subgroup. We have already noted that G/P is a p'–group, so that P is a Sylow p–subgroup of G and then $G = P \times Q$, where Q is a Sylow p'–subgroup of G. Put $C = \zeta(Q)$. Since $C \leq \zeta(G)$, by Lemma 2.4, the factor-module A/L has a simple C–eccentric $\mathbb{Z}C$–submodule B/L. By the choice of L, the natural semidirect product $C \ltimes L$ is a locally nilpotent group. On the other hand, L is a p–group and C is a p'–group, so that $L \leq \zeta_{\mathbb{Z}C}(A)$. By Lemma 2.1, B contains a simple $\mathbb{Z}C$–submodule D such that $B = D \oplus L$. Put $E = D\mathbb{Z}G$. Then E is a sum of simple $\mathbb{Z}C$–submodules Dx ($x \in G$), and hence E is a semisimple $\mathbb{Z}C$–submodule. Since $C \leq \zeta(G)$, $Dx \cong_{\mathbb{Z}C} D$ for all $x \in G$. By Lemma 2.3, $e\mathbb{Z}C \cong_{\mathbb{Z}C} D$ for every $0 \neq e \in E$. Suppose that $E \cap L \neq \{0\}$. We have already noticed that $L \leq \zeta_{\mathbb{Z}C}(A)$. Therefore, if $0 \neq e \in E \cap L$, then $e(g-1) = 0$ for all $g \in C$ and, since $e\mathbb{Z}C \cong_{\mathbb{Z}C} D$, $D(g-1) = \{0\}$. The isomorphism $B/L \cong_{\mathbb{Z}C} D$ yields $B(g-1) \leq L$ if $g \in C$. If $x \in G$ is arbitrary, then

$$Bx(g-1) = B(g-1)x \leq Lx = L.$$

By Lemma 2.4, A/L is a sum of $(B/L)x$ $(x \in G)$. It follows that $A(g-1) \leq L$ for all $g \in C$. Thus $C \leq C_G(A/L)$. Together with thew inclusion $C \leq C_G(L)$ it implies that the natural semidirect product $C \ltimes A$ is a nilpotent group. The equation $\Pi(A) \cap \Pi(C) = \emptyset$ implies that $C \leq C_G(A) = \{1\}$ This is a contradiction. Hence $E \cap L = \{0\}$. Since A/L is a simple $\mathbb{Z}G$–module, $A = E \oplus L$.

Suppose now that the additive group of A is periodic. Since A/L is a p–group, there is no loss if we assume that A is a p–group. The mapping $\rho : a \mapsto pa,\ a \in A$, is a $\mathbb{Z}G$–endomorphism of A. Thus

$$pA = \text{Im } \rho \cong_{\mathbb{Z}G} A/\text{Ker } \rho.$$

We have that Ker $\rho = \Omega_1(A)$ is a lower layer of A. Since A/L is an elementary abelian p–group, $pA \leq L$. By the choice of L, every G–chief factor of $A/\text{Ker } \rho$ is G–central. It follows that $\Omega_1(A)$ has a G–eccentric chief factor. By the arguments given above, we have $\Omega_1(A) = E \oplus K$, where $K = \Omega_1(A) \cap L$ and E is a simple $\mathbb{Z}G$–submodule which is G–eccentric. Then $A = E \oplus L$.

Consider now the next case, and suppose that the additive group of A is torsion–free. We again use the mapping $\rho : a \mapsto pa,\ a \in A$. In this case, Ker $\rho = \{0\}$ and thus $pA = \text{Im } \rho \cong_{\mathbb{Z}G} A$. It follows that $A/pA \cong_{\mathbb{Z}G} pA/p^2A$. Since $pA \leq L$, every G–chief factor of pA/p^2A must be central, a contradiction that shows that this case cannot appear.

Finally, we deal with the case in which A is mixed. Denote by T the periodic part of A. Using arguments similar to those of above, we obtain that A/T is a locally nilpotent $\mathbb{Z}G$–module. It follows that $T/(T \cap L)$ is a simple $\mathbb{Z}G$–module, which is G–eccentric. By the arguments given above, T includes a simple $\mathbb{Z}G$–submodule E, which is G–eccentric. Then $E \cap L = \{0\}$. Since A/L is a simple $\mathbb{Z}G$–module, $A = E \oplus L$. □

Corollary 2.6 *Let G be a periodic hypercentral group, and suppose that A is a $\mathbb{Z}G$–module. If $A/LN_{\mathbb{Z}G}(A)$ is a G–hypereccentric $\mathbb{Z}G$–module then A has the LN–decomposition.*

Proof – Let

$$L = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots A_\gamma = A$$

be an ascending series of $\mathbb{Z}G$–submodules whose factors are $\mathbb{Z}G$–simple and G–eccentric. We proceed by transfinite induction. If $\alpha = 1$, the result follows from Proposition 2.5. Let $\alpha > 1$ and suppose we have already proved that A_β has the LN–decomposition for all $\beta < \alpha$. Assume first that α is a limit ordinal. If $\delta < \beta < \alpha$ then

$$A_\delta = LN_{\mathbb{Z}G}(A_\delta) \bigoplus E^\infty_{\mathbb{Z}G}(A_\delta) \text{ and } A_\beta = LN_{\mathbb{Z}G}(A_\beta) \bigoplus E^\infty_{\mathbb{Z}G}(A_\beta).$$

