On Descartes’ rules of signs and their exactness

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This paper revisits the Descartes’ rules of signs and provides new bounds for the number of complex roots of a polynomial in certain complex regions. We also prove that the Descartes’ rules associated with the Bernstein basis are exact for polynomials whose roots are real.

1. Introduction.

Descartes’ rule of signs states that the number of positive zeros (with multiplicities) of a real polynomial

\[ c_0 + c_1 z + \cdots + c_n z^n \]

is less than or equal to the number of variations of strict sign in the sequence of the coefficients \((c_0, c_1, \ldots, c_n)\) and that, if there is no equality, the difference between both numbers is an even number (see some related questions in [1], [2] and [8]). This rule has been extended to more general systems of functions. As shown in [6], the first part holds if we substitute the monomial basis \((1, z, \ldots, z^n)\) by any extended sign-regular system of univariate functions and the second part also holds if we substitute that basis by a B-basis, concept introduced in [5] in the context of Computer Aided Geometric Design and associated to optimal shape preserving properties (see also [12]).

Given a system of functions \((u_0, \ldots, u_n)\) defined on \(I \subseteq \mathbb{R}\), we may define the matrices

\[
M\left(\begin{array}{c} u_0, u_1, \ldots, u_n \\ t_0, t_1, \ldots, t_m \end{array}\right) := \left( u_j(t_i) \right)_{i=0, \ldots, m; j=0, \ldots, n}, \quad t_0 < t_1 < \cdots < t_m \text{ in } I,
\]

which are usually called **collocation matrices**. If \(I\) is an interval, \(u_0, \ldots, u_n \in C^n(I)\), we may define for any \(t_0 \leq t_1 \leq \cdots \leq t_m\) the corresponding **extended collocation matrices**

\[
M^*\left(\begin{array}{c} u_0, u_1, \ldots, u_n \\ t_0, t_1, \ldots, t_m \end{array}\right) := \left( \lambda_i(u_j) \right)_{i=0, \ldots, m; j=0, \ldots, n},
\]

where \(\lambda_i(\mu) := u(\mu_i)(t_i)\) and \(\mu_i \in \{0, 1, \ldots, n\}\) is the number of indices \(j < i\) such that \(t_j = t_i\).

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A system of functions \((u_0, \ldots, u_n)\) defined on an interval \(I\) is called \textit{extended sign-regular} if all its extended collocation matrices are strictly sign-regular. An \(m \times n\) matrix \(A\) is called \textit{sign-regular} (resp., \textit{strictly sign-regular}) if all minors of order \(k\) have the same nonstrict (resp., strict) sign \(\varepsilon_k\) for each \(k\). If, for each \(r \leq k\), all minors of order \(r\) have the same nonstrict (resp., strict) sign \(\varepsilon_r\), we say that \(A\) is \(\text{SR}_k\) (resp., \(\text{SSR}_k\)) . If all the minors of order \(r \leq k\) have the same nonstrict (resp., strict) sign \(\varepsilon_r\), we say that \(A\) is \(\text{TP}_k\) (resp., \(\text{STP}_k\)) and, if \(k = \min\{m, n\}\) we say that \(A\) is \textit{totally positive} (resp., \textit{strictly totally positive}). The theory of variation-diminishing transformations was originated by Schoenberg [13] and is closely related to these classes of matrices.

For any vector \(\lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}\), \(V(\lambda)\) will denote the number of strict sign changes in \(\lambda\) and \(V^+(\lambda)\) will denote the maximal number of sign changes in \(\lambda\). More precisely,

\[
V(\lambda) := \max \{k \mid \text{there exist } 0 \leq i_0 < \cdots < i_k \leq m, \text{ such that } (-1)^j \lambda_{i_j} > 0 \text{ for all } j \in \{0, \ldots, k\} \text{ or } (-1)^j \lambda_{i_j} < 0 \text{ for all } j \in \{0, \ldots, k\}\},
\]

\[
V^+(\lambda) := \max \{k \mid \text{there exist } 0 \leq i_0 < \cdots < i_k \leq m, \text{ such that } (-1)^j \lambda_{i_j} \geq 0 \text{ for all } j \in \{0, \ldots, k\} \text{ or } (-1)^j \lambda_{i_j} \leq 0 \text{ for all } j \in \{0, \ldots, k\}\}.
\]

The nonsingular sign-regular matrices are characterized by their variation diminishing property (cf. Theorem 5.6 of [3]):

**Proposition 1.1.** Let \(A\) be a nonsingular real matrix. Then the following properties are equivalent:

(i) \(A\) is sign-regular.
(ii) \(V(Ax) \leq V(x)\) for all \(x \in \mathbb{R}^n\).
(iii) \(V^+(Ax) \leq V^+(x)\) for all \(x \in \mathbb{R}^n\).

