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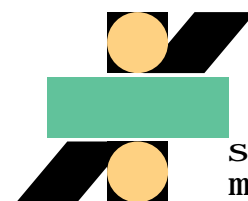
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Direct problem

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On rational transformations of linear functionals: Direct problem

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Abstract

Let u be a quasi-definite linear functional. We find necessary and sufficient conditions in order to the linear functional v satisfying $(x-\tilde{a})u = \lambda(x-a)v$ be a quasi-definite one. Also we analyze some linear relations linking the polynomials orthogonal with respect to u and v .

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1 Introduction

Let u be a linear functional in the linear space \mathbb{P} of polynomials with complex coefficients and denote by $\{u_n\}_{n \geq 0}$ the sequence of the moments associated with u , $u_n = \langle u, x^n \rangle$, $n \geq 0$, where $\langle \cdot, \cdot \rangle$ means the duality bracket.

The linear functional u is said to be quasi-definite if the Hankel matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ is quasi-definite, i.e. the principal submatrices $H_n = (u_{i+j})_{i,j=0}^n$, $n \in \mathbb{N} \cup \{0\}$, are non singular.

The linear functional δ_a given by $\langle \delta_a, P \rangle = P(a)$, for every $P \in \mathbb{P}$, is not a quasi-definite linear functional since $\text{rank } H_n = 1$ for every $n \geq 0$. This linear functional is said to be either the Dirac linear functional or the Dirac mass at the point a .

To the linear functional u we can associate a formal power series $S_u(z) = \sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}}$ which is related with the z -transform of the sequence $\{u_n\}$ of moments of u . S_u is said to be the Stieltjes function of u . For the Dirac linear functional $u = \delta_a$ given as above, we have $S_u(z) = \frac{1}{z-a}$ in a neighbourhood of infinite.

Assuming u quasi-definite, there exists a sequence of monic polynomials $\{P_n\}_{n \geq 0}$ such that (see [2])

- (i) $\deg P_n = n$, $n \geq 0$,
- (ii) $\langle u, P_n P_m \rangle = k_n \delta_{n,m}$ with $k_n \neq 0$.

The sequence $\{P_n\}_{n \geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional u .

If $\{P_n\}_{n \geq 0}$ is an SMOP with respect to the quasi-definite linear functional u , then it is well known (see [2]) that it satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0, \quad (1.1)$$

with $\gamma_n \neq 0$ and $P_{-1}(x) = 0$, $P_0(x) = 1$.

Conversely, given a sequence of monic polynomials generated by a recurrence relation as above, there exists a unique quasi-definite linear functional u such that the family $\{P_n\}_{n \geq 0}$ is the corresponding SMOP. Such a result is known as the Favard theorem (see [2]).

For an SMOP $\{P_n\}_{n \geq 0}$ relative to u , let $\{P_n^{(1)}\}_{n \geq 0}$ the sequence of monic polynomials such that

$$\begin{aligned} P_{n+1}^{(1)}(x) &= (x - \beta_{n+1})P_n^{(1)}(x) - \gamma_{n+1}P_{n-1}^{(1)}(x), \quad n \geq 0, \\ P_{-1}^{(1)}(x) &= 0, \quad P_0^{(1)}(x) = 1. \end{aligned}$$

According to the Favard theorem there exists a quasi-definite linear functional $u^{(1)}$ such that $\{P_n^{(1)}\}_{n \geq 0}$ is the corresponding SMOP. The family $\{P_n^{(1)}\}_{n \geq 0}$ is said to be the sequence of polynomials of first kind associated with the linear functional u .

Another representation of $\{P_n^{(1)}\}_{n \geq 0}$ is given by

$$P_n^{(1)}(y) = \frac{1}{u_0} \langle u, \frac{P_{n+1}(y) - P_{n+1}(x)}{y - x} \rangle,$$

$n \geq 0$ (see [2], Chapter 3).

Notice that $\frac{P_n^{(1)}(z)}{P_{n+1}(z)}$ is the $(n+1)$ -convergent of the continued fraction

$$\frac{1}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \ddots}}.$$

Thus

$$S_u(z) = \frac{u_0}{z - \beta_0 - \frac{\gamma_1}{z - \beta_1 - \ddots}} \quad (1.2)$$

from a formal point of view (see [2]).

For simplicity we will assume $u_0 = 1$.

Let $\{P_n(x, \alpha)\}_{n \geq 0}$ be the sequence of monic polynomials satisfying (1.1) with initial conditions $P_0(x, \alpha) = 1$, $P_1(x, \alpha) = P_1(x) - \alpha$. Taking into account the Favard theorem, there exists a quasi-definite linear functional u_α such that $\{P_n(x, \alpha)\}_{n \geq 0}$ is the corresponding SMOP. This sequence is said to be the co-recursive SMOP of parameter α associated with the linear functional u . It is known see ([2], [7]) that $P_n(x, \alpha) = P_n(x) - \alpha P_{n-1}^{(1)}(x)$.