Note that $LN_{\mathbb{Z}G}(A_\beta) = LN_{\mathbb{Z}G}(A_\delta) = L$. Since A_β has the LN–decomposition, $E^\infty_{\mathbb{Z}G}(A_\beta)$ contains every G–hypereccentric $\mathbb{Z}G$–submodule and, in particular, $E^\infty_{\mathbb{Z}G}(A_\delta) \leq E^\infty_{\mathbb{Z}G}(A_\beta)$. It follows that

$$E^\infty_{\mathbb{Z}G}(A_\alpha) = \bigcup_{\beta<\alpha} E^\infty_{\mathbb{Z}G}(A_\beta) \text{ and } A_\alpha = LN_{\mathbb{Z}G}(A_\alpha) \bigoplus E^\infty_{\mathbb{Z}G}(A_\alpha).$$

Suppose now that $\alpha - 1$ exists. Put $U = A_\alpha$ and $V = A_{\alpha-1}$. Then $V = L \oplus W$, where $W = E^\infty_{\mathbb{Z}G}(V)$. The factor–module U/W has a locally nilpotent $\mathbb{Z}G$–submodule $(L+W)/W$ and

$$(U/W)/((L+W)/W) \cong_{\mathbb{Z}G} U/(L+W) = U/V = A_\alpha/A_{\alpha-1}.$$

By Proposition 2.5, there exists a simple G–eccentric $\mathbb{Z}G$–submodule Y/W such that $U/W = V/W \oplus Y/W$. Clearly $Y = E^\infty_{\mathbb{Z}G}(U)$ and, in particular, $Y \cap L = \{0\}$. It follows that

$$A_\alpha = U = Y \bigoplus L = LN_{\mathbb{Z}G}(A_\alpha) \bigoplus E^\infty_{\mathbb{Z}G}(A).$$

Then the result follows by induction. □

Corollary 2.7 *Let G be a periodic hypercentral group and suppose that A is a $\mathbb{Z}G$–module. If $A/LN_{\mathbb{Z}G}(A)$ is an artinian $\mathbb{Z}G$–module then A has the LN–decomposition.*

Proof – Let $C/LN_{\mathbb{Z}G}(A)$ be a $\mathbb{Z}G$–submodule such that $C_G(C/LN_{\mathbb{Z}G}(A)) = G$. It is no hard to see that C is a locally nilpotent $\mathbb{Z}G$–submodule. In this case $C \leq LN_{\mathbb{Z}G}(A)$. Thus $A/LN_{\mathbb{Z}G}(A)$ has no non-zero G–central submodules. Since $A/LN_{\mathbb{Z}G}(A)$ is artinian, it has the Z–decomposition (see [12]). By the arguments given above, this means that $A/LN_{\mathbb{Z}G}(A)$ is a G–hypereccentric $\mathbb{Z}G$–module. Then it suffices to apply Corollary 2.6. □

3 Periodic Groups with Minimal Condition on Abnormal Subgroups

Let G be a group. We denote by $\mathfrak{L}_{abn}(G)$ the family of all abnormal subgroups of G. We say that G satisfies *the minimal condition for abnormal subgroup* (or shortly G *satisfies Min–abn*) if the ordered by inclusion set $\mathfrak{L}_{abn}(G)$ satisfies the minimal condition.

The following result is obvious.

Lemma 3.1 *Let G be a group satisfying Min–abn. If H is a normal subgroup of G then G/H satisfies Min–abn.*

A normal subgroup A of a group G it is said to satisfy the condition *Min–G* if A satisfies the minimal condition for G–invariant subgroups.

Lemma 3.2 *Let G be a group satisfying Min–abn and let H be an abnormal subgroup of G. If A is an H–invariant subgroup of G such that $A \cap H = \{1\}$ then A satisfies Min–H.*

Proof – In fact, if B is a H–invariant subgroup of A then BH is abnormal in G. □

A subgroup H of a group G is said to be *weakly abnormal* if $N_G(S) = S$ for every subgroup S containing H. Obviously, every abnormal subgroup is weakly abnormal.

Lemma 3.3 *Let G be a group and let B be a weakly abnormal subgroup of G. Suppose that $B = HA$, where A is a normal subgroup of G. If H is a weakly abnormal subgroup of B then H is a weakly abnormal subgroup of G.*

Proof – Let S be an arbitrary subgroup of G containing H. If $x \in N_G(S)$ then it is clear that $xA \in N_{G/A}(SA/A)$. Since $H \leq S$, $B/A = (HA)/A \leq (SA/A)$. B is weakly abnormal in G and $A \leq B$ so that B/A is weakly abnormal in G/A. It follows that$(SA)/A$ is self-normalizing within the factor-group G/A, that is $xA \in (SA)/A$ and $x \in SA$. Hence $x = ya$ for some elements $y \in S$ and $a \in A$. We have

$$S = x^{-1}Sx = a^{-1}(y^{-1}Sy)a = a^{-1}Sa$$

and it follows that $a \in N_G(S)$. Since $a \in A \leq B$, $a \in N_G(B)$. Thus $a \in N_G(S \cap B)$. The inclusion $H \leq S \cap B$ shows that $S \cap B$ is self-normalizing in B. It follows that $a \in S \cap B$ and $x = ya \in S$. Therefore, S is self-normalizing in G. □

