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Hermitian modifications of Toeplitz linear functionals and Orthogonal Polynomials

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Abstract.

In this paper the following situation is considered: A moment functional $\mathcal{L}$ associated with a quasi-definite infinite Hermitian Toeplitz matrix is modified by means of a complex polynomial P of degree one, obtaining a new linear functional $\widetilde{\mathcal{L}} = (P(z) + \overline{P}(z^{-1}))\mathcal{L}$. A characterization of the quasi-definiteness of $\widetilde{\mathcal{L}}$ is obtained and same relations between the sequences of orthogonal polynomials with respect to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, as well as some related questions, are studied.

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§1 - Introduction.

Let $M = \left[c_{i-j}\right]_{i,j=0}^{\infty}$ be an Hermitian Toeplitz matrix, i.e., $c_{-k} = \overline{c}_k$. We will denote by M_n the principal submatrix of size $n+1$. We will assume $\Delta_n = \det M_n \neq 0$ for $n = 0, 1, 2 \ldots$.

It is well know that the sequence of monic polynomials $\left(\Phi_n\right)_{n=0}^{\infty}$ given by

$$\Phi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ \overline{c_1} & c_0 & \cdots & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{c}_{n-1} & \overline{c}_{n-2} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \qquad n = 1, 2 \ldots,$$

$$\Phi_0(z) = 1$$

is a sequence of monic orthogonal polynomials (SMOP) with respect to the inner product

$$(p, q) \;=\; \boldsymbol{L}\left[p(z)\overline{q}(z^{-1})\right] \quad p, q \in \mathbb{P} \tag{1}$$

where $\mathbb{P}$ denotes the linear space of polynomials with complex coefficients and $\boldsymbol{L}$ is the linear functional defined on the linear space of Laurent polynomials Λ in the following way

$$\boldsymbol{L}\left[z^n\right] = c_n, \qquad n \in \mathbb{Z}.$$

A linear functional $\boldsymbol{L}$ on Λ such that $\boldsymbol{L}[z^n] = \overline{\boldsymbol{L}[z^n]}$, $n = 0, 1, \ldots,$ is called Hermitian. The condition $\Delta_n \neq 0$, $n = 0, 1, 2, \ldots,$ defines the so-called quasi-definite or regular Hermitian linear functionals and characterizes the existence of a (unique) SMOP. In particular, if $\Delta_n > 0$, $n = 0, 1 \ldots,$ $\boldsymbol{L}$ is said to be a positive definite linear functional. In such a case, there exists a finite positive Borel measure μ supported on $[-\pi, \pi)$, such that

$$\boldsymbol{L}[p] = \int_{-\pi}^{\pi} p(e^{i\theta}) d\mu(\theta), \quad p \in \mathbb{P}.$$

Taking into account (1) it is straightforward to deduce that the shift operator is isometric with respect to the inner product (1), i.e.,

$$\big(zp,\ zq\big) = \big(p,\ q\big) \qquad p, q \in \mathbb{P}.$$

As a consequence of this fact, we can deduce two equivalent recurrence relations for the SMOP $\left(\Phi_n\right)$. They were obtained by Szegő [14] in the positive definite case and by Geronimus [7] in the general situation stated above.

The SMOP $\left(\Phi_n\right)$ verifies the recurrence relation

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z), \qquad n = 1, 2, \dots. \tag{2}$$

If we define the $*_n$ operation for a polynomial P with deg$P = n$ by $P^*(z) = z^n \overline{P}(\overline{z}^{-1})$, performing this operation on (2) we also get the reverse recurrence relation

$$\Phi_n^*(z) = \overline{\Phi_n(0)} z\Phi_{n-1}(z) + \Phi_{n-1}^*(z), \qquad n = 1, 2, \dots. \tag{3}$$

The values $\Phi_n(0)$ are called reflection (or Schur) parameters of the linear functional $\mathcal{L}$. We will denote $e_n = (\Phi_n, \Phi_n)$. Observe that, in general, $e_n \neq 0$ and when $\mathcal{L}$ is positive definite then $e_n = ||\Phi_n||^2_{L^2(\mu)} > 0$.

A straightforward computation yields

$$1 - |\Phi_n(0)|^2 = \frac{e_n}{e_{n-1}}, \quad e_n = \frac{\Delta_n}{\Delta_{n-1}}. \quad n = 1, 2, \dots. \tag{4}$$

As it is well known, the kernel polynomial $K_n(z, y) = \displaystyle\sum_{k=0}^{n} \frac{\Phi_k(z)\overline{\Phi_k(y)}}{e_k}$ satisfies the Christoffel-Darboux formula (see [6, 7, 14])

$$e_{n+1}(1 - \overline{y}z)K_n(z, y) = \Phi_{n+1}^*(z)\overline{\Phi_{n+1}^*(y)} - \Phi_{n+1}(z)\overline{\Phi_{n+1}(y)}, \tag{5}$$

which, taking into account (2) and (3), can be also written as

$$e_n(1 - \overline{y}z)K_n(z, y) = \Phi_n^*(z)\overline{\Phi_n^*(y)} - \overline{y}z\Phi_n(z)\overline{\Phi_n(y)}. \tag{6}$$

From (4) we see that in the positive definite case $|\Phi_n(0)| < 1$, while in the general case $|\Phi_n(0)| \neq 1$. Conversely, given a sequence of complex numbers $\left(a_n\right)_{n=0}^{\infty}$ with $|a_n| \neq 1$, $n = 1, 2, \dots,$ and $a_0 = 1$, there exists a quasi-definite Hermitian linear functional $\mathcal{L}$ such that $\left(a_n\right)_{n=0}^{\infty}$ is its sequence of reflection parameters, or, equivalently, $a_n = \Phi_n(0)$, where $\left(\Phi_n\right)$ is the SMOP with respect to $\mathcal{L}$. This result is an analog of Favard's theorem (see [5]).