The following variation diminishing property of \(\text{SR}_k\) matrices (see Theorem 2.4 (a) of Chapter 5 of [9]) will be used in the following section.

**Proposition 1.2.** If \(A\) is an \(m \times n\) (\(m > n\)) \(\text{SR}_k\) real matrix and \(c \in \mathbb{R}^n\) satisfies \(V(c) \leq k - 1\), then \(V(Ac) \leq V(c)\).

In Section 2, we apply the previous results to study the concept of variation diminishing polynomials.

Section 3 includes new bounds for the number of complex roots of a polynomial in certain complex regions. We shall use the sets \(L_n, R_n: \)

\[
L_n := \{z \in \mathbb{C} \mid |\arg z - \pi| < \frac{\pi}{n}\}, \quad R_n := \{z \in \mathbb{C} \mid |\arg z| < \frac{\pi}{n}\}
\]
and so their closures $\bar{L}_n, \bar{R}_n$ are given by

$$
\bar{L}_n := \{ z \in \mathbb{C} \mid \arg z - \pi \leq \frac{\pi}{n} \}, \quad \bar{R}_n := \{ z \in \mathbb{C} \mid \arg z \leq \frac{\pi}{n} \}.
$$

Let us observe that 0 does not belong to any of the subsets $L_n, R_n$ because $\arg 0$ is not defined.

Let $S$ be any subset of complex plane $\mathbb{C}$. Given a polynomial $p(z)$, we denote by $Z|_S(p(z))$ the number of zeros (counting multiplicities) of $p(z)$ in $S$. Schoenberg proved in Theorem VI of [15] that, given a polynomial $r(z)$ with $V(r(z)) \geq 2$,

$$
Z|_{\bar{L}_{V(r(z))}}(r(z)) \leq n - V(r(z)).
$$

In Section 3 we prove that a similar inequality holds by replacing $V$ by $V^+$ and $\bar{L}_{V^+(r(z))}$ by $L_{V^+(r(z))}$. On the other hand, Drucker introduced in [7] the concept of measure of a polynomial and proved that this number provides a lower bound of the number of its nonreal roots. We prove in Theorem 3.3 that the measure of a polynomial $p(z)$ provides a lower bound of the number of complex roots of $p(z)$ in $\mathbb{C} \setminus S$, where $S := L_{V^+(p(z))} \cup R_{n-V(p(z))} \cup \{0\}$.

In [6] we characterized the systems of functions whose Descartes’ rules are exact up to an even number by means of the concept of B-basis. For instance, the monomial basis is a B-basis of the space of polynomials of degree less than or equal to $n$ on $(0, +\infty)$, and the Bernstein basis is a B-basis of the same space on $(a, b)$. In fact, the classical Jacobi’s rule for bounding the number of zeros of a polynomial on $(a, b)$ (cf. [14, p. 548]) can be interpreted as the Descartes’ rule associated to the Bernstein basis. In [6] we also proved that Descartes’ rules associated to B-bases are optimal in the sense that they provide optimal bounds for the number of zeros. In Section 4 we prove that these rules are exact for polynomials whose roots are real. We also define a B-measure which provides a necessary condition to recognize such polynomials.

2. Variation diminishing polynomials.

Given a polynomial of real coefficients

$$
p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0,
$$

we denote $V(p(z)) := V(a)$ and $V^+(p(z)) := V^+(a)$, where $a$ is the vector given by the coefficients: $a = (a_0, \ldots, a_n)^T$. 

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Definitions 2.1. We say that a real polynomial $q(z)$ is *variation diminishing* (or, equivalently, *V-diminishing*) if $V(p(z)q(z)) \leq V(p(z))$ for all real polynomial $p(z)$. Analogously, a real polynomial $q(z)$ is *V-increasing* (resp., *V-preserving*) if $V(p(z)q(z)) \geq V(p(z))$ (resp., $V(p(z)q(z)) = V(p(z))$) for all real polynomial $p(z)$. Finally, a real polynomial $q(z)$ is *$V^+$-diminishing* if $V^+(p(z)q(z)) \leq V^+(p(z))$ for all real polynomial $p(z)$.

Let $\mathcal{P}_m$ denote the space of real polynomials of degree less than or equal to $m$.