Proposition 3.4 *Let G be a group and suppose that A is a normal subgroup of G such that G/A has no proper weakly abnormal subgroups. If A is nilpotent then the relation "being a weakly abnormal subgroup" is transitive in G.*

Proof – Let B be a weakly abnormal subgroup of G and let H be a weakly abnormal subgroup of B. If $B = G$, we are done, so suppose that $B \neq G$. Let S be a subgroup including BA. Suppose that $xA \in N_{G/A}(S/A)$. Clearly $x \in NG(S)$ and the inclusion $B \leq S$ implies $x \in S$ and $xA \in S/A$. In other words, BA/A is weakly abnormal in G/A. Since G/A has no proper weakly

abnormal subgroups, $AB = G$. Put $K = A \cap B$ so that K is a normal subgroup of B and, as above, KH/K is weakly abnormal in B/K. Since

$$G/A = AB/A \cong B/(A \cap B) = B/K,$$

B/K has no proper weakly abnormal subgroups. It follows that $B/K = HK/K$, that is $B = HK$. Let

$$\{1\} = Z_0 \leq Z_1 \leq \cdots \leq Z_{n-1} \leq Z_n = A$$

be the upper central series of A. Put $K_1 = KZ_1$ and $B_1 = BZ_1$. Since K is normal in both B and Z_1, K is normal in B_1. By Lemma 3.3 and the equation $B = HK$, H is weakly abnormal in B_1. Suppose that we have already proved that H is weakly abnormal in $BZ_{n-1} = B_{n-1}$. By the arguments given above, AB_{n-1}/A is weakly abnormal in G/A, and, since G/A has no proper weakly abnormal subgroups, $AB_{n-1} = G$. The intersection

$$A \cap B_{n-1} = A \cap BZ_{n-1} = Z_{n-1}(A \cap B) = Z_{n-1}K$$

is a normal subgroup of B_{n-1} and so $(Z_{n-1}K)H/(Z_{n-1}K)$ is weakly abnormal in $B_{n-1}/(Z_{n-1}K)$. However by

$$G/A = AB_{n-1}/A \cong B_{n-1}/(A \cap B_{n-1}) = B_{n-1}/(Z_{n-1}K)$$

we obtain that $B_{n-1}/(Z_{n-1}K)$ has no proper weakly abnormal subgroups. It follows that

$$B_{n-1}/(Z_{n-1}K) = (Z_{n-1}K)H/(Z_{n-1}K),$$

that is $B_{n-1} = (Z_{n-1}K)H$. Since $(Z_{n-1}K)$ is normal in both B_{n-1} and A, the equation $G = AB = AB_{n-1}$ implies that $(Z_{n-1}K)$ is normal in G. By Lemma 3.3 and the equation $B_{n-1} = (Z_{n-1}K)H$, H is weakly abnormal in G. □

Since every weakly abnormal subgroup of an hyperabelian group is abnormal (see [3, Corollary 6.9]), the Proposition 3.4 has the following consequence.

Corollary 3.5 *Let G be an hyperabelian group. If G has a normal nilpotent subgroup A such that G/A is locally nilpotent, then the relation "being an abnormal subgroup" is transitive in G.*

Theorem 3.6 *Let G be a periodic group and suppose that H is a locally nilpotent normal subgroup of G. Assume further that G/H is hypercentral. If G has no proper abnormal subgroups then G is locally nilpotent.*

Proof – Let

$$H = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = G$$

be the reciprocal images of the upper central series of G/H. Choose an arbitrary finite subgroup Z/H of $\zeta(G/H)$, and suppose that Z is not locally nilpotent. Then there is a prime p such that a Sylow p–subgroup S of Z is not normal in Z. Let $g \in G$ and define $Q = \langle S, S^g \rangle$. Since $Q \leq Z$, the finiteness of $\mathrm{Syl}_p(Z)$ implies that $\mathrm{Syl}_p(Q)$ is likewise finite. In this case, all Sylow p–subgroup of Q are conjugate (see [8, §54]). Hence there exists an element $x \in Q$ such that $S^g = S^x$, and thus S is pronormal in G. Therefore $N_G(S)$ is abnormal in G, and hence S is normal in G. This contradiction shows that Z is locally nilpotent. By the choice of Z, it follows that C_1 is locally nilpotent. Then, it suffices to apply transfinite induction. □

Lemma 3.7 *Let G be a periodic hyperabelian group and let A be an abelian normal subgroup of G. Suppose that G/A is locally nilpotent and $G/C_G(A)$ is hypercentral. If G satisfies Min–abn, then either G is locally nilpotent or G satisfies the following conditions:*

(1) *The $\mathbb{Z}G$–module A has the LN–decomposition;*

(2) *$G = H \ltimes E$, where E is the maximal G–hypereccentric $\mathbb{Z}G$–submodule of A;*

(3) *E is an artinian $\mathbb{Z}G$–submodule (that is E satisfies Min–G);*

(4) *H is a locally nilpotent subgroup;*

(5) *H is a minimal abnormal subgroup; and*

(6) *if H_1 is another complement to E in G then the subgroups H and H_1 are conjugate.*

Proof – If G has no proper abnormal subgroup, then by Theorem 3.6, G is locally nilpotent. Therefore we may assume that G has proper abnormal subgroups. Then the family $\mathfrak{L}_{abn}(G)$ is not empty and hence has a minimal member H. By Corollary 3.5, H has no proper abnormal subgroups so that, by Theorem 3.6, H is locally nilpotent. The product HA is an abnormal subgroup of G and so HA/A is abnormal in G/A. However G/A is hypercentral and, in particular, it has no proper abnormal subgroups. This means that $G/A = HA/A$, that is $G = HA$. The intersection $L = H \cap A$ is a normal subgroup of H. Since A is abelian, the equation $G = HA$ yields that L is normal in G. The equation $G = HA$ also shows that every H–invariant subgroup of A is G–invariant. Since H is locally nilpotent, $L \leq LN_{\mathbb{Z}G}(A)$.