From (1.2) we get

$$\begin{aligned} S_{u^{(1)}}(z) &= \frac{1}{\gamma_1} \left[z - \beta_0 - \frac{1}{S_u(z)} \right] \\ S_{u_\alpha}(z) &= \left[\frac{1}{S_u(z)} - \alpha \right]^{-1} = \frac{S_u(z)}{1 - \alpha S_u(z)}. \end{aligned}$$

These two bilinear rational transforms are related to self-similar reductions and spectral transformations in the theory of nonlinear integrable systems (see [12]).

For a linear functional u , a polynomial π , and a complex number a , let πu , $(x - a)^{-1}u$, and Du be the linear functionals defined on \mathbb{P} by

$$\begin{aligned} \langle \pi u, P \rangle &= \langle u, \pi P \rangle, \\ \langle (x - a)^{-1}u, P \rangle &= \langle u, \frac{P(x) - P(a)}{x - a} \rangle, \\ \langle Du, P \rangle &= -\langle u, P' \rangle, \end{aligned}$$

where $P \in \mathbb{P}$.

A Cauchy product of two linear functionals u, v can be defined as the linear functional uv such that $\langle uv, x^n \rangle = \sum_{h=0}^n u_h v_{n-h}$, $n \geq 0$. Obviously, $uv = vu$ and $\delta_0 u = u \delta_0 = u$. Since $u_0 = 1$, there exists a unique linear functional v such that $uv = vu = \delta_0$. This linear functional v is said to be the inverse linear functional of u and it will be denoted by u^{-1} . Notice that $(u^{-1})_0 = 1$ and $(u^{-1})_n = -\sum_{h=0}^{n-1} u_{n-h} (u^{-1})_h$, $n \geq 1$ (see [10]).

Since $z^2 S_{u^{-1}}(z) S_u(z) = 1$, we have $S_{u^{(1)}}(z) = \frac{1}{\gamma_1} [z - \beta_0 - z^2 S_{u^{-1}}(z)]$. Taking into account $(u^{-1})_0 = 1$ and $(u^{-1})_1 = -\beta_0$, we get $u^{(1)} = -\frac{1}{\gamma_1} x^2 u^{-1}$. Concerning the linear functional u_α , it is easy to check that $u_\alpha = (u^{-1} + \alpha \delta'_0)^{-1}$. This is an alternative proof of the result of [10] but notice that there the Stieltjes function has an opposite sign.

In the constructive theory of orthogonal polynomials the so-called direct problem is considered. A direct problem for linear functionals can be stated as follows: Given two linear functionals u, v such that $v = F(u)$, where F is a function defined in \mathbb{P}' , the dual space of \mathbb{P} , to find necessary and sufficient conditions in order to F preserves quasi-definiteness. As a subsequent question, to find the explicit relations between the corresponding SMOP $\{P_n\}$ and $\{Q_n\}$ associated with u and v respectively.

If u is a linear functional defined by a nonnegative measure μ on some interval I of the real line, with an infinite set of increasing points such that the moments exist, i. e. $\langle u, x^n \rangle = \int_I x^n d\mu < \infty$ then we can introduce the linear functional v such that

$$\langle v, x^n \rangle = \int_I x^n \frac{p(x)}{q(x)} d\mu \quad (1.3)$$

where p, q are two polynomials with pairwise distinct zeros that has constant sign on I . If we assume (1.3) is finite for every n , the generalized Christoffel theorem gives the SMOP with respect to v in terms of polynomials of the SMOP with respect to u (see [4], [11]). In terms of linear functionals, the above transform reads $qv = pu$. Notice that $pu = qv$ is a more general transform because of Dirac measures and derivatives of Dirac measures at the zeros of $q(x)$ can be considered for v in addition in such a general problem.

When $q(x) = 1$ and $p(x) = x - \tilde{a}$, the transform for linear functionals is said to be a Christoffel transform (see [12]). Using the Jacobi matrix J associated with the linear functional u , the shifted Darboux transform of J without free parameter yields the Jacobi matrix of v (see [6]).

It is known that v is quasi-definite if and only if $P_n(\tilde{a}) \neq 0, n \geq 1$, and

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) - \frac{P_{n+1}(\tilde{a})}{P_n(\tilde{a})} P_n(x)$$

as well as

$$\frac{Q_n(x)P_n(\tilde{a})}{\langle u, P_n^2 \rangle} = \sum_{k=0}^n \frac{P_k(x)P_k(\tilde{a})}{\langle u, P_k^2 \rangle}.$$

The polynomials $\{Q_n\}_{n \geq 0}$ are said to be the monic kernel polynomials of parameter \tilde{a} associated with the linear functional u (see [2]).