Theorem 2.2. A linear polynomial $z - \alpha$ is V-diminishing and $V^+$-diminishing (resp., V-increasing, V-preserving) if $\alpha < 0$ (resp., $\alpha > 0$, $\alpha = 0$). Besides, for any real polynomial $q(\alpha)$, if $\alpha > 0$ then $V(q(z)) - V(q(z)(z - \alpha))$ is a nonnegative even number and if $\alpha < 0$ then $V(q(z)(z - \alpha)) - V(q(z))$ is an odd positive number.

Proof. Given a polynomial $q(z) = q_nz^{n-1} + q_{n-1}z^{n-2} + \cdots + q_2z + q_1$ ($q_n \neq 0$), we define the vectors $q := (q_1, \ldots, q_n)^T$ and $\bar{q} = (0, q_1, \ldots, q_n)^T$. Let $a := (a_0, \ldots, a_n)^T$ be the coefficients vector with respect to the basis $(1, z, \ldots, z^n)$ of the polynomial $p(z) := q(z)(z - \alpha)$. Then we can write $A_{n+1}^{\alpha} \bar{q} = a$, where $A_{n+1}^{\alpha}$ is the $(n + 1) \times (n + 1)$ matrix given by

$$A_{n+1}^{\alpha} = \begin{pmatrix}
1 & -\alpha & 0 & \cdots & 0 \\
0 & 1 & -\alpha & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -\alpha \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}.$$ 

Observe that the bidiagonal matrix $A_{n+1}^{\alpha}$ is sign-regular (and totally positive) if and only if $\alpha < 0$. Hence, by Proposition 1.1, if $\alpha < 0$ then

$$V(a) \leq V(\bar{q}) = V(q) \quad (2.2)$$

and

$$V^+(a) \leq V^+(\bar{q}) = V^+(q) + 1 \quad (2.3)$$

If $q_i$ ($i \geq 1$) is the first nonzero component of $q$, then $a_{i-1} = -\alpha q_i$ is the first nonzero component of $a$. Taking into account that the signs of $q_i$ and $a_{i-1}$ coincide and that $q_n = a_n$, we conclude that $V(q) - V(a)$ and $V^+(q) - V^+(a)$ are even numbers. From (2.2), we get that $V(q) - V(a)$ is a nonnegative even number and, from (2.3), we can deduce that $V^+(a) \leq V^+(q)$.

If $\alpha = 0$, then it is obvious that $V(a) = V(q)$. Finally, let us assume that $\alpha > 0$. If $J_n$ is the diagonal matrix $J_n := \text{diag}\{1, -1, 1, \ldots, (-1)^n\}$ and $J_n AJ_n$ is a totally positive matrix, then, by Theorem 3.3 c) of [3], $A^{-1}$ is totally positive. Hence, since $J_n + A_{n+1}^{\alpha} J_{n+1}^{\alpha} = A_{n+1}^{\alpha}$
is a totally positive matrix, \((A_{n+1}^\alpha)^{-1}\) is also totally positive (in fact, if we write \(A_{n+1}^\alpha =: I_{n+1} - \alpha N_1\), then \((A_{n+1}^\alpha)^{-1} = \sum_{\nu=0}^n \alpha^\nu N^\nu\)). Since \((A_{n+1}^\alpha)^{-1}\) is totally positive, applying Proposition 1.1 to \((A_{n+1}^\alpha)^{-1}\) we deduce that \(V(q) = V(q) \leq V(a)\). Taking into account that \(q_n = a_n\) but the signs of \(q_i\) and \(a_{i-1}\) are different, we conclude that \(V(a) - V(q)\) is an odd number. ■

Let us observe that the classical Descartes’ rule of signs is a straightforward consequence of applying the previous result for the case \(\alpha > 0\).

**Proposition 2.3.** If \(p(z)\) is a polynomial for which the Descartes’ rule of signs is exact and all the roots of \(q(z)\) are nonpositive real numbers, then \(V(p(z)q(z)) = V(p(z))\) and the Descartes’ rule of signs is exact for \(p(z)q(z)\).