The subgroup H/L is abnormal in G/L and $(H/L) \cap (A/L) = \{1\}$. By Lemma 3.2, A/L satisfies Min–H and hence Min–G. By Corollary 2.7, A has the LN–decomposition $A = C \oplus E$, where $C = LN_{\mathbb{Z}G}(A)$ and $E = E^{\infty}_{\mathbb{Z}G}(A)$.

In particular, CH is a locally nilpotent subgroup. On the other hand, CH is abnormal in G. Since H is an abnormal subgroup of G, $HC = H$, that is $C \leq H \cap A = L$ and then $C = L$. It follows that

$$H \cap E = H \cap A \cap E = L \cap E = \{1\}.$$

By Lemma 3.2, E satisfies Min–H and hence Min–G. Condition (5) follows from [13, Theorem 2]. □

Lemma 3.8 *Let G be a group, and suppose that A is an abelian normal subgroup of G. Assume further that G satisfies the following conditions:*

(a) *G/A is locally nilpotent.*

(b) *$G/C_G(A)$ is hypercentral.*

(c) *A satisfies Min–G.*

If $A = E^{\infty}_{\mathbb{Z}G}(A)$ then G has an abnormal subgroup H such that $G = H \ltimes A$.

Proof – By [13, Theorem 2], G has a subgroup H such that $G = H \ltimes A$. Moreover, every complement to A in G is conjugate to H. Let S be a subgroup of G containing H. Then $S = (A \cap S)H$. Clearly, $A \cap S$ is a G–invariant subgroup. Moreover, every S–invariant (even H–invariant) subgroup of $A \cap S$ is G–invariant. Put $L = N_G(S)$. As above, $L = (A \cap L)H$. If $L \neq S$, then $A \cap S \neq A \cap L$. Inside the factor–group $L/(A \cap S)$ we have

$$S/(A \cap S) = H(A \cap S)/(A \cap S).$$

If $x \in (L \cap A) \setminus S$ then $[x, S] \leq S$ and so

$$[x, S](A \cap S)/(A \cap S) \leq H(A \cap S)/(A \cap S).$$

On the other hand, $[x, S] \leq A$, that is $[x, S] \leq A \cap S$. In other words, the factor $(A \cap L)/(A \cap S)$ is S–central, and therefore G–central. By the choice of A, every G–chief factor of A is G–eccentric, so that A has no G–central factors. This contradiction shows that $S = N_G(S)$, and so H is a weakly abnormal subgroup of G. It follows, by Lemma 6.5 in [3], that H is pronormal and hence even abnormal in G. □

Lemma 3.9 *Let G be a group and suppose that A is an abelian normal subgroup of G such that G/A is locally nilpotent and $G/C_G(A)$ is hypercentral. Assume further that the $\mathbb{Z}G$–module A has the LN–decomposition. If $E = E^{\infty}_{\mathbb{Z}G}(A)$ satisfies Min–G then G satisfies Min–abn. Moreover every complement to E is a minimal abnormal subgroup of G.*

Proof – By [13, Theorem 2], G has a subgroup H such that $G = H \ltimes E$. By Lemma 3.8, H is abnormal. On the other hand, H is locally nilpotent and therefore has no proper abnormal subgroups. This means that H is a minimal abnormal subgroup of G. Let

$$L_1 \geq L_2 \geq \cdots \geq L_n \geq \cdots$$

be a descending chain of abnormal subgroups. Since G/E has no proper abnormal subgroups, $G = L_n E$ for all $n \geq 1$. As $L_n \cap E$ is a G–invariant subgroup of E ($n \geq 1$), we have a descending chain

$$E \cap L_1 \geq E \cap L_2 \geq \cdots \geq E \cap L_n \geq \cdots$$

of G–invariant subgroups of E. Since E satisfies Min–G, there exists some $m \geq 1$ such that $E \cap L_t = E \cap L_r$ for all $t, r > m$. This equation together with $G = L_t E = L_r E$ yields that $L_t = L_r$. It follows that G satisfies the minimal condition for abnormal subgroups. □