In comparison with the real case (see [4]) few explicit examples of SMOP with respect to an Hermitian linear functional are known in the literature.

A way to generate a new SMOP from a given one $\left(\Phi_n\right)$ is to consider a sieving process. Unfortunately, there is a strong constraint. For instance, and this is a basic difference with the real case, there exists a unique SMOP $\left(\Psi_n\right)$ such that $\Psi_{2n+1}(z) = z\Phi_n(z^2)$. Furthermore, $\Psi_{2n}(z) = \Phi_n(z^2)$ (see [9,11]). In terms of the reflection parameters, this can be seen as a linear transformation T in the space of sequences of reflection parameters given by

$$\begin{aligned} T(a_{2n}) &= a_n, \qquad n = 0, 1, 2, \cdots \\ T(a_{2n+1}) &= 0, \qquad n = 0, 1, 2, \cdots \end{aligned}$$

with $\Phi_n(0) = a_n$.

A different way to obtain new SMOP's consists in considering modifications of the functional $\mathcal{L}$. There exists the posibility that some of these modifications do not preserve the Hermitian character of the functional and, so, we have right and left orthogonal polynomials (see [3]).

In this paper we will study modifications of an Hermitian linear functional $\mathcal{L}$ by means of a complex polynomial P of degree one, in the following way

$$\widetilde{L} := \big(P(z) + \overline{P}(z^{-1})\big)L. \tag{7}$$

These modifications preserve the Hermitian character of the moment functionals (see [1]), so, if $\widetilde{L}$ is a quasi-definite one, we can assure the existence of a unique SMOP $\big(\Psi_n\big)$ with respect to $\widetilde{L}$.

The aim of our contribution is to characterize the quasi-definiteness of $\widetilde{L}$ under the assumption that L is quasi-definite. We also deduce the expression of the SMOP $\big(\Psi_n\big)$ related to the functional $\widetilde{\mathcal{L}}$ in terms of the SMOP $\big(\Phi_n\big)$ corresponding to the functional $\mathcal{L}$, as well as relations between the sequences of Schur parameters. The particular situation of positive definite functionals will be considered. Finally we illustrate the preceding results with some examples.

§2 - Quasi-definite Hermitian modified functionals.

The modifications of a linear functional we are studying are related to the so-called self-reciprocal polynomials of degree two. Recall that a polynomial A is said to be a self-reciprocal polynomial if $A = A^*$. It is easy to prove the following:

2.1 Lemma

A is a self-reciprocal polynomial of degree 2 if and only if it has the form

$$A(z) = z\big(P(z) + \overline{P}(z^{-1})\big), \tag{8}$$

with P a polynomial of degree 1.

Remark that if ζ is a root of a self-reciprocal polynomial A, then $\zeta \neq 0$ and $\overline{\zeta}^{-1}$ is also a root of A. Hence, for the roots ζ_1, ζ_2 of a self-reciprocal polynomial of degree two there are two different possibilities: $\zeta_2 = \overline{\zeta_1}^{-1}$ or $\zeta_1 \neq \zeta_2$, $|\zeta_1| = |\zeta_2| = 1$. Without loss of generality we can write $P(z) = \alpha z + \beta$, $\alpha \neq 0$, $\beta \in \mathbb{R}$. Then, $A(z) = \alpha z^2 + 2\beta z + \overline{\alpha}$ and $\zeta_{1,2} = \alpha^{-1}\big(-\beta \pm \sqrt{\beta^2 - |\alpha|^2}\big)$, which gives

$$\zeta_2 = \overline{\zeta_1}^{-1} \iff |\beta| \geq |\alpha|,$$
$$\zeta_1 \neq \zeta_2, \;\; |\zeta_1| = |\zeta_2| = 1 \iff |\beta| < |\alpha|.$$

As a consequence, we get the following inmediate result.

2.2 Corollary

Let L *be a positive definite linear functional and* $P(z) = \alpha z + \beta$, $\alpha \neq 0$, $\beta \in \mathbb{R}$. *Then, the Hermitian functional* $\widetilde{\mathcal{L}} = (P(z) + \overline{P}(z^{-1}))\mathcal{L}$ *is positive definite (up to a factor) if and only if* $|\beta| \geq |\alpha|$.

2.3 Definition ([1])

Let L be an Hermitian linear functional and $\big(\Psi_n\big)$ a sequence of polynomials with $\deg \Psi_n = n$. We will say that $\big(\Psi_n\big)$ is quasi-orthogonal of order s, $s \in \mathbb{N}$, with respect to L if:

(i) $L[\Psi_n(z) z^{-k}] = 0$ for all k with $s \leq k \leq n - s - 1$ and for all $n \geq 2s + 1$.