**Proof.** By hypothesis, the number of positive roots of \(p(z)\) coincides with \(V(p(z))\) and with the number of positive roots of \(p(z)q(z)\). Applying the Descartes’ rule of signs to \(p(z)q(z)\), we now get \(V(p(z)) \leq V(p(z)q(z))\). Since \(q(z)\) can be factorized as a product of linear polynomials \((z - \alpha)\) with negative roots and a factor \(z^k\) \((k \geq 0)\), we deduce from Theorem 2.2 that \(V(p(z)q(z)) \leq V(p(z))\) and so \(V(p(z)q(z)) = V(p(z))\) and the Descartes’ rule of signs is also exact for \(p(z)q(z)\). ■

**Remark 2.4.** Given a polynomial whose roots are real, we can factorize it as a product of a polynomial \(p(z)\) whose roots are positive (and so the Descartes’ rule of signs is exact for \(p(z)\)) and a polynomial whose roots are nonpositive. Thus, we derive from the previous result the following well known property: the Descartes’ rule of signs is exact for polynomials whose roots are real.

Let us observe that we cannot state a result similar to Proposition 2.3 by replacing \(V^+\) by \(V\). If we consider the polynomial \(p(z) = z^2 + 1\), then the Descartes’ rule of signs is exact for \(p(z)\) and \(V^+(p(z)) = 2\). Taking \(q(z) = z + 1\), we have \(p(z)q(z) = z^3 + z^2 + z + 1\) and so \(V^+(p(z)q(z)) = 0 < V^+(p(z))\).

**Definition 2.5.** A real polynomial \(q(z)\) is \(V\)-**diminishing** for \(\mathcal{P}_m\) if \(V(p(z)q(z)) \leq V(p(z))\) for all \(p(z) \in \mathcal{P}_m\).

The following result is a consequence of Theorem 7.1 of Chapter 8 of [9] and Proposition 1.2.

**Proposition 2.6.** If the roots of a polynomial \(r(z)\) belong to \(\bar{L}_{k+2}\), then \(r(z)\) is \(V\)-diminishing for \(\mathcal{P}_k\).

**Proof.** Let \(n\) be such that \(n - k\) is the degree of \(r(z)\). Then \(r(z) = r_{n-k}z^{n-k} + r_{n-k-1}z^{n-k-1} + \cdots + r_1z + r_0\) \((r_{n-k} \neq 0)\) and let \(r := (r_{n-k}, r_{n-k-1}, \ldots, r_1, r_0)^T\). Given
a polynomial $q(z) = q_n z^k + q_{n-1} z^{k-1} + \cdots + q_{n-k+1} z + q_{n-k} \in \mathcal{P}_k$, we define the vectors $q := (q_n, \ldots, q_0)^T$ and $\bar{q} = (0, \ldots, 0, q_{n-k}, \ldots, q_0)^T \in \mathbb{R}^{n+1}$. Let $a := (a_0, \ldots, a_n)^T$ be the coefficients vector with respect to the basis $(1, z, \ldots, z^n)$ of the polynomial $p(z) := q(z)r(z)$. Then we can write

$$A_{n+1}^r \bar{q} = a,$$  \hspace{1cm} (2.4)

where $A_{n+1}^r$ is the $(n + 1) \times (n + 1)$ matrix given by

$$A_{n+1}^r = \begin{pmatrix}
    r_{n-k} & r_{n-k-1} & \cdots & r_0 & 0 & \cdots & 0 \\
    0 & r_{n-k} & r_{n-k-1} & \cdots & r_0 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & & \ddots & \ddots & \ddots & \ddots & r_0 \\
    \vdots & & & \ddots & \ddots & \ddots & \vdots \\
    \vdots & & & & \ddots & \ddots & \vdots \\
    \vdots & & & & & \ddots & \vdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & r_{n-k-1} \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & r_{n-k}
\end{pmatrix}.$$

By Theorem 7.1 of Chapter 8 of [9], we derive from the hypothesis on the roots of $r(z)$ that, using its coefficients, we obtain the Polya frequency

$$(\ldots, 0, r_{n-k}, r_{n-k-1}, \ldots, r_1, r_0, 0, \ldots)$$

of order $k + 1$ and so the matrix $A_{n+1}^r$ is TP$_{k+1}$ (see p. 393 of [9]). Let $\bar{A}$ be the $(n + 2) \times (n + 1)$ matrix obtained from $A_{n+1}^r$ by incorporating the new first row $(1, 0, \ldots, 0)$. Clearly, $\bar{A}$ is also TP$_{k+1}$ and, from (2.4), we deduce that

$$\bar{A}\bar{q} = \bar{a},$$  \hspace{1cm} (2.5)

where $\bar{a} = (0, a_0, \ldots, a_n)^T$. Taking into account $V(\bar{q}) = V(q) \leq k$, we can deduce from Proposition 1.2 and formula (2.5) that $V(a) = V(\bar{a}) \leq V(\bar{q}) = V(q)$ and the result follows. \hfill \qed