Proof of Theorem 1.1 Let

$$\{1\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$$

be the upper central series of L. We proceed by induction on n. If $n = 1$, the result follows from Lemma 3.7. Suppose that $n > 1$. Clearly, $G/C_G(L_1)$ is hypercentral, so that, if G/L_1 is locally nilpotent, the statement follows from Lemma 3.7. Therefore, we may suppose by induction that $G/L_1 = (L/L_1)(Y/L_1)$, where Y/L_1 is a locally nilpotent abnormal subgroup of G/L_1 (in particular, Y/L_1 is a minimal abnormal subgroup of G/L_1). By Corollary 3.5, every abnormal subgroup of Y is abnormal in G. Therefore Y satisfies Min–abn. By Lemma 3.7, L_1 has the LN–decomposition, that is

$$L_1 = Z_1 \times E_1,$$

where $Z_1 = LN_{\mathbb{Z}G}(L_1)$ and $E_1 = E^{\infty}_{\mathbb{Z}G}(L_1)$, and

$$Y = H \ltimes E_1,$$

where H is a locally nilpotent abnormal subgroup of Y (in particular, H is a minimal abnormal subgroup of Y). Since $Z_1 H$ is locally nilpotent and abnormal, then $Z_1 \leq H$. Therefore, $L_1/(H \cap L_1)$ is G–hypereccentric and satisfies Min–G. Since $G = LY$ and $Y = L_1 H$, then $G = L_1 H$. Suppose U/V is one of the factors L_{j+1}/L_j ($0 \leq j \leq n-1$). If $j = 0$, $U = L_1$ and we have already seen that L_1 satisfies (3) and (4). If $j > 0$, U/V satisfies (3)and (4) by the hypothesis of induction, because $HL_1/L_1 = Y/L_1$.

It remains to prove the condition (5). For, let H_1 be another minimal abnormal subgroup of G. Since G/L is locally nilpotent and H_1L/L is an abnormal subgroup of G/L, $G/L = H_1L/L$, that is $G = H_1L$, because locally nilpotent groups have no proper abnormal subgroups. As $L_1 \leq \zeta(L)$, every H_1–invariant subgroup of L_1 is also G–invariant. In particular, $Z_1 = LN_{\mathbb{Z}H_1}(L_1)$ and $E_1 = E^{\infty}_{\mathbb{Z}H_1}(L_1)$. It follows that Z_1H_1 is locally nilpotent. Since Z_1H_1 is abnormal, $Z_1H_1 = H_1$, that is $Z_1 \leq H_1$. We claim that $H_1 \cap E_1 = \{1\}$. Otherwise, since E_1 has an ascending series of G–invariant subgroups whose factors are G–chief and G–eccentric, $H_1 \cap E_1$ has a minimal G–invariant subgroup M. From the equation $G = H_1L$ and the inclusion $L_1 \leq \zeta(L)$, we obtain that M is a minimal H_1–invariant subgroup. As H_1 is locally nilpotent, $M \leq \zeta(H_1)$. It follows that $M \leq \zeta(G)$. On the other hand, $M \leq E_1$ and therefore $C_G(M) \neq G$, a contradiction which shows our claim. Thus, $H_1 \cap L_1 = Z_1$ and then

$$H_1L_1 = H_1 \ltimes E_1.$$

In the factor-group G/L_1 the subgroup HL_1/L_1 is locally nilpotent and abnormal, so in particular, is a minimal abnormal subgroup. By induction, there exists some $x \in G$ such that $(H_1L_1/L_1)^{xL_1} = HL_1/L_1$. Thus $(L_1H_1)^x = HL_1 = Y$ and then

$$(L_1H_1)^x = L_1^xH_1^x = (E_1Z_1)^xH_1^x = E_1^xZ_1^xH_1^x = E_1^xH_1^x = H_1^x \ltimes E_1.$$

By Lemma 3.7, there exists some $y \in Y$ such that $(H_1^x)^y = H$. Therefore, the proof is now complete.

Conversely, by condition (3) the center L_1 of L has the LN–decomposition. Put $E = E^{\infty}_{\mathbb{Z}G}(L_1)$. Clearly, L_1/E is a $\mathbb{Z}(G/C_G(L_1))$–locally nilpotent module, and hence G/E is locally nilpotent. On the other hand, E satisfies Min-G by (4), and Lemma 3.9 completes the proof. □

4 FC–nilpotent Groups with Minimal Condition on Abnormal Subgroups

We return to our module theoretical setting.

Lemma 4.1 *Let G be a hypercentral group. Let A be a $\mathbb{Z}G$–module and let E be a $\mathbb{Z}G$–submodule of A satisfying the following conditions:*

(a) *E is an artinian $\mathbb{Z}G$–submodule.*

(b) *E is G–hypereccentric.*

(c) $C_G(A/E) = G$.

Then A has a $\mathbb{Z}G$–submodule C such that $A = C \oplus E$.

Proof – Let

$$\mathfrak{M} = \{U \mid U \text{ is a } \mathbb{Z}G - \text{submodule of } A \text{ such that } A = U + E\}$$

and

$$\mathfrak{N} = \{U \cap E \mid U \in \mathfrak{M}\}$$

Since E is an artinian $\mathbb{Z}G$–submodule, then $\mathfrak{N}$ has a minimal element M. Let C be an element of $\mathfrak{M}$ such that $M = C \cap E$. If $M = \{0\}$, we are done. Suppose that $M \neq \{0\}$. There is no loss if we assume that $C_G(C) = \{1\}$. Let $1 \neq z \in \zeta(G)$. Then the mapping $\phi_z : a \mapsto a(z-1), a \in A$, is a $\mathbb{Z}G$–endomorphism of A. Since $C_G(A/E) = G$, $C(z-1) \leq E \cap C = M$. On the other hand, $C(z-1)/M(z-1)$ is an epimorphic image of

$$C/M = C/(E \cap C) \cong_{\mathbb{Z}G} (C+E)/E = A/E.$$

We claim that $C(z-1) = M(z-1)$. Otherwise, E has a non–zero G–central factor, contradicting (b). Thus, for each element $c \in C$ there exists some element $e \in M$ such that

