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NONLINEAR STABILITY UNDER A LOGARITHMIC GRAVITY FIELD

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Abstract

In this paper we analyze the orbital stability of the equilibria corresponding to the motion of point mass under the gravitational field of a massive finite segment. This potential is expressed in closed form as a logarithmic function, depending on the distance of the particle to the end-points of the segment. This model may be considered as an approximation to the potential generated by elongated bodies. For this motion, we determine the equilibria, their stability both the linear and the nonlinear stability, even for the case of resonances.

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1 Introduction

The searching of orbits around minor celestial bodies, like asteroids, minor bodies, comet nucleii, is attracting the interest of researches and spatial agencies have planned missions to these bodies. For instance, the ROSETTA project [24] of the European Space Agency (ESA), or the NEAR mission [18] fo the NASA. In such missions an orbiter moves around the celestial body, either to map the surface, to determine the density, or even to land on its surface. The main force is originated by the gravity field, but in most of the cases (small asteroids, comet nucleus) the main characteristic of the bodies is its shape, elongated and irregular.

Traditionally, in the case of planets (see e.g.[11, 13]), the potential function is represented in spherical variables (r, λ, β) as series of spherical harmonics in the form

$$U = -\frac{\mu}{r} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n \sum_{0 \le m \le n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\cos \beta),$$

where μ is the Gaussian constant, α the mean equatorial radius, and P_{nm} are the associated Legendre polynomials of degree n and order m. The coefficients C_{nm} and S_{nm} are dimensionless quantities depending on the mass distribution of the body.

This expression is widely used for computing motions around the Earth, Moon of other planets, since almost sphericity is a common characteristic of such bodies. However, when the body is far to be spherical, such expansion are no longer useful, and in some cases the convergence is not guaranteed. Indeed, the series' rate of convergence slows down as the field point approaches the circumscribing sphere; further, the series usually diverges where the field point is within the sphere. Thus, spherical harmonic approach is not convenient to express the gravity field near a non-spherical body.

To overcome these difficulties, different models have been proposed, for instance, the use of polyhedrons that fit the shape of the asteroid [23, 26, 27]. For those bodies, it is possible to obtain the potential function in closed form (see e.g. [26]).

We find other approaches in the literature. Thus, Prieto and Gómez–Tierno [19] model this type of bodies by a massive dipole; they find also that an axial symmetric body can be replaced by a massive wire lying in the axis of symmetry with appropriate mass distribution. Halamek [10] and Riaguas *et al.*[21] also studied the gravitational field of a finite massive straight segment. This model fits very well elongated bodies (let us mention, for instance, that the asteroid *Geographos* is a tube of 5.1×1.8 km).

The problem may be considered between two classical ones, on the one hand the Two Center Problem, when the segment is fixed in the space, and on the other, the Restricted Three Body Problem, when the segment is rotating. Albeit its simplicity, the dynamics it generates is very rich, the problem is no-integrable [2]; and since the potential depends on some parameters, there are bifurcations as well as regions of chaoticity [21].

In this paper, we consider the gravitational field originated by a finite massive straight segment rotating uniformly about an axis perpendicular to it. The paper is organized as follows: in Section 2 we derive the potential function in closed from as a logarithmic function (which is not common in Dynamics) depending on the distance to the end-points of the segment. For this potential, we obtain the equilibria (Section 3) and their linear stability (Section 4) in a synodic reference frame. To determine the nonlinear stability (Section 5) we need to apply the Arnold's theorem for non definite quadratic forms [1], which requires the determining of the normal form. Lastly, the resonant cases are analyzed separately.

2 Potential function

Let us consider a finite straight segment of length 2ℓ and mass M. The gravitational potential per unit mass created by this *one* dimensional body at a certain point P, exterior to the segment, is given by the line integral

$$U(P) = -\mathcal{G} \int_L \frac{\mathrm{d}m}{r}.$$

where $\mathcal{G}$ denotes the Gaussian constant. Assuming constant the linear mass density (σ), this quadrature is not difficult to solve, and appears in textbooks on potential theory [13]. Here we will use its expression in terms of intrinsic quantities.