The proof of the previous result uses the fact that the matrix $A_{n+1}^r$ is TP$_{k+1}$, where $r$ is given by the coefficients of a polynomial whose zeros belong to $\bar{L}_{k+2}$. The following examples show (for the cases $n = 3, 4$ and $k = n - 2$) that if the zeros of $r(z)$ do not belong to $\bar{L}_n$, then the matrix $A_{n+1}^r$ is not TP$_{n-1}$. In these examples, we shall use the following matricial notation. Given an $m \times n$ matrix $A$ and $\alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_l)$ with $1 \leq \alpha_1 < \cdots < \alpha_k \leq m$, $1 \leq \beta_1 < \cdots < \beta_l \leq n$, $A[\alpha|\beta]$ denotes the $k \times l$ submatrix of $A$ containing rows $\alpha$ and columns $\beta$. 

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Example 2.7. If \( r = (1, -2\alpha, \alpha^2 + \beta^2)^T \) (i.e., \( r(z) \) has roots \( \alpha \pm \beta i \)) with \( |\beta| = \sqrt{3}|\alpha| \) and \( \alpha < 0 \), then \( A^r_4 \) is TP_2 and \( \det A^r_4[1, 2|2, 3] = 4\alpha^2 - \alpha^2 - \beta^2 = 0 \). However, if \( |\beta| > \sqrt{3}|\alpha| \), then \( \det A^r_4[1, 2|2, 3] < 0 \) and \( A^r_4 \) is not TP_2 (and is not SR_2).

Example 2.8. If \( r = (1, -2\alpha, \alpha^2 + \beta^2)^T \) with \( |\beta| = |\alpha| \) and \( \alpha < 0 \), then \( A^r_5 \) is TP_3 and \( \det A^r_5[1, 2, 3|2, 3, 4] = -8\alpha^3 + \alpha^3 = 0 \). However, if \( |\beta| > |\alpha| \), then \( \det A^r_5[1, 2, 3|2, 3, 4] < 0 \) and \( A^r_5 \) is not TP_3 (and is not SR_3).

The following example shows that Proposition 2.6 cannot be improved, in the sense that, under its hypotheses, we cannot guarantee that \( r(z) \) is V-diminishing for \( \mathcal{P}_{k+1} \).

Example 2.9. The polynomial \( r(z) = z^2 + 2z + 4 \) has its roots in \( \bar{L}_3 \). Let \( q(z) = z^2 - 3z + 4 \). Then \( p(z) = q(z)r(z) = z^4 - z^3 + 2z^2 - 4z + 16 \) satisfies \( V(p(z)) > V(q(z)) \).

Reasoning as in Proposition 2.3, but replacing Theorem 2.2 by Proposition 2.6, we can deduce the following result.

Proposition 2.10. If all the roots of \( q(z) \in \mathcal{P}_{n-k} \) belong to \( \bar{L}_{k+2} \) and the Descartes’ rule of signs is exact for \( p(z) \in \mathcal{P}_k \), then \( V(p(z)q(z)) = V(p(z)) \) and the Descartes’ rule of signs is exact for \( p(z)q(z) \).

3. Bounds for the number of roots in complex regions.

The following bounds

\[
Z|_{R_{n-V(r(z)}}(r(z)) \leq V(r(z)), \quad V(r(z)) \leq n - 1, \tag{3.1}
\]

and

\[
Z|_{\bar{L}_{V(r(z)}}(r(z)) \leq n - V(r(z)), \quad V(r(z)) \geq 2, \tag{3.2}
\]

are given in Corollary 7.2 of Chapter 8 of [9] and Corollary 7.1 of Chapter 8 of [9], respectively. In fact, (3.1) is due to Obreschkoff and is known as Obreschkoff-Descartes rule (see also [11] and Theorem VII of [15]) and (3.2) also can be seen in Theorem VI of [15]. Let us observe that (3.2) also can be derived from Proposition 2.6.