$$c(z-1) = e(z-1).$$

This means that $c - e \in C_C(z)$ and then $c \in M + C_C(z)$. Since $z \in \zeta(G)$, $Z = C_C(z)$ is a $\mathbb{Z}G$–submodule. Moreover $C = Z + (E \cap C)$. This and $A = C + E$ give

$$A = C + E = Z + (E \cap C) + E = Z + E,$$

which shows that $Z \in \mathcal{M}$. We remark that Z is a proper $\mathbb{Z}G$–module of C. For, otherwise $z \in C_G(Z) = C_G(C) = \{1\}$, a contradiction. By the choice of C, we must have $Z \cap E = C \cap E$. But then

$$C = C \cap (Z + E) = Z + (C \cap E) = Z + (Z \cap E) = Z.$$

This final contradiction shows that $M = \{0\}$ and then $A = C \oplus E$. □

Corollary 4.2 *Let G be a hypercentral group and let A be a $\mathbb{Z}G$–module. Suppose that A has a finite series of $\mathbb{Z}G$–submodules*

$$\{0\} = E_0 \leq E_1 \leq \cdots \leq E_t = E \leq A$$

such that E_{j+1}/E_j is a G–eccentric simple $\mathbb{Z}G$–module ($0 \leq j \leq t-1$) and $C_G(E/E_t) = G$. Then $A = \zeta_{\mathbb{Z}H}(A) \oplus E$.

This and Corollary 2.2 immediately give the next result.

Corollary 4.3 *Let G be a hypercentral group and let A be a $\mathbb{Z}G$–module. Suppose that A has a finite series of $\mathbb{Z}G$–submodules*

$$\langle 0\rangle = A_0 \leq A_1 \leq \cdots \leq A_n = A$$

whose G–eccentric factors are simple $\mathbb{Z}G$–modules. Then A has the Z–decomposition.

Corollary 4.4 *Let G be a hypercentral group and let A be a $\mathbb{Z}G$–module. Suppose that A has a G–hypereccentric artinian $\mathbb{Z}G$–submodule E. If A/E is a $\mathbb{Z}G$–hypercentral module then A has the Z–decomposition.*

Proof – Let

$$E = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots A_\gamma = A$$

be the reciprocal images of the terms of the upper $\mathbb{Z}G$–central series of A/E. We proceed by induction on α. If $\alpha = 1$, the result follows from Lemma 4.1. Suppose that $\alpha > 1$ and we have already seen that A_β has the Z–decomposition for all $\beta < \alpha$.

Suppose first that α is a limit ordinal. If $\delta < \beta < \alpha$ then, by the induction hypothesis,

$$A_\beta = \zeta^\infty_{\mathbb{Z}G}(A_\beta) \bigoplus E \text{ and } A_\delta = \zeta^\infty_{\mathbb{Z}G}(A_\delta) \bigoplus E.$$

Since $A_\delta \leq A_\beta$, $\zeta^\infty_{\mathbb{Z}G}(A_\delta) \leq \zeta^\infty_{\mathbb{Z}G}(A_\beta)$. It follows that

$$\zeta^\infty_{\mathbb{Z}G}(A_\alpha) = \bigcup_{\beta<\alpha} \zeta^\infty_{\mathbb{Z}G}(A_\beta) \text{ and } A_\alpha = \zeta^\infty_{\mathbb{Z}G}(A_\alpha) \bigoplus E.$$

Suppose now that α is not limit so that $A_{\alpha-1} = \zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1}) \oplus E$. Then the factor–module $A_\alpha/\zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1})$ satisfies all conditions of Lemma 4.1 and therefore

$$A_\alpha/\zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1}) = (E + \zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1}))/\zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1})) \bigoplus B/\zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1}),$$

where $B/\zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1})$ is a $\mathbb{Z}G$–submodule such that $C_G(B/\zeta^\infty_{\mathbb{Z}G}(A_{\alpha-1})) = G$. Thus, $B \leq \zeta^\infty_{\mathbb{Z}G}(A_\alpha)$. Clearly $B \cap E = \{0\}$ and so $A_\alpha = B \oplus E$. Since E is G–hypereccentric, $B = \zeta^\infty_{\mathbb{Z}G}(A_\alpha)$, that is $A_\alpha = E \oplus \zeta^\infty_{\mathbb{Z}G}(A_\alpha)$. The result follows by induction. □

Let G be a group and let A be a $\mathbb{Z}G$–module. We define

$$FC_{\mathbb{Z}G}(A) = \{a \in A \mid |G : C_G(a\mathbb{Z}G)| < \infty\}.$$

Clearly, $FC_{\mathbb{Z}G}(A)$ is a $\mathbb{Z}G$–submodule of A called the $\mathbb{Z}G$–*FC-center* of A. By induction we define the n–th $FC_{n,\mathbb{Z}G}(A)$ for every non-negative integer n. We remark that if G is a group and A is a normal abelian subgroup whose elements have finite conjugacy classes in G, then for A considered as $\mathbb{Z}G$–module, we have $A = FC_{\mathbb{Z}G}(A)$.

Lemma 4.5 *Let G be a hypercentral group. Suppose that A is a $\mathbb{Z}G$–module and C is a $\mathbb{Z}G$–submodule of A such that the following conditions hold:*

(a) $C \leq FC_{\mathbb{Z}G}(A)$.