If $p(x) = 1$ and $q(x) = \lambda(x - a)$ then the transform is said to be the Geronimus transform of the linear functional u (see [12], [10]). The Jacobi matrix of v is the shifted Darboux transform with free parameter of the Jacobi matrix of u (see [6]).

Notice that in such a case, $v = \lambda^{-1}(x - a)^{-1}u + \delta_a$ is a quasi-definite linear functional if and only if $P_n(a, -\lambda^{-1}) \neq 0, n \geq 1$, and then

$$Q_n(x) = P_n(x) - \frac{P_n(a, -\lambda^{-1})}{P_{n-1}(a, -\lambda^{-1})} P_{n-1}(x)$$

(see [9]).

In our contribution, we analyze the direct problem stated as above for the case $p(x) = (x - \tilde{a})$ and $q(x) = \lambda(x - a)$. For $a \neq \tilde{a}$ this situation has not been studied in the literature as far as we know up to in the so-called positive definite case (see [4]).

In Section 2, given a quasi-definite linear functional u and complex numbers a, \tilde{a} , and λ with $a \neq \tilde{a}$ and $\lambda \neq 0$, we characterize the quasi-definiteness of the linear functional $v = \frac{1}{\lambda}(x - a)^{-1}(x - \tilde{a})u + (1 - \frac{1}{\lambda})\delta_a$. Instead of the analysis of the quasi-definiteness of the linear functional v in two steps (first, the rational perturbation and, second, the addition of the Dirac linear functional) we consider the whole transformation taking into account the first one cannot preserve the quasi-definiteness of the linear functional u . Indeed in [4] this constraint must be emphasized when polynomial perturbations are introduced. Further, we show that $(x - \tilde{a})Q_n$ is a linear combination of three consecutive polynomials of the SMOP $\{P_n\}_{n \geq 0}$.

Notice that the confluent case $a = \tilde{a}$ yields a perturbation of u via the addition of a Dirac mass at the point $x = a$. This corresponds to the Uvarov transform of the linear functional u (see [12]). The direct problem has been solved in [8]. We point out that the results for $a \neq \tilde{a}$ extend in a natural way those already known for $a = \tilde{a}$.

In Section 3, under the thesis of Section 2 we characterize when the relation between $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, obtained there, can be reduced to a relation $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$ for every $n \geq 1$, and $s_1 \neq t_1$. This last type of relation, as an inverse problem, has been analyzed in [1]. The motivation for such a kind of problems is reflected in [3] when an extension of the concept of coherent pairs of measures associated with Sobolev inner products is considered.

We also observe that there is an important difference for the cases $a = \tilde{a}$ and $a \neq \tilde{a}$. Namely, if $a = \tilde{a}$ then $s_n \neq t_n$ for every $n \geq 1$ while if $a \neq \tilde{a}$ both situations, i.e. either $s_n \neq t_n$ for every $n \geq 1$ or $s_n = t_n$ for some values of n , can appear as we show in some examples.

2 Direct problem

In this section, we study the direct problem for $v = \frac{1}{\lambda}(x - a)^{-1}(x - \tilde{a})u + (1 - \frac{1}{\lambda})\delta_a$ where u is a given quasi-definite linear functional, and $a, \tilde{a}, \lambda \in \mathbb{C}$ with $a \neq \tilde{a}$, $\lambda \neq 0$.

Theorem 2.1 *Let u, v be two linear functionals related by*

$$(x - \tilde{a})u = \lambda(x - a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}. \quad (2.1)$$

Assume $u_0 = 1 = v_0$ and $a \neq \tilde{a}$. If u is a quasi-definite linear functional with corresponding SMOP $\{P_n\}_{n \geq 0}$ then, the linear functional v is quasi-definite if and only if

$$\Delta_n = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{vmatrix} \neq 0, \quad n \geq 1,$$

where $R_n(x) = (\lambda - 1)P_n(x) + (a - \tilde{a})P_{n-1}^{(1)}(x)$. Furthermore, if $\{Q_n\}_{n \geq 0}$ is the SMOP associated with v then

$$(x - \tilde{a})Q_n(x) = \Delta_n^{-1} \begin{vmatrix} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geq 1. \quad (2.2)$$

Proof. Assume v is a quasi-definite linear functional and $\{Q_n\}_{n \geq 0}$ is its corresponding SMOP.