Let us consider an inertial reference frame $Oxyz$, and let $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$ be the position vectors of the end-points of the straight segment. The position vector of a point on the segment is

$$\boldsymbol{x} = \boldsymbol{x}_1 + \nu\,(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{x}_1 + \nu\,\boldsymbol{x}_{12} \quad \text{with} \quad 0 \le \nu \le 1,$$

hence, $r^2 = \boldsymbol{x} \cdot \boldsymbol{x} = 4\nu^2\ell^2 + \nu\,(r_2^2 - r_1^2 - 4\ell^2) + r_1^2$, and the length element is $\mathrm{d}r = 2\ell\,\mathrm{d}\nu$, where we denote $r_1 = \|\boldsymbol{x}_1\|$, $r_2 = \|\boldsymbol{x}_2\|$, and $2\ell = \|\boldsymbol{x}_{12}\|$.

Thus, for the origin O, the above quadrature is

$$U(O) = -2\,\mathcal{G}\,\sigma\,\ell \int_0^1 \frac{\mathrm{d}\nu}{r} = -\mathcal{G}M \int_0^1 \left(4\nu^2\ell^2 + \nu(r_2^2 - r_1^2 - 4\ell^2) + r_1^2\right)^{-1/2} \mathrm{d}\nu,$$

but is known that

$$\int \frac{\mathrm{d}\nu}{\sqrt{a\nu^2 + b\nu + c}} = \frac{1}{\sqrt{a}} \log\left(\sqrt{a}(\nu + \frac{b}{2a}) + \sqrt{a\nu^2 + b\nu + c}\right),$$

hence, there results that

$$U(O) = -\frac{\mathcal{G}M}{2\ell} \log\left(\frac{r_1 + r_2 + 2\ell}{r_1 + r_2 - 2\ell}\right), \tag{1}$$

expression that depends only on the distances (2ℓ, r_1 and r_2).

As it was expected, equation (1) shows that the potential is symmetric with respect to the line containing the segment.

Thus, we can formulate the following

Theorem 1 *The potential on an exterior point Q, originated by a finite straight segment (with constant density and homogeneous distribution of mass) is the one given by the Equation (1), that depends on its mass, its length and the distance to the end-points.*

Corollary 1 *The zero velocity surfaces (equipotential surfaces) of the potential given by Eq. (1) are ellipsoids of revolution, which focii are the end-points*

Proof. It is enough to consider the axial symmetry and that

$$U(P) = \text{constant} \Longleftrightarrow r_1 + r_2 = \text{constant}, \qquad \blacksquare$$

3 Rotating segment. Equilibria

Asteroids and planetary satellites belongs to the class of natural elongated bodies that are in pure rotation. They are old objects in the solar system and have reached the state of lowest energy for a given angular momentum, i.e., pure rotation about the principal axis of highest moment of inertia; any primeval nutation faded away because nutation induces time-varying internal stresses that dissipate mechanical energy through hysteresis cycles (see e.g. [19]).

Since in our model we approximate the elongated body by a straight segment, we shall assume that the segment uniformly rotates about an axis z (perpendicular to it and fixed in the space) with constant angular velocity $\boldsymbol{\omega}$. With this, we define a synodic reference frame $Oxyz$, with origin at the center of mass O, and such that the segment lies on the axis Ox. In this synodic system, the equations of motion of a point mass attracted by the segment are

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x - U_x, \quad \ddot{y} + 2\omega\dot{x} = \omega^2 y - U_y, \quad \ddot{z} = -U_z. \tag{2}$$

As usual in Mechanics, we can define an effective potential as

$$W(x, y, z) = U(x, y, z) - \frac{\omega^2}{2}(x^2 + y^2), \tag{3}$$

and the Lagrangian function becomes

$$\mathcal{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \omega(x\dot{y} - y\dot{x}) - W(x, y, z). \tag{4}$$

In this function, there appear several parameters, namely the length of the segment, the Gaussian constant and the angular velocity of rotation, however, it is possible to get rid of some of them by an appropriate choice of units. In words of Meyer [16], *scaling and changing units are essentially the same thing,* hence, let us make a scaling of the Lagrangian, as the equations of the motion are not modified when the whole Lagrangian function is multiplied by a constant [7].