(ii) There exists $n_0 \geq 2s$ such that $L[\Psi_{n_0}(z) z^{-n_0+s}] \neq 0$.

Given a quasi-definite linear functional L, its corresponding SMOP satisfies orthogonality properties with respect to the modified functional $\widetilde{L} = \big(P(z) + \overline{P}(z^{-1})\big)L$.

2.4 Proposition

Let L *be quasi-definite and* $\big(\Phi_n\big)$ *the corresponding SMOP. Then,* $\big(\Phi_n\big)$ *is a quasi-orthogonal sequence of order* 1 *with respect to the linear functional* $\widetilde{L}$.

Proof

We suppose $P(z) = \alpha z + \beta$. So,

$$\widetilde{L}\big[\Phi_n(z) z^{-k}\big] = L\big[(P(z) + \overline{P}(z^{-1}))\Phi_n(z) z^{-k}\big] = 0, \quad 1 \leq k \leq n-2, \quad n \geq 3,$$

and

$$\widetilde{L}\big[\Phi_n(z) z^{-n+1}\big] = L\big[\big(P(z) + \overline{P}(z^{-1})\big)\Phi_n(z) z^{-n+1}\big] = \\ = \overline{\alpha} L\big[\Phi_n(z) z^{-n}\big] = \overline{\alpha} e_n \neq 0, \quad n \geq 2.$$

◇◇

2.5 Theorem

Given a quasi-definite linear functional L *and a polynomial* P *of degree one, the modified fuctional* $\widetilde{L} = \big(P(z) + \overline{P}(z^{-1})\big)L$ *is quasi-definite if and only if, for each* $n = 0, 1, 2 \ldots$, *there exist unique polynomials* Ψ_n *and* B_n, *with* Ψ_n *monic,* $\deg B_n = 1$, *and* $B_n(0) \neq 0$, *and a unique constant* c_n, *such that*

$$A(z)\Psi_n(z) = B_n(z)\Phi_{n+1}(z) + c_n \Phi^*_{n+1}(z), \tag{9}$$

where $A(z) = z\big(P(z) + \overline{P}(z^{-1})\big)$ *and* $\big(\Phi_n\big)$ *denotes the SMOP related to the functional* $\boldsymbol{L}$*. Moreover,* $\big(\Psi_n\big)$ *is the SMOP with respect to* $\widetilde{\boldsymbol{L}}$*.*

Proof

Let $\widetilde{\boldsymbol{L}}$ be quasi-definite and let $\big(\Psi_n\big)$ be the SMOP with respect to $\widetilde{\boldsymbol{L}}$. Then,

$$\widetilde{\boldsymbol{L}}\big[\Psi_n(z)z^{-k}\big] = \boldsymbol{L}\big[A(z)\Psi_n(z)z^{-k-1}\big] = 0, \quad 0 \le k \le n-1, \quad n \ge 1.$$

Thus, $A\Psi_n$ belongs to the subspace of $\mathbb{P}_{n+2}$ which is orthogonal to $\big\{z, z^2, \dots, z^{n-1}\big\}$. A basis of this orthogonal complement is given by (see [1])

$$\big\{\Phi_{n+1}(z), z\Phi_{n+1}(z), \Phi^*_{n+1}(z)\big\},$$

so that, $A\Psi_n$ can be expressed in a unique way as

$$A(z)\Psi_n(z) = b_{n0}\Phi_{n+1}(z) + b_{n1}z\Phi_{n+1}(z) + c_n\Phi^*_{n+1}(z), \quad n \ge 1.$$

This proves equality (9) for $n \ge 1$, being $B_n(z) = b_{n0} + b_{n1}z$.

For $n = 0$, it is sufficient to identify coefficients in

$$A(z) = B_0(z)\Phi_1(z) + c_0\Phi^*_1(z), \qquad (\deg B_0 = 1).$$

Conversely, let us suppose that Ψ_n satisfies (9). Then, deg $\Psi_n = n$ and the orthogonality conditions for (Φ_n) give

$$\widetilde{\boldsymbol{L}}\big[\Psi_n(z)z^{-k}\big] = \boldsymbol{L}\big[(B_n(z)\Phi_{n+1}(z) + c_n\Phi^*_{n+1}(z))z^{-k-1}\big] = 0, \quad 0 \le k \le n-1, \quad n \ge 1,$$

and

$$\widetilde{\boldsymbol{L}}\big[\Psi_n(z)z^{-n}\big] = \boldsymbol{L}\big[(B_n(z)\Phi_{n+1}(z) + c_n\Phi^*_{n+1}(z))z^{-n-1}\big] = B_n(0)e_{n+1} \ne 0, \quad n \ge 1,$$

and, so, (Ψ_n) is a SMOP with respect to the functional $\widetilde{\boldsymbol{L}}$ which, therefore, is quasi-definite.

The uniqueness for the solution of (9) is just a consequence of the uniqueness for the SMOP related to a quasi-definite linear functional. ◇◇

Remarks

1.- Notice that $\widetilde{e}_n := \widetilde{\boldsymbol{L}}\big[\Psi_n(z)z^{-n}\big] = B_n(0)e_{n+1}$, and then, $B_n(0) \in \mathbb{R}$, $\forall n \ge 0$.