(b) $C \leq \zeta^{\infty}_{\mathbb{Z}G}(A)$.

(c) *A/C is a finite simple $\mathbb{Z}G$–module.*

(d) $C_G(A/C) \neq G$.

Then $A = C \oplus E$, where E is a finite simple $\mathbb{Z}G$–submodule.

Proof – Suppose first that A is $\mathbb{Z}$–periodic. Since A/C is finite, $G/C_G(A/C)$ is finite. Therefore we may choose a finitely generated subgroup K such that $G = KC_G(A/C)$. There is no loss if we assume that $A = a\mathbb{Z}G$ for some element $a \in A \setminus C$. Let $\mathfrak{K}$ be the set of all finitely generated subgroup of G containing K. Choose an arbitrary subgroup $L \in \mathfrak{K}$. Being a finitely generated subgroup of a hypercentral group, L is nilpotent and so there exists a finite subnormal series

$$K = K_0 \lhd K_1 \lhd \cdots \lhd K_n = L$$

between K and L. Put $U = a\mathbb{Z}K$ and $V = a\mathbb{Z}K_1$. From conditions (a) and (c), it follows that the natural semidirect product $G \ltimes A$ is an FC–hypercentral group. In turn, it follows that every finitely generated subgroup of $G \ltimes A$ is nilpotent–by–finite (see [9], Part 1, p.133). Consequently, if F is a finitely generated subgroup of G and M is a finite subset of A, then the $\mathbb{Z}G$–submodule $M\mathbb{Z}F$ is finite. In particular, U and V are finite. By Corollary 2.2,

$$U = C_1 \bigoplus E,$$

where $C_1 = \zeta^{\infty}_{\mathbb{Z}G}(U)$ and $E = E^{\infty}_{\mathbb{Z}G}(U)$. From the choice of K we have

$$E \cong U/C_1 = U/(U \cap C) \cong (U + C)/C = A/C,$$

so that E is a simple $\mathbb{Z}G$–submodule of U, and moreover $|E| = |A/C|$. Put $E_1 = E\mathbb{Z}K_1$. Since K is normal in K_1, then Eg is a simple $\mathbb{Z}K$–submodule for each element $g \in K_1$ and E_1 is a semisimple $\mathbb{Z}K$–submodule of V. We claim that $E_1 \cap C = \{0\}$. Otherwise, $E_1 \cap C$ must contain some of the simple $\mathbb{Z}G$–submodules Eg for some $g \in K_1$. It is not hard to see that in this case the $\mathbb{Z}K_1$–center of V contains too Eg. Then $Eg(x-1) = \{0\}$ for each $x \in K_1$. In particular,

$$\{0\} = Eg(g^{-1}xg - 1) = E(xg - g) = E(x-1)g$$

and it follows that $E(x-1)=\{0\}$ for all $x \in K_1$, what is impossible. Thus $E_1 \cap C = \{0\}$ as claimed. Since

$$E_1 \cong E_1/(E_1 \cap C) \cong (E_1 + C)/C = A/C,$$

$|E_1| = |A/C| = |E|$ and hence $E = E_1$. In other words, E is a $\mathbb{Z}K_1$–submodule. Proceeding in this way, after finitely many steps we obtain that E is a $\mathbb{Z}L$–submodule. Since this holds for every $L \in \mathfrak{L}$, E is a $\mathbb{Z}G$–submodule. But E is a simple G–eccentric $\mathbb{Z}G$–submodule, so that $A = C \oplus E$.

Consider now the case when the additive group of A is torsion-free. Since A/C is finite, then it is an elementary abelian p–group for some prime p. The mapping $\rho : a \mapsto pa,\ a \in A$ is a $\mathbb{Z}G$–endomorphism of A. For this case, $\mathrm{Ker}\ \rho = \{0\}$, and thus $pA = \mathrm{Im}\ \rho \cong_{\mathbb{Z}G} A$. It follows that $A/pA \cong_{\mathbb{Z}G} pA/p^2A$. Since $pA \leq C$, every G–chief factor of pA/p^2A must be central by condition (b), a contradiction that shows that this case cannot appear.

Finally, consider the case in which A is mixed. Denote by T the periodic part of A. Using arguments similar to those of above, we obtain that A/T is a $\mathbb{Z}G$–hypercentral $\mathbb{Z}G$–module. It follows that $T/(T \cap C)$ is a simple $\mathbb{Z}G$–module, which is G–eccentric. By the arguments given above, T includes a simple $\mathbb{Z}G$–submodule E, which is G–eccentric. Then $E \cap C = \{0\}$. Since A/L is a simple $\mathbb{Z}G$–module, $A = E \oplus L$. □

The following are easy consequences of Lemma 4.5.

Corollary 4.6 *Let G be a hypercentral group. Suppose that A is a $\mathbb{Z}G$–module and C is a $\mathbb{Z}G$–submodule of A such that the following conditions hold:*

(a) $C \leq FC_{n,\mathbb{Z}G}(A)$ *for some* n.

(b) $C \leq \zeta^{\infty}_{\mathbb{Z}G}(A)$.

(c) A/C *is a finite simple* $\mathbb{Z}G$*–module.*

(d) $C_G(A/C) \neq G$.

Then $A = C \oplus E$, where E is a finite simple $\mathbb{Z}G$–submodule.

