PRE-PUBLICACIONES del seminario matematico "garcia de galdeano"

2003

Stability of discrete liftings

V. Domínguez
F. J. Sayas

n. 16

seminario matemático
garcía de galdeano
Universidad de Zaragoza

Stability of discrete liftings

Víctor Domínguez
Dep. Matemática e Informática, Univ. Pública de Navarra
Campus de Arrosadía, 31006, Pamplona, Spain
victor.dominguez@unavarra.es
&
Francisco–Javier Sayas
Dep. Matemática Aplicada, C.P.S.
Universidad de Zaragoza, 50018 Zaragoza, Spain
jsayas@unizar.es

Abstract

In this short note we prove the equivalence between having a discrete lifting of Dirichlet boundary conditions for (abstract) finite element spaces and having a Scott–Zhang type operator in the space, i.e., a stable projection preserving homogeneous boundary conditions. Both results are equivalent to the possibility of obtaining a Céa estimate where approximation of the boundary conditions is separated from the approximation capabilities of the space.

1 Statement of the problem

Let V and M be Hilbert spaces, $\gamma : V \to M$ be a bounded surjective linear operator (the abstract trace) and $V^0 = \ker \gamma$. Let $a : V \times V \to \mathbb{C}$ be a bounded sesquilinear form. We consider the following problem: given $\eta \in M$, find the solution to

$$\left| \begin{array}{l} u \in V, \quad \gamma u = \eta, \\ a(u,v) = 0, \qquad \forall v \in V^0. \end{array} \right. \tag{1}$$

Remark 1 *To keep notations as light as possible, in the following we will be using the same symbol for norms, $\|\cdot\|$, and inner products, $(\,\cdot\,,\,\cdot\,)$, of M and V. It is the notation for the elements (Greek letters for elements of M and Latin for those of V) that will make the context clear.*

To ensure well–posedness of (1) we assume the following:

Hypothesis I. *The operator $A_0 : V^0 \to V^0$ defined by the relation $(A_0 u, v) = a(u,v)$, for all $u, v \in V^0$, is invertible.*

If this hypothesis holds, then (1) has a unique solution. We define $R : M \to V$ to the operator such that $Rg := u$, the solution of (1). It is clear that R is bounded and is a right–inverse for γ. We will call it a lifting. In particular, if we take the inner product of V as sesquilinear form, the associated lifting is just the pseudoinverse of γ, which we denote γ^+ (see [2]).

Remark 2 *The standard example for this abstract setting consists of taking* $V = H^1(\Omega)$, $M = H^{1/2}(\Gamma)$, γ *the trace operator (and thus* $V_0 = H^1_0(\Omega)$*) and* $a(\,\cdot\,,\,\cdot\,)$ *being the sesquilinear form associated to an elliptic operator, for instance,* $a(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v}$.

Let now $V_h \subset V$ be a family of finite dimensional subspaces of V and consider the spaces $V_h^0 := V_h \cap V^0$ and $M_h := \gamma V_h$. We then consider the discretized version of (1): given $\eta_h \in M_h$ (in practice one takes $\eta_h \approx \eta$ in some way), solve

$$\left| \begin{array}{ll} u_h \in V_h, \quad \gamma u_h = \eta_h, & \\ a(u_h, v_h) = 0, & \forall v_h \in V_h^0. \end{array} \right. \tag{2}$$

The discretized version of Hypothesis I is:

Hypothesis II. *There exists* $\alpha > 0$ *such that*

$$\sup_{0 \neq u_h \in V_h^0} \frac{|a(u_h, v_h)|}{\|u_h\|} \geq \alpha \|v_h\|, \qquad \forall v_h \in V_h^0. \tag{3}$$

If this hypothesis holds, it is very simple to prove that (2) has a unique solution and that there exists $C_0 > 0$ such that

$$\|u - u_h\| \leq C_0 \inf\{\|u - v_h\| \,|\, v_h \in V_h, \gamma v_h = \eta_h\}. \tag{4}$$

The operator mapping η_h to u_h will be denoted $R_h : M_h \to V_h$. Again, in case the sesquilinear form is the inner product, Hypothesis II trivially holds and the operator, denoted by γ_h^+, is just the pseudoinverse of $\gamma_h := \gamma|_{V_h} : V_h \to M_h$.