Consider the Fourier expansion of $(x - \tilde{a})Q_n$ in terms of the polynomials P_n , that is

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + \sum_{j=0}^n \alpha_{n,j} P_j(x), \quad n \geq 1,$$

where $\alpha_{n,j} = \langle u, P_j^2 \rangle^{-1} \langle u, (x - \tilde{a})Q_n P_j \rangle$. From formula (2.1) we get

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + \alpha_{n,n} P_n(x) + \alpha_{n,n-1} P_{n-1}(x) \quad (2.3)$$

with $\alpha_{n,n-1} = \lambda \frac{\langle v, Q_n^2 \rangle}{\langle u, P_{n-1}^2 \rangle} \neq 0$.

For $x = \tilde{a}$

$$0 = P_{n+1}(\tilde{a}) + \alpha_{n,n} P_n(\tilde{a}) + \alpha_{n,n-1} P_{n-1}(\tilde{a}). \quad (2.4)$$

On the other hand,

$$(a - \tilde{a})Q_n(a) = P_{n+1}(a) + \alpha_{n,n} P_n(a) + \alpha_{n,n-1} P_{n-1}(a). \quad (2.5)$$

Subtracting (2.5) to (2.3) and dividing by $x - a$, we can apply u in order to get

$$\langle u, \frac{(x - \tilde{a})Q_n(x) - (a - \tilde{a})Q_n(a)}{x - a} \rangle = P_n^{(1)}(a) + \alpha_{n,n} P_{n-1}^{(1)}(a) + \alpha_{n,n-1} P_{n-2}^{(1)}(a). \quad (2.6)$$

The left hand side becomes

$$\langle u, (x - \tilde{a}) \frac{Q_n(x) - Q_n(a)}{x - a} \rangle + Q_n(a) = \lambda \langle v, Q_n(x) - Q_n(a) \rangle + Q_n(a) = (1 - \lambda) Q_n(a)$$

and therefore

$$(1 - \lambda) Q_n(a) = P_n^{(1)}(a) + \alpha_{n,n} P_{n-1}^{(1)}(a) + \alpha_{n,n-1} P_{n-2}^{(1)}(a). \quad (2.7)$$

Thus, (2.5) and (2.7) yield

$$0 = R_{n+1}(a) + \alpha_{n,n} R_n(a) + \alpha_{n,n-1} R_{n-1}(a). \quad (2.8)$$

Since the system of equations (2.4) and (2.8) in $\alpha_{n,n}$ and $\alpha_{n,n-1}$ has a non-zero solution, then we get $\Delta_n \neq 0$ for every $n \geq 1$.

Besides, from (2.3), (2.4) and (2.8) we obtain (2.2).

Conversely, if $\Delta_n \neq 0$ for every $n \geq 1$ we will prove that the polynomials Q_n defined by

$$(x - \tilde{a}) Q_n(x) = \Delta_n^{-1} \begin{vmatrix} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geq 1,$$

are orthogonal with respect to v . Indeed, for $0 \leq j \leq n-2$,

$$\lambda \langle v, Q_n(x)(x-a)P_j(x) \rangle = \langle u, (x-\tilde{a})Q_n(x)P_j(x) \rangle = 0$$

and for $j = n-1$,

$$\lambda \langle v, Q_n(x)(x-a)P_{n-1}(x) \rangle = \langle u, (x-\tilde{a})Q_n(x)P_{n-1}(x) \rangle = \Delta_{n+1} \Delta_n^{-1} \langle u, P_{n-1}^2 \rangle \neq 0.$$

Thus, we only need to prove that $\langle v, Q_n \rangle = 0$ for every $n \geq 1$. In order to do this, observe that

$$\begin{aligned} \lambda \langle v, Q_n \rangle &= \lambda \left[\langle v, (x-a) \frac{Q_n(x) - Q_n(a)}{x-a} \rangle + Q_n(a) \right] \\ &= \langle (x-\tilde{a})u, \frac{Q_n(x) - Q_n(a)}{x-a} \rangle + \lambda Q_n(a) \\ &= \langle u, \frac{(x-\tilde{a})Q_n(x) - (a-\tilde{a})Q_n(a)}{x-a} \rangle + (\lambda - 1) Q_n(a). \end{aligned}$$

Applying the expression of $(x-\tilde{a})Q_n(x)$ in terms of the polynomials $P_n(x)$ and (2.7) we get

$$\begin{aligned} &\langle u, \frac{(x-\tilde{a})Q_n(x) - (a-\tilde{a})Q_n(a)}{x-a} \rangle \\ &= \Delta_n^{-1} \begin{vmatrix} P_n^{(1)}(a) & P_{n-1}^{(1)}(a) & P_{n-2}^{(1)}(a) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix} = (1 - \lambda) Q_n(a). \end{aligned}$$

So $\langle v, Q_n \rangle = 0$ for every $n \geq 1$.