Proof – Clearly we may suppose the $\mathbb{Z}G$–submodule C to be not trivial. Let m be the minimum positive integer such that $C \leq FC_{m,\mathbb{Z}G}(A)$. If $m = 1$, then C is contained in $F = FC_{\mathbb{Z}G}(A)$, and Lemma 4.5 yields the thesis. Let now $m > 1$. By induction we may suppose that the $\mathbb{Z}G$–module A/F contains a finite simple $\mathbb{Z}G$–submodule D/F such that $A/F = C/F \oplus D/F$. Moreover, $C_G(D/F)$ is contained in $C_G(A/C)$ and the factor module D/F is G–eccentric. Therefore, the $\mathbb{Z}G$–module D satisfies all the conditions of Lemma 4.5, so that it includes a finite simple $\mathbb{Z}G$–submodule E such that

$D = F \oplus E$. Since $A = C + E$, the $\mathbb{Z}G$–modules E and A/C are isomorphic. Thus, $C_G(E) = C_G(A/C) \neq G$. On the other hand, E is a simple $\mathbb{Z}G$–module, thus it is not contained in C, by condition (b). It follows that $C \cap E = \{0\}$. □

Corollary 4.7 *Let G be a hypercentral group. Suppose that A is a $\mathbb{Z}G$–module and C is a $\mathbb{Z}G$–submodule of A such that the following conditions hold:*

(a) $A = FC_{n,\mathbb{Z}G}(A)$ *for some* n.

(b) $C \leq \zeta^{\infty}_{\mathbb{Z}G}(A)$.

(c) A/C *is a G–hypereccentric $\mathbb{Z}G$–module.*

Then A has the Z–decomposition.

Corollary 4.8 *Let G be a hypercentral group and let A be a $\mathbb{Z}G$–module satisfying the following conditions:*

(a) $A = FC_{n,\mathbb{Z}G}(A)$ *for some* n.

(b) *The $\mathbb{Z}G$–hypercenter $\zeta^{\infty}_{\mathbb{Z}G}(A)$ contains a $\mathbb{Z}G$–submodule C such that A/C is artinian.*

Then A has the Z–decomposition.

Proof – Indeed A/C has the Z–decomposition since it is artinian (see [12]). In particular, $\zeta^{\infty}_{\mathbb{Z}G}(A)$ has a $\mathbb{Z}G$–submodule D such that A/D is G–hypereccentric. Now it suffices to apply Corollary 4.7. □

Let G be a group without proper abnormal subgroups, and denote by F_n the n-th FC–center of G for some non-negative integer n. Recall that, if G/F_n is hypercentral, then G is likewise hypercentral (see [7]).

Lemma 4.9 *Let G be a group and let A be an abelian normal subgroup of G. Suppose that A lies into the n-th FC–center of G and G/A is hypercentral. If G satisfies Min–abn, then either G is hypercentral or G satisfies the following conditions:*

(1) *The $\mathbb{Z}G$–module A has the Z–decomposition;*

(2) $G = H \ltimes E$, *where E is a maximal G–hypereccentric $\mathbb{Z}G$–submodule of A;*

(3) *E is an artinian $\mathbb{Z}G$–submodule, that is E satisfies Min–G;*

(4) *H is a hypercentral subgroup;*

(5) *H is a minimal abnormal subgroup;*

(6) *if H_1 is another complement to E in G, then H and H_1 are conjugate.*

Conversely, if G satisfies the conditions (1)–(6), then G has Min–abn.

Proof – If G has no proper abnormal subgroup then, by the quoted result, G is hypercentral. Thus, we may assume that G has proper abnormal subgroups. Then $\mathfrak{L}_{abn}(G)$ is not empty and, by hypothesis, there exists a minimal abnormal subgroup H. On the other hand, the relation of abnormality is transitive in G by Corollary 3.5. Thus H has no proper abnormal subgroups, so that it is hypercentral. The product HA is an abnormal subgroup of G, and hence $HA/A = G/A$ by hypothesis. The intersection $Z = H \cap A$ is a normal subgroup of H, and the equation $G = HA$ also proves that every H–invariant subgroup of A is G–invariant. Since H is hypercentral, $Z \leq \zeta^{\infty}_{\mathbb{Z}G}(A)$. The subgroup H/Z is abnormal in G/Z and $(H/Z) \cap (A/Z) = \{1\}$. By Lemma 3.2, A/Z satisfies Min–H (and hence Min–G). By Corollary 4.8, A has the Z–decomposition

$$A = C \bigoplus E$$

where $C = \zeta^{\infty}_{\mathbb{Z}G}(A)$ and $E = E^{\infty}_{\mathbb{Z}G}(A)$. In particular, CH is a hypercentral subgroup. On the other hand, CH is abnormal in G, so that the minimality of H gives that $HC = H$. In other words, $C \leq H \cap A = Z$ and then $C = Z$. We have

$$H \cap E = H \cap E \cap A = (H \cap A) \cap E = C \cap E = \{1\}.$$

By Lemma 3.2, E satisfies Min–H and hence Min–G. Since the (6) is ensured by [13, Theorem 2], the first part of theorem is proved.

The converse follows from Lemma 3.9. □

Proof of Theorem 1.1 The above lemma and the argument used in the proof of the Theorem 1.1 allow to show the following description of groups satisfying Min–abn which are FC–nilpotent and nilpotent-by-hypercentral.

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