Remark 3 *There are two simple cases where both hypotheses hold.*

(a) *The sesquilinear form is* V_0-*elliptic, i.e., there exists* $\alpha > 0$ *such that* $\operatorname{Re} a(u,u) \geq \alpha \|u\|^2$ *for all* $u \in V_0$.

(b) *There exists a Hilbert space* H*, such that* $V \subset H$ *with dense compact inclusion,* $a(\,\cdot\,,\,\cdot\,)$ *satisfies a Garding inequality (here* $\alpha, \kappa > 0$*)*

$$\operatorname{Re} a(u,u) \geq \alpha \|u\|^2 - \kappa \|u\|_H^2, \qquad \forall u \in V^0 \tag{5}$$

and the homogeneous version of (1) does not admit but the trivial solution, then the hypothesis is satisfied. Then (3) holds for h *small enough provided that for all* $u \in V^0$, $\inf_{v_h \in V_h^0} \|u - v_h\| \to 0$.

In the remainder of the paper, we will assume that Hypotheses I and II hold.

2 Main results

Theorem 2.1 *The following statements are equivalent:*

(1) R_h *is uniformly bounded.*

(2) γ_h^+ *is uniformly bounded.*

(3) *There exists* $L_h : M_h \to V_h$ *linear uniformly bounded satisfying* $\gamma\, L_h \eta_h = \eta_h$ *for all* $\eta_h \in M_h$.

Proof. Notice that $\|\gamma_h^+ \eta_h\| \le \|v_h\|$ for any $v_h \in V_h$ such that $\gamma v_h = \eta_h$. Then we just have to prove that uniform boundedness of R_h is implied by that of γ_h^+. This last is, however equivalent to the following discrete uniform Babuška–Brezzi type condition (see [1]): there exists $\beta > 0$ such that

$$\sup_{0 \neq v_h \in V_h} \frac{|(\gamma v_h, \mu_h)|}{\|v_h\|} \ge \beta \|\mu_h\|, \qquad \forall \mu_h \in M_h. \tag{6}$$

Since $(\gamma v_h, \mu_h) = 0$ for all $\mu_h \in M_h$ implies that $\gamma v_h = 0$, then $u_h := R_h g_h$ solves

$$\left| \begin{array}{lllll} u_h \in V_h, \lambda_h \in M_h, & & & & \\ a(u_h, v_h) & + & (\lambda_h, \gamma v_h) & = & 0, & \forall v_h \in V_h, \\ (\gamma u_h, \mu_h) & & & = & (\eta_h, \mu_h), & \forall \mu_h \in M_h. \end{array} \right.$$

Then (3) and (6) show that $\|u_h\| + \|\lambda_h\| \le C\|\eta_h\|$, with a constant C depending on α and β. □

> **Hypothesis III.** *For all* h, *there exists an operator* $\Pi_h : V \to V_h$, *such that: it is uniformly bounded, it is a projection onto* V_h *and if* $u \in V^0$, *then* $\Pi_h u \in V_h^0$ *(it respects the boundary condition* $\gamma u = 0$*).*

Two of these operators have been studied in [4] and [3], for particular choices of finite element spaces.

Theorem 2.2 *If Hipothesis III holds, then* R_h *is uniformly bounded.*

Proof. Let $\eta_h \in M_h$ and consider $u := R\eta_h$ (i.e. problem (1) with $\eta = \eta_h$) and $u_h := R_h \eta_h$, the solution of (2). Since $u - u_h \in V^0$, then $\Pi_h u - u_h = \Pi_h(u - u_h) \in V_h^0$ and we can take $\Pi_h u$ in (4):

$$\|u - u_h\| \le C_0 \|u - \Pi_h u\| \le C_0 (1 + \|\Pi_h\|) \inf_{v_h \in V_h} \|u - v_h\| \le C_1 \|u\|.$$

This easily gives the result. □

Remark 4 *Notice that existence of* Π_h *satisfying Hypothesis III allows to prove a variant of the Céa estimate (4)*

$$\|u - u_h\| \le C_2 \inf_{v_h \in V_h} \|u - v_h\| + C_3 \|\eta - \eta_h\|. \tag{7}$$

This allows for a simple approach to the analysis of the approximation of (1) by (2), even with non–homogeneous right–hand side.