As a conclusion, $\langle v, Q_n^2 \rangle = \langle v, Q_n(x-a)P_{n-1} \rangle \neq 0$, and $\langle v, Q_n p \rangle = 0$ for every polynomial p of degree less than n . \square

Corollary 2.2 *Under the conditions of Theorem 2.1 the linear functional v is quasi-definite if and only if $1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2 \rangle} \neq 0$, for every $n \geq 1$.*

Furthermore, we have

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + a_n(a, \tilde{a})P_n(x) + b_n(a, \tilde{a})P_{n-1}(x), \quad n \geq 1 \quad (2.9)$$

with

$$a_n(a, \tilde{a}) = \beta_n - \tilde{a} + (a - \tilde{a})\Delta_n^{-1}P_{n-1}(\tilde{a})R_n(a) \quad (2.10)$$

and

$$b_n(a, \tilde{a}) = \gamma_n + (\tilde{a} - a)\Delta_n^{-1}P_n(\tilde{a})R_n(a). \quad (2.11)$$

Proof. From the expression of Δ_n , using the Christoffel–Darboux formula (see [2]), we have for $n \geq 1$

$$\Delta_n = (a - \tilde{a}) [(1 - \lambda)K_{n-1}(a, \tilde{a}; u)\langle u, P_{n-1}^2 \rangle + B_n(a, \tilde{a})],$$

where $K_n(x, y; u)$ denotes the reproducing kernel of degree n associated with u and

$$B_n(a, \tilde{a}) = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ P_{n-1}^{(1)}(a) & P_{n-2}^{(1)}(a) \end{vmatrix}.$$

Inserting the three-term recurrence relation for both polynomials P_n and $P_{n-1}^{(1)}$, we get

$$\frac{B_n(a, \tilde{a})}{\langle u, P_{n-1}^2 \rangle} = (\tilde{a} - a) \frac{P_{n-1}(\tilde{a})P_{n-2}^{(1)}(a)}{\langle u, P_{n-1}^2 \rangle} + \frac{B_{n-1}(a, \tilde{a})}{\langle u, P_{n-2}^2 \rangle}, \quad n \geq 2.$$

Iteration yields

$$\frac{B_n(a, \tilde{a})}{\langle u, P_{n-1}^2 \rangle} = (\tilde{a} - a) \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})P_{j-1}^{(1)}(a)}{\langle u, P_j^2 \rangle} - 1, \quad n \geq 1. \quad (2.12)$$

Therefore

$$\begin{aligned} \Delta_n &= (\tilde{a} - a)\langle u, P_{n-1}^2 \rangle \left[1 + (\lambda - 1)K_{n-1}(a, \tilde{a}; u) + (a - \tilde{a}) \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})P_{j-1}^{(1)}(a)}{\langle u, P_j^2 \rangle} \right] \\ &= (\tilde{a} - a)\langle u, P_{n-1}^2 \rangle \left[1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2 \rangle} \right], \end{aligned} \quad (2.13)$$

and the first part of the Corollary follows from Theorem 2.1.

On the other hand, we can write formula (2.2) as follows

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + a_n(a, \tilde{a})P_n(x) + b_n(a, \tilde{a})P_{n-1}(x), \quad n \geq 1.$$

Using the three-term recurrence relation for $P_{n+1}(\tilde{a})$ and $R_{n+1}(a)$ we get

$$\begin{aligned} a_n(a, \tilde{a}) &= \beta_n - \Delta_n^{-1}[\tilde{a}P_n(\tilde{a})R_{n-1}(a) - aP_{n-1}(\tilde{a})R_n(a)] \\ &= \beta_n - \tilde{a} + (a - \tilde{a})\Delta_n^{-1}P_{n-1}(\tilde{a})R_n(a). \end{aligned}$$

Besides, from (2.13) we obtain

$$\frac{\Delta_{n+1}}{\langle u, P_n^2 \rangle} = \frac{\Delta_n}{\langle u, P_{n-1}^2 \rangle} + (\tilde{a} - a) \frac{P_n(\tilde{a})R_n(a)}{\langle u, P_n^2 \rangle}$$

and, since $b_n(a, \tilde{a}) = \frac{\Delta_{n+1}}{\Delta_n}$ and $\gamma_n = \frac{\langle u, P_n^2 \rangle}{\langle u, P_{n-1}^2 \rangle}$, then

$$b_n(a, \tilde{a}) = \gamma_n + (\tilde{a} - a)\Delta_n^{-1}P_n(\tilde{a})R_n(a). \quad \square$$