The following result provides an inequality resembling (3.2), but using \( V^+ \) instead of \( V \):

Proposition 3.1. If \( r(z) \in \mathcal{P}_n \) and \( V^+(r(z)) \geq 1 \), then

\[
Z|_{L_{V^+(r(z))}}(r(z)) \leq n - V^+(r(z)). \tag{3.3}
\]

Proof. Let \( \tilde{V}(r(z)) := V(r(-z)) \). Then, applying (3.1) to \( r(-z) \), we derive

\[
Z|_{L_{n-\tilde{V}(r(z))}}(r(z)) = Z|_{R_{n-\tilde{V}(r(z))}}(r(-z)) \leq \tilde{V}(r(z)) \tag{3.4}
\]
whenever $\tilde{V}(r(z)) \leq n - 1$. It can be checked that
\[ V^+(p(z)) + \tilde{V}(p(z)) = n. \quad (3.5) \]

Then (3.3) follows from (3.4) and (3.5). ■

If in formula (3.3) we could write $\bar{L}V^+(r(z))$ instead of $LV^+(r(z))$, the analogy with (3.2) would be complete. However, the following example shows that in that case the inequality (3.3) does not hold.

**Example 3.2.** The roots of the polynomial $r(z) = z^4 + 4 = (z^2 + 2z + 2)(z^2 - 2z + 2)$ are $-1 \pm i$ and $1 \pm i$ and so satisfy $|\arg z - \pi| = \pi/4$ and $|\arg z| = \pi/4$, respectively. Observe that $V^+(r(z)) = 4$ and $Z|_{L_4} = 0$ so that (3.3) holds. However, two roots belong to $\bar{L}_4$ and so $2 = Z|_{L_4} > 4 - V^+(r(z)) = 0$.

The following result will provide a lower bound for the number of roots of a polynomial in a complex region. Let us first recall another result of this type. Let $p(z)$ be the polynomial given in (2.1) and let $a_e$ and $a_0$ denote the vectors formed by the even coefficients $a_{2i}$ and the odd coefficients $a_{2i+1}$ ($i \geq 0$) of $p(z)$, respectively. In [10], Kemperman proved that the number of roots in each half plane $\Re z > 0$ and $\Re z < 0$ is greater than or equal to $\max\{V(a_e), V(a_o)\}$. We now recall some definitions given in [7]. Given a real polynomial (2.1), a gap in $p(z)$ consists of two nonzero coefficients, say $a_i$ and $a_j$ ($i < j$) such that $a_k = 0$ whenever $i > k > j$. When the signs of $a_i$ and $a_j$ agree, the gap is called a permanence, and a variation when the signs are different. A gap is even (resp., odd) when the number of missing coefficients between $a_j$ and $a_i$ is even (resp., odd). The measure of an even gap is the number $r$ of missing coefficients and the measure of an odd gap is $r + 1$ when the gap is a permanence and $r - 1$ when the gap is a variation. Finally, the measure $M(p(z))$ of a polynomial $p(z)$ is the sum of the measures of all gaps in $p(z)$.

Let us see that $M(p(z))$ provides a lower bound of the number of roots of $p(z)$ in $\mathbb{C} \setminus S$, where $S$ is the complex set
\[ S := L_{V^+(p(z))} \cup R_{n-V(p(z))} \cup \{0\}. \quad (3.6) \]

**Theorem 3.3.** Let $S$ be the complex region given by (3.6), $p(z)$ a real polynomial (2.1) of degree $n$ and $M(p(z))$ its measure. Then the number of conjugate pairs of roots of $p(z)$ in $\mathbb{C} \setminus S$ is at least $M(p(z))/2$.

**Proof.** If $k(\geq 0)$ is the multiplicity of 0 as a zero of $p(z)$, one clearly has
\[ M(p(z)) = V^+(p(z)) - V(p(z)) - k. \quad (3.7) \]
Then, from (3.7), we derive

\[ n = V^+(p(z)) + n - V^+(p(z)) = M(p(z)) + V(p(z)) + k + n - V^+(p(z)) \]

and so

\[
Z|_{C \setminus S}(p(z)) = Z|_{C \setminus (L_{V^+(p(z))} \cup R_{n - V^+(p(z))} \cup \{0\})}(p(z)) \\
= M(p(z)) + (V(p(z)) - Z|R_{n - V^+(p(z))}(p(z))) + (n - V^+(p(z)) - Z|_{L_{V^+(p(z))}})(p(z))).
\]

(3.8)

Since, by formulae (3.1) and (3.3), \( V(p(z)) - Z|R_{n - V^+(p(z))}(p(z)) \) and \( n - V^+(p(z)) - Z|_{L_{V^+(p(z))}}(p(z)) \) are nonnegative numbers, we conclude that \( M(p(z)) \leq Z|_{C \setminus S}(p(z)) \). \( \square \)

From the previous result we derive a necessary condition in order that a real polynomial has all its roots in \( S = L_{V^+(p(z))} \cup R_{n - V^+(p(z))} \cup \{0\} \).