2.- Using the $*_{n+2}$ operator in (9) we get the reverse relation

$$A(z)\Psi_n^*(z) = \overline{c}_n z\Phi_{n+1}(z) + B_n^*(z)\Phi_{n+1}^*(z), \quad n \geq 0. \tag{10}$$

2.6 Theorem

Let L be a quasi-definite linear functional and (Φ_n) the corresponding SMOP. If P is a polynomial of degree one, the functional $\widetilde{L} = (P(z) + \overline{P}(z^{-1}))L$ is quasi-definite if and only if the roots ζ_1, ζ_2 of the polynomial $A(z) = z(P(z) + \overline{P}(z^{-1}))$ satisfy any of the following conditions:

(a) $\zeta_1 = \overline{\zeta}_2^{-1}$.

(b) $\zeta_1 \neq \zeta_2, \quad |\zeta_1| = |\zeta_2| = 1, \qquad \dfrac{\Phi_n(\zeta_1)}{\Phi_n^*(\zeta_1)} \neq \dfrac{\Phi_n(\zeta_2)}{\Phi_n^*(\zeta_2)}, \quad \forall n \in \mathbb{N}.$

Proof

Theorem 2.5 implies that, to decide the quasi-definiteness of $\widetilde{L}$, we just have to study the existence and uniqueness of the solutions for the equations

$$A(z)\Psi_n(z) = B_n(z)\Phi_{n+1}(z) + c_n\Phi_{n+1}^*(z), \quad n \geq 0,$$

with Ψ_n monic and $\deg B_n = 1$, $B_n(0) \neq 0$.

Let us suppose $P(z) = \alpha z + \beta$, $\alpha \neq 0$, $\beta \in \mathbb{R}$. The existence of the solutions for the above equations is equivalent to state that $A(z) = \alpha z^2 + 2\beta z + \overline{\alpha}$ divides $B_n(z)\Phi_{n+1}(z) + c_n\Phi_{n+1}^*(z)$ for some $B_n(z) = \alpha z + b_n$ and c_n. The uniqueness of the solutions means the uniqueness of the parameters b_n, c_n.

If A has two different roots ζ_1, ζ_2 this is equivalent to say that the sistem

$$\begin{cases} (\alpha\zeta_1 + b_n)\,\Phi_{n+1}(\zeta_1) + c_n\Phi_n^*(\zeta_1) = 0 \\ (\alpha\zeta_2 + b_n)\,\Phi_{n+1}(\zeta_2) + c_n\Phi_n^*(\zeta_2) = 0 \end{cases}$$

has a unique solution in the parameters b_n, c_n for each $n \geq 0$. This happens if and only if, for $n \geq 0$,

$$\begin{vmatrix} \Phi_{n+1}(\zeta_1) & \Phi_{n+1}^*(\zeta_1) \\ \Phi_{n+1}(\zeta_2) & \Phi_{n+1}^*(\zeta_2) \end{vmatrix} \neq 0.$$

Using the Christoffel-Darboux formula (5) we see that the above conditions are equivalent to $K_n(\zeta_1, \overline{\zeta}_2^{-1}) \neq 0$ for $n \geq 0$.

Now, we have the following possibilities:

If $\zeta_1 \neq \zeta_2$ and $\zeta_1 = \overline{\zeta_2}^{-1}$, then $K_n(\zeta_1, \overline{\zeta_2}^{-1}) = K_n(\zeta_1, \zeta_1) > 0$ and the functional $\widetilde{L}$ is quasi-definite.

If $\zeta_1 \neq \zeta_2$ and $|\zeta_1| = |\zeta_2| = 1$, we can assure nothing about $K_n(\zeta_1, \overline{\zeta_2}^{-1}) = K_n(\zeta_1, \zeta_2)$. Then, $\widetilde{L}$ is quasi-definite if and only if $\Phi_n(\zeta_1)\Phi_n^*(\zeta_2) \neq \Phi_n(\zeta_2)\Phi_n^*(\zeta_1)$ for $n \geq 1$.

Finally, when $\zeta_1 = \zeta_2 = \zeta$, $|\zeta| = 1$, the parameters b_n, c_n are determined by the system

$$\begin{cases} (\alpha\zeta + b_n)\Phi_{n+1}(\zeta) + c_n\Phi_{n+1}^*(\zeta) = 0 \\ \alpha\Phi_{n+1}(\zeta) + (\alpha\zeta + b_n)\Phi_{n+1}'(\zeta) + c_n\Phi_{n+1}^{*'}(\zeta) = 0, \end{cases}$$

which has a unique solution if and only if

$$\begin{vmatrix} \Phi_{n+1}(\zeta) & \Phi_{n+1}^*(\zeta) \\ \Phi_{n+1}'(\zeta) & \Phi_{n+1}^{*'}(\zeta) \end{vmatrix} \neq 0. \tag{11}$$

Using again the Christoffel-Darboux formula (5) for the kernel polynomials, we obtain that this condition is equivalent to $K_n(\zeta, \zeta) \neq 0$, which is always true. Therefore $\widetilde{L}$ is quasi-definite in this case. $\diamond\diamond$

Remark.