We make the following scaling: $r \longrightarrow 2\ell\, r, \quad t \longrightarrow t/\omega$, which is equivalent to choose 2ℓ (the length of the segment) as the unit of length, and $P/2\pi$ (with P the period of the rotation of the segment) as the unit of time. After this scaling, the Lagrangian (4) is converted into

$$\mathcal{L} = \omega^2(2\ell)^2 \left[\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x\dot{y} - y\dot{x}) + \tfrac{1}{2}(x^2 + y^2) + k \log\left(\frac{s+1}{s-1}\right)\right],$$

where $k = \mathcal{G}M/(\omega^2(2\ell)^3)$. The dimensionless parameter k is the ratio of the gravitational acceleration to centrifugal acceleration. $k < 1$ means fast rotation of the segment, whereas $k > 1$ means slow rotation. Hence, after the scaling, our problem only contains one parameter.

By introducing the auxiliary quantities

$$\begin{aligned} s &= \sqrt{y^2+z^2+(x-\tfrac{1}{2})^2}+\sqrt{y^2+z^2+(x+\tfrac{1}{2})^2}, \\ d &= \sqrt{y^2+z^2+(x-\tfrac{1}{2})^2}-\sqrt{y^2+z^2+(x+\tfrac{1}{2})^2}, \\ p &= \sqrt{y^2+z^2+(x-\tfrac{1}{2})^2}\sqrt{y^2+z^2+(x+\tfrac{1}{2})^2}. \end{aligned}$$

and computing the gradient of the effective potential, the equations of the motion (2) become

$$\begin{aligned} \ddot{x}-2\dot{y} &= -W_x = x\left(1-\frac{2k}{sp}\right), \\ \ddot{y}+2\dot{x} &= -W_y = y\left(1-\frac{2ks}{(s^2-1)p}\right), \\ \ddot{z} &= -W_z = -\frac{2kzs}{(s^2-1)p}. \end{aligned} \tag{5}$$

As it happens with problems formulated in a synodic frame, we have that

Proposition 1 *The dynamical system defined by equations (5) has the Jacobian integral*

$$C = 2W(x,y,z)+2T = 2U(x,y,z)-(x^2+y^2)+(\dot{x}^2+\dot{y}^2+\dot{z}^2), \tag{6}$$

where T stands for the kinetic energy in the rotating frame.

As the kinetic energy is $T \geq 0$, the condition $C-2W \geq 0$ defines the regions of possible motions. The equality $2W(x,y,z)=C$ is precisely the so called zero velocity curves, the limit where the motion can take place.

Theorem 2 *The effective potential W defined by (3) has four critical points for whatever value of the parameter k; two of them on the axis Ox and the other two on the axis Oy.*

Proof. Equilibria are found by zeroing the right hand part of equations (5). Obviously, the third equation vanishes only when $z=0$, hence the equilibria are on the xy-plane, if any. There are two possible solutions, either $x=0$ and $2ks=(s^2-1)p$, or $y=0$ and $2k=s\,p$. Let us determine the position of these equilibria.

In the first case, $x=0$ and $2ks=(s^2-1)p$. Since $x=0$, there follows that the distances r_1, r_2 to the end-points of the segment are equal $r_1=r_2=r$ and besides, $r>1/2$. With this, the condition $2k\,s=p\,(s^2-1)$ reads