Theorem 2.3 *If* γ_h^+ *is uniformly bounded, then there exists* Π_h *in the conditions of Hypothesis III.*

Proof. Let $P_h^0 : V \to V_h^0$ and $T_h : M \to M_h$ be the orthogonal projections onto V_h^0 and M_h respectively. Let then

$$\Pi_h u := P_h^0 u + \gamma_h^+ T_h \gamma u.$$

It is clear that Π_h is uniformly bounded and that if $u \in V^0$ (that is, $\gamma u = 0$) then $\Pi_h u = P_h^0 u \in V_h^0$. If $u_h \in V_h$, then $T_h \gamma u_h = \gamma u_h$ and thus

$$\Pi_h u_h = P_h^0 u_h + \gamma_h^+ \gamma u_h = v_h^0 + v_h^1,$$

where

$$\left| \begin{array}{l} v_h^0 \in V_h, \quad \gamma v_h^0 = 0, \\ (v_h^0, v_h) = (u_h, v_h), \qquad \forall v_h \in V_h^0 \end{array} \right. \qquad \text{and} \qquad \left| \begin{array}{l} v_h^1 \in V_h, \quad \gamma v_h^1 = \gamma u_h, \\ (v_h^1, v_h) = 0, \qquad \forall v_h \in V_h^0. \end{array} \right.$$

Therefore $\Pi_h u_h$ satisfies

$$\left| \begin{array}{l} \Pi_h u_h \in V_h, \quad \gamma \Pi_h u_h = \gamma u_h, \\ (\Pi_h u_h, v_h) = (u_h, v_h), \qquad \forall v_h \in V_h^0, \end{array} \right.$$

and by uniqueness of solution $\Pi_h u_h = u_h$. □

Remark 5 *The theory of mixed methods gives also some additional insight into this matter. Assume there exists an operator* $\Pi_h : V \to V_h$ *satisfying the requirements of Fortin's lemma: uniform boundedness and compatibility*

$$(\gamma \Pi_h u, \mu_h) = (\gamma u, \mu_h), \qquad \forall \mu_h \in M_h.$$

Then, if this operator is a projection onto V_h*, it also satisfies Hypothesis III.*

3 Two simple consequences

The first by–product of these results is a simplified version of the Céa estimate, provided that the choice $\eta_h \approx \eta$ is stable. Obviously, if the sequence V_h satisfies an approximation property in V, then this implies convergence of the solutions of (2) to that of (1).

Corollary 3.1 *Assume that $N_h : M \to M_h$ is a uniformly bounded projection onto M_h. If R_h is uniformly bounded and we take $\eta_h = N_h\eta$ in (2), then*

$$\|u - u_h\| \le C_4 \inf_{v_h \in V_h} \|u - v_h\|.$$

Proof. Let w_h the the best approximation of u in V_h, i.e., $\|u - w_h\| = \inf_{v_h \in V_h} \|u - v_h\|$. Then

$$\|\eta - \eta_h\| \le (1 + \|N_h\|) \inf_{\rho_h \in M_h} \|\eta - \rho_h\| \le (1 + \|N_h\|)\|\eta - \gamma w_h\| \le (1 + \|N_h\|)\|\gamma\| \, \|u - w_h\|.$$

The result then follows by (4). □

The associated Dirichlet–to–Neumann operator in this abstract setting is the mapping $M \to M'$ given by

$$\eta \;\mapsto\; a(R\eta, R\,\cdot\,) = a(R\eta, \gamma^+\,\cdot\,) : M \to \mathbb{C}.$$

The final result proves uniform boundedness of the discretization of this operator between abstract Cauchy data. Its proof is straightforward.

Corollary 3.2 *If R_h is uniformly bounded, then the discrete operator $M_h \to M_h'$ given by*

$$\eta_h \;\mapsto\; a(R_h\eta_h, R_h\,\cdot\,) : M_h \to \mathbb{C}$$

is uniformly bounded.

Acknowledgments. The authors are partially supported by MCYT/FEDER Projects BFM2001-2521 and MAT2002-04153 and by Gobierno de Navarra Ref. 134/2002.

References

[1] Brezzi, F.; Fortin, M., Mixed and hybrid finite element methods. Springer-Verlag, New York, 1991.

[2] Engl, H.W.; Hanke, M.; Neubauer, A., Regularization of inverse problems. Kluwer Academic Publishers Group, Dordrecht, 1996.

[3] Girault, V.; Scott, L. R., Hermite interpolation of nonsmooth functions preserving boundary conditions. Math. Comp. **71** (2002), 1043–1074.

[4] Scott, L. R.; Zhang, S., Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comp. **54** (1990), 483–493.