In Theorem 2.1 and Corollary 2.2 we have assumed $a \neq \tilde{a}$. Notice that if $a = \tilde{a}$ the relation (2.1) between the linear functionals u and v becomes $u = \lambda v + (1 - \lambda)\delta_a$. In this situation it is well known (see [8]) that v is quasi-definite if and only if for every $n \geq 1$

$$1 + (\lambda - 1)K_n(a, a; u) \neq 0$$

and then

$$(x - a)Q_n(x) = P_{n+1}(x) + a_n(a)P_n(x) + b_n(a)P_{n-1}(x), \quad n \geq 1 \quad (2.14)$$

holds, where

$$a_n(a) = \beta_n - a - \frac{(\lambda - 1)P_{n-1}(a)P_n(a)}{\langle u, P_{n-1}^2 \rangle [1 + (\lambda - 1)K_{n-1}(a, a; u)]}$$

and

$$b_n(a) = \gamma_n \frac{1 + (\lambda - 1)K_n(a, a; u)}{1 + (\lambda - 1)K_{n-1}(a, a; u)}.$$

Notice that, these results can be recovered from Corollary 2.2, when \tilde{a} tends to a .

3 Linear relations between the polynomials $\{P_n\}$ and $\{Q_n\}$

Let u and v be quasi-definite linear functionals with corresponding SMOP $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, respectively. In Section 2, we have obtained that if u and v satisfy the relation $(x - \tilde{a})u = \lambda(x - a)v$ with $a, \tilde{a}, \lambda \in \mathbb{C}$ then an expression of the form

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \quad n \geq 1 \quad (3.1)$$

holds (see formulas (2.9) and (2.14)). That is, a linear combination of three consecutive polynomials P_n coincides with a linear combination of three consecutive polynomials Q_n .

On the other hand, in [1], it was proved that if the linear functionals u and v are quasi-definite and they are related as above, then there exists a relation

$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$, $n \geq 1$, and $s_1 \neq t_1$ if and only if for every $n \geq 1$, $P_n \neq Q_n$.

Thus, at the present, we have two expressions linking the polynomials P_n and Q_n , the last quoted and the one given in formula (3.1).

We see below that if $P_n \neq Q_n$, $n \geq 1$, then both formulas are not independent. In fact, one of them can be reduced to the other.

Theorem 3.1 *Let u, v be two different quasi-definite linear functionals normalized by $u_0 = 1 = v_0$ and related by*

$$(x - \tilde{a})u = \lambda(x - a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}.$$

Let $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ be their corresponding SMOP. The following conditions are equivalent:

i) *Formula (3.1) can be reduced to an expression*

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \quad (3.2)$$

with $s_n t_n \neq 0$ for every $n \geq 1$ and $s_1 \neq t_1$.

ii) *For all $n \geq 1$, $R_n(a) = (\lambda - 1)P_n(a) + (a - \tilde{a})P_{n-1}^{(1)}(a) \neq 0$.*

Proof. Suppose that i) holds. In [1, Theorem 2.4] it has been proved that whenever such a relation (3.2) is satisfied then $P_n \neq Q_n$, for every n , and besides $P_n(x) = Q_n(x) + \lambda^{-1} R_n(a) K_{n-1}(x, a; v)$, $n \geq 1$ (see formula 2.24 in [1]). So, ii) follows.

In order to derive the converse result we will first consider the case $a \neq \tilde{a}$. Inserting the three-term recurrence relation in (3.1) successively for P_{n+1} and P_n we get, for $n \geq 2$,

$$\begin{aligned} (x - \tilde{a})Q_n(x) &= (x - \tilde{a})P_n(x) + (\tilde{a} - \beta_n + a_n)P_n(x) + (b_n - \gamma_n)P_{n-1}(x) \\ &= (x - \tilde{a})[P_n(x) + (\tilde{a} - \beta_n + a_n)P_{n-1}(x)] \\ &\quad + [(\tilde{a} - \beta_n + a_n)(\tilde{a} - \beta_{n-1}) + b_n - \gamma_n]P_{n-1}(x) \\ &\quad - \gamma_{n-1}(\tilde{a} - \beta_n + a_n)P_{n-2}(x). \end{aligned} \quad (3.3)$$

The first part of the formula (3.3) for $n - 1$ reads:

$$\begin{aligned} (x - \tilde{a})Q_{n-1}(x) &= (x - \tilde{a})P_{n-1}(x) + (\tilde{a} - \beta_{n-1} + a_{n-1})P_{n-1}(x) \\ &\quad + (b_{n-1} - \gamma_{n-1})P_{n-2}(x). \end{aligned} \quad (3.4)$$