$$4r^3-r-4k=0; \tag{7}$$

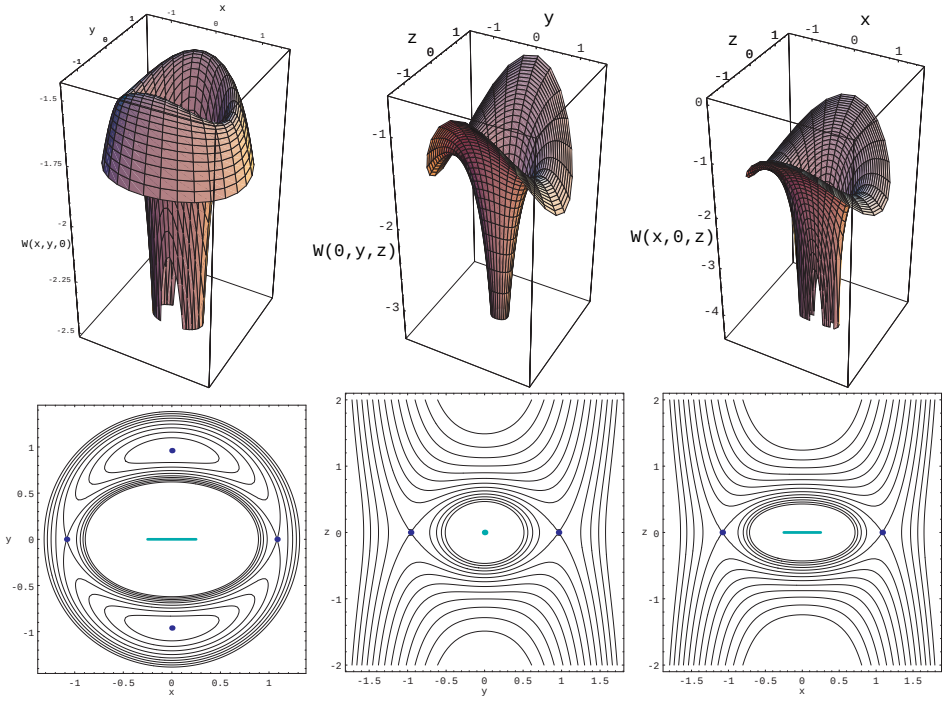

Figure 1: Above: Projection of the effective potential $W(x, y, z)$ onto the coordinate planes $z = 0$ (left), $x = 0$ (center), $y = 0$ (right). Bellow: Contour levels (zero velocity curves) for the previous projections. Note the presence of elliptic and hyperbolic equilibria.

but this cubic has only one root in the interval $(1/2, \infty)$; hence, there are two *isosceles* equilibria, E_2 $(y > 0)$ and E_4 $(y < 0)$, symmetrical each other with respect to the x-axis. It is possible to obtain an explicit formula for the solution of a cubic equation; in this case, the real root is

$$r = \frac{3^{\frac{1}{3}} + \left(36\,k + \sqrt{-3 + 1296\,k^2}\right)^{\frac{2}{3}}}{3^{\frac{2}{3}}\,2\,\left(36\,k + \sqrt{-3 + 1296\,k^2}\right)^{\frac{1}{3}}}.$$

For the second condition, $y = 0$ and $2k = s\,p$. Let us assume that the point is outside the segment, and let us denote by ζ the distance from the equilibrium to the closer end-point of the segment. In this situation, the condition $2k = s\,p$ is converted into

$$2\zeta^3 + 3\zeta^2 + \zeta - 2k = 0, \tag{8}$$

either the possible equilibrium is on the positive or negative semi axis. This cubic has only one positive root, and it is zero (that is to say, is exactly at the end-point of the segment) for $k = 0$ (no rotation). Hence, there are two equilibria, E_1 $(x > 0)$ and E_3 $(x < 0)$, symmetrical each other with respect to the y-axis, that will be denoted *collinear* points, ∎

4 Linear stability

Equilibria and their stability give an insight of the dynamics governed by the differential equations. In order to determine the stability of the equilibria above found, one needs the variational equations of the system (5). By defining a vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$ as the variations

$$\boldsymbol{\xi} = (\delta x, \delta y, \delta \dot{x}, \delta \dot{y}),$$

the variational equations of (5) are

$$\dot{\boldsymbol{\xi}} = A \cdot \boldsymbol{\xi} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -W_{xx} & -W_{xy} & 0 & 2 \\ -W_{xy} & -W_{yy} & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix},$$

where the second derivatives

$$\begin{aligned} W_{xx} &= -1 + \frac{2k}{ps} - x\,k\,\Big(\frac{d}{p^3} + x(\frac{2(s^2-1)}{p^2 s^3} + \frac{s^2+d^2}{p^3 s})\Big), \\ W_{xy} &= \frac{2k(p-s^2)}{p^3 s} xy, \\ W_{yy} &= \frac{p+2ks-ps^2}{p(s^2-1)} - y^2\,ks\,\frac{2p(s^2+1)+(s^2-1)(s^2+d^2)}{p^3(s-1)^2(s+1)^2}, \end{aligned} \tag{9}$$

must be evaluated at the corresponding equilibria.

Since the problem is symmetric with respect to the two coordinate axes, x and y, the stability of the point E_1 is the same as its symmetric E_3, and the same happens for the points E_2 and E_4.