**Corollary 3.4.** If a real polynomial \( p(z) \) has all its roots in \( L_{V^+(p(z))} \cup R_{n - V^+(p(z))} \cup \{0\} \) then \( M(p(z)) = 0 \).

We finish this section with a result on measure diminution of polynomials.

**Proposition 3.5.** If \( p(z) \) is a polynomial for which the Descartes’ rule of signs is exact and all the roots of \( q(z) \) are nonpositive real numbers, then \( M(p(z)q(z)) \leq M(p(z)) \).

**Proof.** The polynomial \( q(z) \) can be factorized as a product of linear polynomials \((z - \alpha)\) with negative roots and a factor \( z^k \) \((k \geq 0)\). This last factor is obviously M-preserving. Since, by Theorem 2.2, the linear factors \((z - \alpha)\) of \( q(z) \) are \( V^+\)-diminishing and, by Proposition 2.3, are \( V\)-preserving when we multiply their product by \( p(z) \), we can deduce the result by applying formula (3.7) to the polynomial \( p(z)q(z) \). \( \square \)

We cannot remove the hypothesis that the Descartes’ rule of signs is exact for \( p(z) \) in the previous result, as the following example shows. If \( p(z) = z^2 - z + 1 \) and \( q(z) = z + 1 \), then \( p(z)q(z) = z^3 + 1 \) and \( M(p(z)q(z)) = 2 > 0 = M(p(z)) \).

4. **On polynomials whose roots are real and B-Descartes’ rules of signs.**

In Corollary 4.4 of [6] we characterized all systems (of a given space of functions defined on an interval) satisfying a Descartes’ rule of signs which is exact up to an even number, which in turn coincide with all systems of functions whose Descartes’ rule of signs is optimal (in the sense that it provides optimal bounds for the number of zeros in the interval). The concept of B-basis plays a crucial role in that characterization. In fact, if \((u_0, \ldots, u_n)\) is a B-basis, taking into account Corollary 4.4 and Proposition 3.3 of [6] we have that all
systems whose Descartes’ rule of signs satisfy the previous properties are precisely the systems

\[(d_0u_0, \ldots, d_nu_n), (-d_0u_0, \ldots, -d_nu_n), (d_nu_n, \ldots, d_0u_0), (-d_nu_n, \ldots, -d_0u_0), \quad (4.1)\]

where \(d_i > 0\) for all \(i\). The Descartes’ rule of signs corresponding to any of such systems \((4.1)\) will be called \(B\)-Descartes’ rule of signs. Given a \(B\)-basis \((b_0, \ldots, b_n)\) and a function \(f(z) = \sum_{i=0}^{n} k_ib_i(z)\), we denote by \(a := (k_0, \ldots, k_n)^T\), \(V(f(z)) := V(a)\) and \(V^+(f(z)) := V^+(a)\). Let us observe that all systems \((b_0, \ldots, b_n)\) of \((4.1)\) lead to the same \(V(f(z))\) and to the same \(V^+(f(z))\).

In the case of the space of polynomials of degree less than or equal to \(n\) on an interval \((a, b)\), a \(B\)-basis is given by

\[((b - z)^n, (b - z)^{n-1}(z - a), \ldots, (b - z)(z - a)^{n-1}, (z - a)^n) \quad (4.2)\]

(see [4] or [5]). From now on, we shall consider the \(B\)-Descartes’ rule of signs with respect to this basis and, analogously to Definition 2.1, we can define the concepts of \(V_B\)-diminishing, \(V_B^+\)-diminishing, \(V_B\)-increasing or \(V_B\)-preserving polynomials.

**Theorem 4.1.** A linear polynomial \(z - \alpha\) is \(V_B\)-diminishing and \(V_B^+\)-diminishing (resp., \(V_B\)-increasing, \(V_B\)-preserving) if \(\alpha \in (-\infty, a) \cup (b, \infty)\) (resp., \(\alpha \in (a, b)\), \(\alpha \in \{a, b\}\)).