It is known that the polynomials $q_n = \Phi_n + u\Phi_n^*$, $n \in \mathbb{C}$, called para-orthogonal polynomials associated to L, have simple roots lying on the unit circle when $|u| = 1$. The quasi-definiteness of $\widetilde{L}$ fails exactly when ζ_1, ζ_2 are different common roots of the same para-orthogonal polynomial (see [9]) $q_n = \Phi_n + u\Phi_n^*$, $|u| = 1$, for some n. It is clear that this can happen only for $n \geq 2$.

The preceding results give us the possiblity to obtain the modified polynomials Ψ_n in terms of the polynomials Φ_{n+1}. If $P(z) = \alpha z + \beta$, $\beta \in \mathbb{R}$, then $\Psi_n(z) = B_n(z)\Phi_{n+1}(z) + c_n\Phi_{n+1}^*(z)$, with $B(z) = \alpha z + b_n$, and we only need to calculate b_n, c_n.

When the roots ζ_1, ζ_2 of $A(z) = \alpha z^2 + 2\beta z + \overline{\alpha}$ are different, the system that gives b_n, c_n is

$$\begin{cases} b_n\Phi_{n+1}(\zeta_1) + c_n\Phi_{n+1}^*(\zeta_1) = -\alpha\zeta_1\Phi_{n+1}(\zeta_1) \\ b_n\Phi_{n+1}(\zeta_2) + c_n\Phi_{n+1}^*(\zeta_2) = -\alpha\zeta_2\Phi_{n+1}(\zeta_2), \end{cases}$$

with solutions

$$b_n = -\alpha\zeta_2 \frac{K_{n+1}(\zeta_1, \overline{\zeta_2}^{-1})}{K_n(\zeta_1, \overline{\zeta_2}^{-1})}, \qquad c_n = \frac{\alpha\zeta_2^{-n}}{e_{n+1}} \frac{\Phi_{n+1}(\zeta_1)\Phi_{n+1}(\zeta_2)}{K_n(\zeta_1, \overline{\zeta_2}^{-1})}. \tag{12}$$

If $\zeta_1 = \zeta_2 = \zeta$, the parameters b_n, c_n are determined by

$$\begin{cases} b_n\Phi_{n+1}(\zeta) + c_n\Phi_{n+1}^*(\zeta) = -\alpha\zeta\Phi_{n+1}(\zeta) \\ b_n\Phi_{n+1}'(\zeta) + c_n\Phi_{n+1}^{*'}(\zeta) = -\alpha\Phi_{n+1}(\zeta) - \alpha\zeta\Phi_{n+1}'(\zeta) \end{cases}$$

and then, we have the solutions

$$b_n = -\alpha\zeta \frac{K_{n+1}(\zeta,\zeta)}{K_n(\zeta,\zeta)}, \qquad c_n = \frac{\alpha\overline{\zeta}^n}{e_{n+1}} \frac{\Phi^2_{n+1}(\zeta)}{K_n(\zeta,\zeta)}. \tag{13}$$

Notice that formulas (12) can be obtained from formulas (13), taking limits $\zeta_1 \to \zeta$ and $\zeta_2 \to \zeta$. Hence, the expressions for b_n, c_n in terms of the kernel polynomials given by (12), can be considered as general expressions valid in any case.

The expressions obtained, not only provide the orthogonal polynomials $\big(\Psi_n\big)$ in terms of $\big(\Phi_n\big)$ but also give relations between the reflection coefficients. Now, we will search for new relations between the Schur parameters.

2.7 Lemma

Let $Q \in \mathbb{P}$, with $degQ > m \geq 0$ and such that $g.c.d.(Q,Q^*) = 1$. Let M, N be the square polynomic matrices of order two

$$M(z) = \begin{pmatrix} M_1(z) & M_2(z) \\ M_2^{*m}(z) & M_1^{*m}(z) \end{pmatrix}, \qquad N(z) = \begin{pmatrix} N_1(z) & N_2(z) \\ N_2^{*m}(z) & N_1^{*m}(z) \end{pmatrix},$$

where $\max\{degM_1, degM_2, degN_1, degN_2\} \leq m$. Then, we have that

$$M(z)\begin{pmatrix} Q(z) \\ Q^*(z) \end{pmatrix} = N(z)\begin{pmatrix} Q(z) \\ Q^*(z) \end{pmatrix} \iff M(z) = N(z).$$

Proof

If

$$M_1(z)Q(z) + M_2(z)Q^*(z) = N_1(z)Q(z) + N_2(z)Q^*(z),$$

then

$$\big(M_1(z) - N_1(z)\big)Q(z) = \big(M_2(z) - N_2(z)\big)Q^*(z),$$

with $\deg\big(M_1 - N_1\big) \leq m$, $\deg\big(M_2 - N_2\big) \leq m$. Since g.c.d. $\big(Q,Q^*\big) = 1$, it follows that

$$M_1 - N_1 = M_2 - N_2 = 0.$$

The converse is obvious. ◇◇

2.8 Theorem

The polynomials B_n and the constants c_n in (9) satisfy

$$\begin{pmatrix} B_{n+1}(z) & c_{n+1} \\ \overline{c}_{n+1} z & B^*_{n+1}(z) \end{pmatrix} \begin{pmatrix} \frac{1}{\Phi_{n+2}(0)} & \Phi_{n+2}(0) \\ & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\Psi_{n+1}(0)} & \Psi_{n+1}(0) \\ & 1 \end{pmatrix} \begin{pmatrix} B_n(z) & c_n z \\ \overline{c}_n & B^*_n(z) \end{pmatrix}, \qquad n \geq 0. \tag{14}$$