4.1 Stability of the collinear points E_1 and E_3.

The coordinates of the collinear point E_1 are $(x_0, y_0) = (1/2 + \zeta, 0)$, hence

$$s = 2\zeta + 1, \quad d = -1, \quad p = \zeta(1+\zeta).$$

With this, the second derivatives (9) of the potential W evaluated at the point E_1 are

$$W_{xx} = -1 - \frac{k(1+2\zeta)}{\zeta^2(1+\zeta)^2}, \quad W_{xy} = 0, \quad W_{yy} = -1 + \frac{k(1+2\zeta)}{2\zeta^2(1+\zeta)^2},$$

but k and ζ satisfy the equation (8), that is to say, $k = (2\zeta^3 + 3\zeta^2 + \zeta)/2$. Hence, there results

$$W_{xx} = -3 - \frac{1}{2\zeta(1+\zeta)}, \quad W_{xy} = 0, \quad W_{yy} = \frac{1}{4\zeta(1+\zeta)}$$

and the matrix A of the variational equations is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (3+2\gamma) & 0 & 0 & 2 \\ 0 & -\gamma & -2 & 0 \end{pmatrix},$$

where we put the shortcut $\gamma = 1/[4\zeta(1+\zeta)]$. Note that $\gamma = \gamma(\zeta)$ is monotone in the interval $(0, +\infty)$.

The characteristic equation of this matrix is

$$\lambda^4 + (1-\gamma)\lambda^2 - (3+2\gamma)\gamma = 0,$$

which discriminant, $\Delta = 9\gamma^2 + 10\gamma + 1$, always is positive in the domain where γ is defined.

The eigenvalues of the matrix

$$\begin{aligned} \lambda_{1,2} &= \pm\frac{1}{\sqrt{2}}\Big(-1+\gamma+\sqrt{9\gamma^2+10\gamma+1}\Big)^{1/2}, \\ \lambda_{3,4} &= \pm\frac{1}{\sqrt{2}}\Big(-1+\gamma-\sqrt{9\gamma^2+10\gamma+1}\Big)^{1/2}, \end{aligned}$$

are such that $\lambda_{1,2}^2 > 0$ and $\lambda_{3,4}^2 < 0$, and since $\lambda_1 > 0$, we can conclude that

Proposition 2 *The collinear equilibria E_1 y E_3 are always linearly unstable and hence they are also orbitally unstable.*

4.2 Stability of the isosceles points E_2 and E_4.

The isosceles point E_2 is located at the coordinates $(x_0, y_0) = (0, \sqrt{r^2 - 1/4})$, where the distance r is the only real root of the cubic (7), $4r^3 - r - 4k = 0$. In this case, there results that

$$s = 2r, \quad d = 0, \quad p = r^2,$$

and the second derivatives (9) of the potential W, evaluated at the point E_2, are

$$W_{xx} = -1 + \frac{k}{r^3}, \quad W_{xy} = 0, \quad W_{yy} = \frac{k(1-8r^2) + r^3 - 4r^5}{r^3(-1+4r^2)};$$

or equivalently, since $4k = 4r^3 - r$,

$$W_{xx} = -\frac{1}{4r^2}, \quad W_{xy} = 0, \quad W_{yy} = -3 + \frac{1}{4r^2}.$$

With this, the matrix A of the variational equations is now

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta & 0 & 0 & 2 \\ 0 & (3-\delta) & -2 & 0 \end{pmatrix},$$

where we called $\delta = 1/(4r^2)$. N.B. since $1/2 < r < \infty$ then $0 < \delta < 1$.

The eigenvalues of this matrix are

$$\begin{aligned} \lambda_{1,2} &= \pm\frac{1}{\sqrt{2}}\Big(-1+\sqrt{4\delta^2-12\delta+1}\Big)^{1/2}, \\ \lambda_{3,4} &= \pm\frac{1}{\sqrt{2}}\Big(-1-\sqrt{4\delta^2-12\delta+1}\Big)^{1/2}. \end{aligned} \tag{10}$$

The discriminant, $\Delta = 4\delta^2 - 12\delta + 1$, vanishes at the value $\delta_c = (3-2\sqrt{2})/2$. Thus,

$$\begin{cases} 0 < \delta < \delta_c & \Longrightarrow \quad 1 > \Delta > 0, \\ \delta = \delta_c & \Longrightarrow \quad \Delta = 0, \\ \delta_c < \delta < 1 & \Longrightarrow \quad 0 > \Delta > -7. \end{cases}$$

One should think of expressing the regions of instability in terms of the distance r rather than in terms of δ. In that case, the critical distance is $r_c^2 = (3 + 2\sqrt{2})/2$, hence

$$\begin{cases} 1/2 < r < r_c & \Longrightarrow \quad \lambda_1 > 0, \\ r_c \le r < \infty & \Longrightarrow \quad \lambda_{1,2}, \lambda_{3,4} \text{ pure imaginary.} \end{cases}$$

For $r = r_c$, although the eigenvalues are pure imaginary, their multiplicity is 2. However, the matrix $(A - \lambda_{1,2}\mathbb{I})$ has rank 3, which means that is not diagonalizable, and hence, the equilibrium is linear unstable.