**Proof.** Given a polynomial

\[q(z) = q_n(z-a)^{n-1}+q_{n-1}(b-z)(z-a)^{n-2}+\cdots+q_2(b-z)^{n-2}(z-a)+q_1(b-z)^{n-1}, \quad q_n \neq 0,\]

we define the vectors \(q := (q_1, \ldots, q_n)^T\) and \(\bar{q} = (0, q_1, \ldots, q_n)^T\). Let \(a := (a_0, \ldots, a_n)^T\) be the coefficients vector with respect to the basis \((4.2)\) of the polynomial \(p(z) := q(z)(z - \alpha)\). Then, taking into account that

\[z - \alpha = \frac{b - \alpha}{b - a}(z - a) + \frac{a - \alpha}{b - a}(b - z),\]

it can be checked that \(B^\alpha_{n+1}\bar{q} = a\), where \(B^\alpha_{n+1}\) is the \((n + 1) \times (n + 1)\) matrix given by

\[
B^\alpha_{n+1} = \begin{pmatrix}
\frac{b - \alpha}{b - a} & \frac{a - \alpha}{b - a} & 0 & \cdots & 0 \\
0 & \frac{b - \alpha}{b - a} & \frac{a - \alpha}{b - a} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \frac{b - \alpha}{b - a}
\end{pmatrix}.
\]
Observe that the bidiagonal matrix $B_{n+1}^\alpha$ is sign-regular if and only if its entries have constant sign, which in turn is equivalent to $\alpha \in (-\infty, a) \cup (b, \infty)$, and that $(B_{n+1}^\alpha)^{-1}$ is sign-regular if $\alpha \in (a, b)$.

We now can reason in a similar way to the proof of Theorem 2.2 in order to prove the result.

Let us observe that the B-Descartes’ rule of signs is a straightforward consequence of applying the previous result to the case $\alpha \in (a, b)$. Analogously to Proposition 2.3 and Remark 2.4 we can derive the following results:

**Proposition 4.2.** If $p(z)$ is a polynomial for which a B-Descartes’ rule of signs is exact and all the roots of $q(z)$ are real numbers outside $(a, b)$, then $V(p(z)q(z)) = V(p(z))$ and the B-Descartes’ rule of signs is exact for $p(z)q(z)$.

**Corollary 4.3.** A B-Descartes’ rule of signs is exact for polynomials whose roots are real.

Analogously to the definition in Section 3 of the measure $M(p(z))$ of a polynomial $p(z)$ expressed with respect to the monomial basis $(1, z, z^2, \ldots, z^n)$, we could define the B-measure $M_B(p(z))$ of a polynomial given with respect to a B-basis $(u_0, \ldots, u_n)$ on an interval $(a, b)$. Observe that the measure would be the same if we consider another basis of (4.1). In [7] it was proved that $M(p(z)) = 0$ is a necessary condition for a polynomial with only real roots and in the previous section we have improved such result. A similar result for polynomials whose roots are real can be derived for any B-measure. Previously, we need some auxilliary results.

If we have a B-basis on $(a, b)$ and $k_1, k_2 (\geq 0)$ are the respective multiplicities of $a$ and $b$ as zeros of a polynomial $p(z)$, one clearly has

$$M_B(p(z)) = V_B^+(p(z)) - V_B(p(z)) - k_1 - k_2. \quad (4.3)$$

**Proposition 4.4.** If $p(z)$ is a polynomial for which a B-Descartes’ rule of signs associated to a B-basis on $(a, b)$ is exact and all the roots of $q(z)$ are real numbers outside $(a, b)$, then $M_B(p(z)q(z)) \leq M_B(p(z))$.

**Proof.** The polynomial $q(z)$ can be factorized as a product of linear polynomials $(z - \alpha)$ with roots outside $[a, b]$ and factors $(z - a)^{k_1}, (z - b)^{k_2}$ ($k_1, k_2 \geq 0$). These last factors are obviously M-preserving. Since, by Theorem 4.1, the linear factors $(z - \alpha)$ of $q(z)$ are $V^+$-diminishing and, by Proposition 4.2, are $V$-preserving when we multiply their product by $p(z)$, we can deduce the result from applying formula (4.3) to the polynomial $p(z)q(z)$.

\[\blacksquare\]
Corollary 4.5. If a real polynomial $r(z)$ has all its roots real, then $M_B(r(z)) = 0$.

Proof. Given a polynomial whose roots are real, we can factorize it as $r(z) = p(z)q(z)$, where all the roots of $q(z)$ are real numbers outside $(a, b)$ and all the roots of $p(z)$ belong to $(a, b)$. By Corollary 4.3, the B-Descartes’ rule of signs is exact for $p(z)$. Hence, $V_B(p(z))$ coincides with the degree of $p(z)$ and so with $V_B^+(p(z))$. Then, by (4.3), $M_B(p(z)) = 0$ and the result follows from Proposition 4.4. ■

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