Proof

Using the recurrence relations (2), (3) and taking into account the expresions (9), (10) we get

$$A(z) \begin{pmatrix} \Psi_{n+1}(z) \\ \Psi^*_{n+1}(z) \end{pmatrix} = \begin{pmatrix} B_{n+1}(z) & c_{n+1} \\ \overline{c}_{n+1} z & B^*_{n+1}(z) \end{pmatrix} \begin{pmatrix} \Phi_{n+2}(z) \\ \Phi^*_{n+2}(z) \end{pmatrix} =$$

$$= \begin{pmatrix} B_{n+1}(z) & c_{n+1} \\ \overline{c}_{n+1} z & B^*_{n+1}(z) \end{pmatrix} \begin{pmatrix} \frac{z}{\Phi_{n+2}(0)\, z} & \Phi_{n+2}(0) \\ & 1 \end{pmatrix} \begin{pmatrix} \Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{pmatrix}.$$

Now, since $\left(\Psi_n\right)$ is also a SMOP, again by (2), (3) and (9), (10) we can write

$$A(z) \begin{pmatrix} \Psi_{n+1}(z) \\ \Psi^*_{n+1}(z) \end{pmatrix} = A(z) \begin{pmatrix} \frac{z}{\Psi_{n+1}(0)\, z} & \Psi_{n+1}(0) \\ & 1 \end{pmatrix} \begin{pmatrix} \Psi_n(z) \\ \Psi^*_n(z) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{z}{\Psi_{n+1}(0)\, z} & \Psi_{n+1}(0) \\ & 1 \end{pmatrix} \begin{pmatrix} B_n(z) & c_n \\ \overline{c}_n z & B^*_n(z) \end{pmatrix} \begin{pmatrix} \Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} B_{n+1}(z) & c_{n+1} \\ \overline{c}_{n+1} z & B^*_{n+1}(z) \end{pmatrix} \begin{pmatrix} \frac{1}{\Phi_{n+2}(0)} & \Phi_{n+2}(0) \\ & 1 \end{pmatrix} \begin{pmatrix} z\Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{\Psi_{n+1}(0)} & \Psi_{n+1}(0) \\ & 1 \end{pmatrix} \begin{pmatrix} B_n(z) & c_n z \\ \overline{c}_n & B^*_n(z) \end{pmatrix} \begin{pmatrix} z\Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{pmatrix},$$

and, applying Lemma 2.7, the result follows. ⋄⋄

2.9 Corollary

If the functional $\widetilde{L}$ *is quasi-definite, its Schur parameters can be obtained by means of any of the relations*

$$\Psi_{n+1}(0) = \frac{\alpha\Phi_{n+2}(0) - c_n}{b_n} = \frac{b_{n+1}\Phi_{n+2}(0) + c_{n+1}}{\overline{\alpha}} = \frac{b_{n+1} - b_n + c_{n+1}\Phi_{n+2}(0)}{\overline{c}_n}, \qquad n \geq 0$$

Proof

It is suffices to take $z = 0$ in (14). $\diamond\diamond$

§**3 - Examples.**

1.- In the first example we consider the functional L related to the Lebesgue normalized measure $d\mu(\theta) = \dfrac{d\theta}{2\pi}$. The corresponding moments are $c_0 = 1$ and $c_n = 0$, $n \neq 0$, while the related SMOP is $\Phi_n(z) = z^n$.

If $P(z) = \alpha z + \beta$, $\alpha \neq 0$, $\beta \in \mathbb{R}$, then $\widetilde{L} := \left(\alpha z + 2\beta + \overline{\alpha} z^{-1}\right) L$. We will study the quasi-definiteness of $\widetilde{L}$, both, directly and using our approach, to show the simplicity of the last one.

The moments $\widetilde{c}_n$ of $\widetilde{L}$ are $\widetilde{c}_n = \alpha c_{n+1} + 2\beta c_n + \overline{\alpha} c_{n-1}$, $\quad n \in \mathbb{Z}$, that is, $\widetilde{c}_0 = 2\beta$, $\quad \widetilde{c}_1 = \overline{\alpha}$, $\quad \widetilde{c}_n = 0$ for $n \geq 2$. Notice that, if $\widetilde{L}$ is quasi-definite, then $\beta \neq 0$. We normalize $\widetilde{L}$ by dividing by 2β. Then,

$$\widetilde{c}_0 = 1, \quad \widetilde{c}_1 = \frac{\overline{\alpha}}{2\beta} =: a.$$

The measure related to $\widetilde{L}$ (in general, a signed measure), is given by

$$\begin{aligned} d\widetilde{\mu}(\theta) &= \left(\overline{a} e^{i\theta} + 1 + a e^{-i\theta}\right) d\theta/2\pi \\ &= (1 + 2|a| \cos(\theta - \theta_0))\, d\theta/2\pi, \end{aligned}$$

where θ_0 =arg(a). If $|a| \leq \frac{1}{2}$, $d\widetilde{\mu}$ is a positive measure (see [7]) and, hence, $\widetilde{L}$ is positive definite. Thus, we just have to study the case $|a| > \frac{1}{2}$ (that is $|\frac{\beta}{\alpha}| < 1$) in order to give necessary and sufficient conditions for the linear functional $\widetilde{L}$ be quasi-definite.