Taking into account (2.10) and (2.11), the above two formulas can be written

$$(x - \tilde{a})Q_n(x) = (x - \tilde{a}) \left[P_n(x) + \frac{(a - \tilde{a})}{\Delta_n} R_n(a) P_{n-1}(\tilde{a}) P_{n-1}(x) \right]$$

$$+ \frac{(a-\tilde{a})}{\Delta_n} R_n(a) \gamma_{n-1} [P_{n-2}(\tilde{a})P_{n-1}(x) - P_{n-1}(\tilde{a})P_{n-2}(x)],$$

$$(x - \tilde{a})Q_{n-1}(x) = (x - \tilde{a})P_{n-1}(x) + \frac{(a-\tilde{a})}{\Delta_{n-1}} R_{n-1}(a) [P_{n-2}(\tilde{a})P_{n-1}(x) - P_{n-1}(\tilde{a})P_{n-2}(x)].$$

Thus, for any $t_n \in \mathbb{R}$, $n \geq 2$

$$\begin{aligned} & (x - \tilde{a})[Q_n(x) + t_n Q_{n-1}(x)] \\ &= (x - \tilde{a}) \left[P_n(x) + \left(\frac{(a-\tilde{a})}{\Delta_n} R_n(a) P_{n-1}(\tilde{a}) + t_n \right) P_{n-1}(x) \right] \\ &+ (a - \tilde{a}) \left[\frac{R_n(a)}{\Delta_n} \gamma_{n-1} + \frac{R_{n-1}(a)}{\Delta_{n-1}} t_n \right] [P_{n-2}(\tilde{a})P_{n-1}(x) - P_{n-1}(\tilde{a})P_{n-2}(x)]. \end{aligned}$$

Now, since by hypothesis $R_n(a) \neq 0$ for all n , if we take

$$t_n = -\frac{R_n(a)}{R_{n-1}(a)} \frac{\Delta_{n-1}}{\Delta_n} \gamma_{n-1}, \quad n \geq 2,$$

we get $t_n \neq 0$ as well as

$$Q_n(x) + t_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x),$$

where $s_n = (a - \tilde{a}) \Delta_n^{-1} R_n(a) P_{n-1}(\tilde{a}) + t_n$.

Observe that, using (2.11), we can obtain

$$s_n = -\frac{R_n(a)}{R_{n-1}(a)} \neq 0, \quad n \geq 2.$$

For $n = 1$, from the values of a_1 and b_1 , the first part of formula (3.3) becomes $Q_1(x) = P_1(x) + \frac{(a-\tilde{a})}{\Delta_1} R_1(a)$. Then $P_1(x) + s_1 = Q_1(x) + t_1$ holds with $s_1 t_1 \neq 0$ and $s_1 - t_1 \neq 0$.

Finally, notice that the case $a = \tilde{a}$ can be derived in a similar way. \square

Remarks. 1) In Section 2, we have seen that the linear functional v is quasi-definite if and only if $1 + \sum_{j=0}^n \frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2 \rangle} \neq 0$, $n \geq 1$. It is worth noticing that the parameters $\{R_n(a)\}_{n \geq 0}$, which appear in the above result, also characterize the existence of formula (3.2).

2) In terms of the linear functionals, we have that $R_n(a) \neq 0$ ($n \geq 1$) if and only if the linear functional $(x - a)w$ is quasi-definite, where w is either the linear functional u (case $a = \tilde{a}$, $\lambda \neq 1$), or the linear functional $u^{(1)}$ (case $a \neq \tilde{a}$, $\lambda = 1$) or the linear functional associated with the co-recursive polynomials (case $a \neq \tilde{a}$, $\lambda \neq 1$).

3) If $a \neq \tilde{a}$ and $\lambda \neq 1$ it was proved in [9] that $R_n(a) \neq 0$ for every $n \geq 1$ if and only if the linear functional $\frac{a-\tilde{a}}{\lambda-1}(x-a)^{-1}u + \delta_a$ is quasi-definite. When u and v are related as in Theorem 3.1, this last condition is equivalent to the

quasi-definiteness of the linear functional $\lambda v - u$. Moreover, in this case the SMOP associated with $\lambda v - u$ is $\{P_n - \frac{R_n(a)}{R_{n-1}(a)}P_{n-1}\}_{n \geq 0}$.

Next, we want to point out that a difference appears between the cases $a = \tilde{a}$ and $a \neq \tilde{a}$ with respect to the parameters s_n and t_n in formula (3.2).