By direct calculations, the moment matrix $\widetilde{M} = (\widetilde{c}_{i-j})$ is

$$\widetilde{M} = \begin{pmatrix} 1 & \overline{a} & 0 & 0 & \dots & 0 \\ a & 1 & \overline{a} & 0 & \dots & 0 \\ 0 & a & 1 & \overline{a} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

If D_n is the principal minor of $\widetilde{M}$ of order $n+1$, we have the following recurrence relation

$$D_{n+2} = D_{n+1} - |a|^2 D_n, \quad n \geq 0, \tag{15}$$

with the initial conditions

$$D_0 = 1, \quad D_1 = 1 - |a|^2. \tag{16}$$

Notice that $D_n \in \mathbb{R}$.

The general solution of (15) is given by

$$D_n = A_1 r_1^n + A_2 r_2^n,$$

where r_1, r_2 are the roots of the corresponding characteristic equation,

$$r^2 - r + |a|^2 = 0,$$

which, under the hypothesis $|a| > \frac{1}{2}$, are the conjugated complex numbers

$$r_1 = |a| e^{i\omega}, \qquad r_2 = |a| e^{-i\omega}, \qquad \omega = \arccos\left(\frac{1}{2|a|}\right) \in (0, \frac{\pi}{2}).$$

The conditions (16) determine A_1, A_2 as the conjugated complex coefficients $A_1 = \rho e^{i\lambda}$, $A_2 = \rho e^{-i\lambda}$ with

$$\lambda = 2\omega \pm \frac{\pi}{2}, \qquad \rho = \frac{1}{2\cos\lambda},$$

where the sign $+$ or $-$ is determined by the condition $\rho > 0$.

So,

$$D_n = |a|^n \rho \left(e^{i(n\omega+\lambda)} + e^{-i(n\omega+\lambda)}\right) = 2|a|^n \rho \cos(n\omega + \lambda) =$$

$$= \frac{\operatorname{sen}(n+2)\omega}{2^{n+1}\cos^{n+1}\omega \operatorname{sen}\omega} = \frac{U_{n+1}(\cos\omega)}{2^{n+1}\cos^{n+1}\omega},$$

where $U_{n+1}(x)$ are the Tchebyshev polynomials of second kind.

As a consequence, the functional $\widetilde{L}$ is not quasi-definite if and only if $n\omega + \lambda = \frac{\pi}{2} + k\pi$, for some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, that is,

$$(n+2)\omega = m\pi, \qquad n \in \mathbb{N}, \qquad m \in \mathbb{Z}.$$

So, $\widetilde{L}$ is quasi-definite if and only if $|\frac{\beta}{\alpha}| < 1$ and $\omega = \arccos(|\frac{\beta}{\alpha}|) \notin \mathbb{Q}\pi$, or $|\frac{\beta}{\alpha}| \geq 1$.

If we use the results developped in the paper, the same conclusion is achieved, but in a much simpler way. Theorem 2.6 implies that $\widetilde{L}$ is not quasi-definite if and only if the roots ζ_1, ζ_2 of $A(z) = \alpha\zeta^2 + 2\beta z + \overline{\alpha}$ are different points of the unit circle satisfying $\dfrac{\Phi_n(\zeta_1)}{\Phi_n^*(\zeta_1)} = \dfrac{\Phi_n(\zeta_2)}{\Phi_n^*(\zeta_2)}$ for some n. Since $\Phi_n(z) = z^n$ we find that the conditions in order to the linear functional $\widetilde{L}$ be non quasi-definite are $|\beta| < |\alpha|$ and $\dfrac{1}{2}\arg(\dfrac{\zeta_1}{\zeta_2}) = \arccos(\dfrac{\beta}{|\alpha|}) \notin \mathbb{Q}\pi$, which is equivalent to the result above found.

Moreover, when $\widetilde{L}$ is quasi-definite, we can also determine the related SMOP (Ψ_n) by using the expressions (12) and (13). From the Christoffel-Darboux formula (5) we get

$$K_n(\zeta_1, \overline{\zeta_2}^{-1}) = \zeta_2^{-n}\frac{\zeta_1^{n+1} - \zeta_2^{n+1}}{\zeta_1 - \zeta_2} = (-1)^n \frac{\overline{\alpha}^n}{|\alpha|^n}\zeta_2^{-n} U_n\left(\frac{\beta}{|\alpha|}\right),$$

where we have used that $\zeta_{1,2} = -\dfrac{\overline{\alpha}}{|\alpha|}e^{\pm i\vartheta}$, $\vartheta = \arccos(\dfrac{\beta}{|\alpha|})$.