In Theorem 3.1, it has been shown that there exists a relation of the form

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \quad (3.5)$$

with $s_n t_n \neq 0, n \geq 1$, and $s_1 \neq t_1$ if and only if $R_n(a) \neq 0, n \geq 1$. Moreover, we get for every $n \geq 1$

$$t_n - s_n = \frac{P_{n-1}(\tilde{a})R_n(a)}{\langle u, P_{n-1}^2 \rangle \left[1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2 \rangle} \right]}.$$

Then, whenever $a = \tilde{a}$ and $\lambda \neq 1$, (3.5) holds if and only if the linear functional $(x - \tilde{a})u$ is quasi-definite. Besides $s_n \neq t_n$, for $n \geq 1$.

However, if $a \neq \tilde{a}$, even if the condition $R_n(a) \neq 0$ is satisfied for all $n \geq 1$ then both situations either $(x - \tilde{a})u$ is quasi-definite or $(x - \tilde{a})u$ is not quasi-definite can appear. In fact, an example of the first situation was given in [1] being u and v the Jacobi linear functionals with parameters $\alpha - 1, \beta$ and $\alpha, \beta - 1$ ($\alpha, \beta > 0$), respectively, and $a = -1, \tilde{a} = 1, \lambda = -\alpha\beta^{-1}$. In this case, also $s_n \neq t_n$ for every $n \geq 1$.

Next, we are going to show an example of the second situation, that is, when the linear functional $(x - \tilde{a})u$ is not quasi-definite and, as a consequence, the condition $s_n \neq t_n$ is not satisfied for every $n \geq 1$.

Let u be the Chebyshev linear functional of second kind, that is, the Jacobi linear functional with parameters $\alpha = \beta = 1/2$, and take $a = 1, \tilde{a} = 0$, and $\lambda = 3$. We denote by $\{P_n\}$ the monic polynomials associated with u whose recurrence coefficients are $\beta_n = 0$ and $\gamma_n = 1/4$ (see [2]). Observe that the linear functional xu is not quasi-definite.

With these conditions the co-recursive polynomials R_n are given by

$$R_n(x) = 2 \left[P_n(x) + \frac{1}{2}P_{n-1}(x) \right]. \quad (3.6)$$

Notice that $\frac{1}{2}R_n(x)$ are the monic Chebyshev polynomials of fourth kind, that is the monic Jacobi polynomials with parameters $\alpha = 1/2$ and $\beta = -1/2$, see [5].

First, we check that the linear functional v defined by $xu = 3(x - 1)v$ is quasi-definite. As we have introduced in Theorem 2.1

$$\Delta_n = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geq 1,$$

and since $P_{2n}(0) = \frac{(-1)^n}{4^n}$, $P_{2n+1}(0) = 0$, and $R_n(1) = \frac{2n+1}{2^{n-1}}$ we get

$$\Delta_{2n} = (-1)^n \frac{4n-1}{4^{2n-1}} \quad \text{and} \quad \Delta_{2n+1} = (-1)^{n+1} \frac{4n+3}{4^{2n}}.$$

Therefore, $\Delta_n \neq 0$ for every $n \geq 1$, and thus v is quasi-definite. Observe that $v = -\frac{x}{3}w + \delta_1$ where w denotes the Chebyshev linear functional of third kind.

As $R_n(1) \neq 0$, for $n \geq 1$, from Theorem 3.1 a relation of the form (3.5) holds with

$$s_n = -\frac{R_n(1)}{R_{n-1}(1)} = -\frac{2n+1}{2(2n-1)}, \quad n \geq 2,$$

and

$$t_n = \frac{\Delta_{n-1}}{4\Delta_n} s_n, \quad n \geq 2.$$

Therefore, taking into account $P_1(x) = Q_1(x) + 1$, we deduce

$$P_{2n}(x) - \frac{4n+1}{2(4n-1)} P_{2n-1}(x) = Q_{2n}(x) - \frac{4n+1}{2(4n-1)} Q_{2n-1}(x), \quad n \geq 1,$$

$$P_{2n+1}(x) - \frac{4n+3}{2(4n+1)} P_{2n}(x) = Q_{2n+1}(x) + \frac{4n-1}{2(4n+1)} Q_{2n}(x), \quad n \geq 0.$$

Notice that in this case $s_{2n} = t_{2n}$, $n \geq 1$.

Eventually, from the values of the recurrence coefficients of $\{P_n\}$ and Theorem 2.2 in [1], we can deduce that the recurrence parameters for $\{Q_n\}$ are $\tilde{\beta}_n = (-1)^n$, $n \geq 0$, and

$$\tilde{\gamma}_{2n+1} = -\frac{4n-1}{4(4n+3)}, \quad n \geq 0, \quad \text{and} \quad \tilde{\gamma}_{2n} = -\frac{4n+3}{4(4n-1)}, \quad n \geq 1.$$

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