If $\zeta_1 \neq \zeta_2$,

$$b_n = -\alpha\frac{\zeta_1^{n+2} - \zeta_2^{n+2}}{\zeta_1^{n+1} - \zeta_2^{n+1}} = |\alpha|\frac{U_{n+1}(\frac{\beta}{|\alpha|})}{U_n(\frac{\beta}{|\alpha|})},$$

$$c_n = \alpha\zeta_1^{n+1}\zeta_2^{n+1}\frac{\zeta_1 - \zeta_2}{\zeta_1^{n+1} - \zeta_2^{n+1}} = (-1)^n\frac{\overline{\alpha}^{n+1}}{|\alpha|^n}\frac{1}{U_n(\frac{\beta}{|\alpha|})}.$$

On the other hand, if $\zeta_1 = \zeta_2 = \zeta$, then $\zeta = -\dfrac{\beta}{\alpha}$ and

$$b_n = \beta\frac{n+2}{n+1}, \qquad c_n = \frac{(-1)^n}{(n+1)}\frac{\beta^{n+2}}{\alpha^{n+1}}.$$

From this we obtain the modified SMOP using (9). Also, from Corollary 2.9, for $n \geq 1$, we have

$$\Psi_n(0) = \frac{b_n\Phi_{n+1}(0) + c_n}{\overline{\alpha}} = \frac{c_n}{\overline{\alpha}}.$$

2.- In the second example we consider the sequence of monic orthogonal polynomials (Φ_n) with sequence of Schur parameters $\Phi_0(0) = 1, \Phi_1(0) = a$, $|a| \neq 1$, and $\Phi_n(0) = 0$, $\forall n \geq 2$. This SMOP (Φ_n) is a Bernstein-Szegő type sequence, given by

$$\Phi_n(z) = z^{n-1}(z + a), \qquad \forall n \geq 1.$$

The linear functional $\widetilde{\boldsymbol{L}} = (\alpha z + 2\beta + \overline{\alpha}\zeta^{-1})\boldsymbol{L}$ only is non quasi-definite when $|\beta| < |\alpha|$ and $\dfrac{\zeta_1^n(\zeta_1 + a)}{1 + \overline{a}\zeta_1} = \dfrac{\zeta_2^n(\zeta_2 + a)}{1 + \overline{a}\zeta_2}$ for some $n \geq 0$, where $\zeta_{1,2} = -\dfrac{\overline{\alpha}}{|\alpha|}e^{\pm i\vartheta}$, $\vartheta = \arccos(\frac{\beta}{|\alpha|})$ are the roots of $A(z) = \alpha z^2 + 2\beta z + \overline{\alpha}$. Therefore, $\widetilde{\boldsymbol{L}}$ is quasi-definite if and only if either $|\beta| \geq \alpha$, or $|\beta| < |\alpha|$ and

$$\arg(\zeta_1 + a) - \arg(\zeta_2 + a) + (n-1)\vartheta \neq k\pi, \qquad \forall n \in \mathbb{N}, \qquad \forall k \in \mathbb{Z}.$$

In this situation, using again (12) and (13) and the fact that

$$K_n(\zeta_1, \overline{\zeta_2}^{-1}) = \frac{\zeta_2^{-n}}{1-|a|^2} \frac{\zeta_1^n(\zeta_1 + a)(1 + \overline{a}\zeta_2) - \zeta_2^n(\zeta_2 + a)(1 + \overline{a}\zeta_1)}{\zeta_1 - \zeta_2} =$$

$$= \frac{(-1)^n}{1-|a|^2} \frac{\overline{\alpha}^n}{|\alpha|^n} \zeta_2^{-n} \left[U_n\left(\frac{\beta}{|\alpha|}\right) - 2\mathrm{Re}\left(\frac{\alpha a}{|\alpha|}\right) U_{n-1}\left(\frac{\beta}{|\alpha|}\right) + |a|^2 U_{n-2}\left(\frac{\beta}{|\alpha|}\right)\right],$$

we obtain

$$b_n = |\alpha| \frac{U_{n+1}(\frac{\beta}{|\alpha|}) - 2\mathrm{Re}(\frac{\alpha a}{|\alpha|})U_n(\frac{\beta}{|\alpha|}) + |a|^2 U_{n-1}(\frac{\beta}{|\alpha|})}{U_n(\frac{\beta}{|\alpha|}) - 2\mathrm{Re}(\frac{\alpha a}{|\alpha|})U_{n-1}(\frac{\beta}{|\alpha|}) + |a|^2 U_{n-2}(\frac{\beta}{|\alpha|})},$$

$$c_n = (-1)^n \frac{\overline{\alpha}^n}{|\alpha|^n} \frac{A(-a)}{U_n(\frac{\beta}{|\alpha|}) - 2\mathrm{Re}(\frac{\alpha a}{|\alpha|})U_{n-1}(\frac{\beta}{|\alpha|}) + |a|^2 U_{n-2}(\frac{\beta}{|\alpha|})}.$$

In particular, if $\zeta_1 = \zeta_2 = \zeta$, we have the expresions

$$b_n = \beta \frac{(n+2) - 2\mathrm{Re}(\frac{\alpha a}{\beta})(n+1) + |a|^2 n}{(n+1) - 2Re(\frac{\alpha a}{\beta})n + |a|^2(n-1)},$$

$$c_n = (-1)^n \frac{\beta^n}{\alpha^n} \frac{A(-a)}{(n+1) - 2\mathrm{Re}(\frac{\alpha a}{\beta})n + |a|^2(n-1)}.$$

From here it is possible using (9), to obtain the modified polynomials (Ψ_n).

Also, from Corollary 2.9, we get $\Psi_n(0) = \dfrac{c_n}{\overline{\alpha}}$ for $n \geq 1$.

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