In terms of the natural parameter k of the problem, the critical value is

$$k_c = \frac{1}{8}(18 + 13\sqrt{2}) = 4.548097039.$$

Proposition 3 *For $0 < k \le k_c$ the isosceles equilibria are unstable, and for $k_c < k$ the isosceles equilibria are linearly stable.*

5 Nonlinear stability

When the Hamiltonian is expressed in terms of action-angle variables $(\Psi_1, \Psi_2, \psi_1, \psi_2)$, and the quadratic part of the Hamiltonian (corresponding to the expansion of the Hamiltonian in a neighborhood of the equilibria) is not sign defined, that is to say, is in the form

$$\mathcal{H} = (\omega_1\Psi_1 - \omega_2\Psi_2) + \sum_{j>2} \mathcal{H}_j(\Psi_1, \Psi_2, \psi_1, \psi_2),$$

Liapunov's stability theorem [25] can not be applied. In that case, the linear motion in the neighborhood of the origin is equivalent to the subtraction of two harmonic oscillators with frequencies $\omega_1, \omega_2 > 0$. But in this situation, Arnold's theorem [1] can be used for studying the non linear stability of two degrees of freedom Hamiltonian systems.

Arnold's theorem was already used by Leontovich [14], Deprit and Deprit-Bartholomé [4], Markeev [15] to study the equilibria of the planar restricted problem of three bodies (RTBP). Later on, Meyer and Schmidt [17] gave a new proof and reformulated the theorem. Deprit and López-Moratalla [5] applied it to the problem of the stationary satellites and automatized the normalization of the Hamiltonian facilitating the use of this result. More recently, it has been applied to some generalizations of the RTBP [9, 12, 22]. The theorem, as formulated by Meyer and Schmidt [17] reads

Theorem 3 (Arnold) *Consider a Hamiltonian $\mathcal{H}$ expressed, in the action and angle variables $(\psi_1, \psi_2, \Psi_1, \Psi_2)$, as*

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \ldots + \mathcal{H}_{2n} + \tilde{\mathcal{H}}, \tag{11}$$

where:

1. *$\mathcal{H}$ is real analytic in a neighborhood of the origin $\mathbb{R}^4$,*
2. *$\mathcal{H}_{2k}$, $1 \leq k \leq n$, is a homogeneous polynomial of degree k in Ψ_i, with real coefficients. In particular,*

$$\begin{aligned} \mathcal{H}_2 &= \omega_1 \Psi_1 - \omega_2 \Psi_2, \quad 0 < \omega_1,\, 0 < \omega_2; \\ \mathcal{H}_4 &= \tfrac{1}{2}\left(A\Psi_1^2 - 2B\Psi_1\Psi_2 + C\Psi_2^2\right). \end{aligned}$$

3. *$\tilde{\mathcal{H}}$ has a power expansion in Ψ_i which starts with terms at least of order $n + 1/2$.*

Under this assumptions, the origin is a stable equilibrium provided for some k, $2 \leq k \leq n$, $\mathcal{H}_2$ does not divide $\mathcal{H}_{2k}$ or equivalently, provided $D_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1) \neq 0$ and for $2 \leq j < k$, $D_{2j} = \mathcal{H}_{2j}(\omega_2, \omega_1) = 0$.

The second condition implies that $\mathcal{H}$ is in the Birkhoff's normal form up to terms of degree $2n$ and hence some non–resonance assumption on ω_i.

Thus, in order to analyze the nonlinear stability, we have to express the Hamiltonian of the problem in the form of the above theorem.

Let us point out first, that colinear points here obtained are everywhere lineraly unstable, hence they are also nonlinearly unstable. The same happens with isosceles equilibria for $0 < k < k_c$, consequently we have to deal only with the isosceles points, and restric ourselves to the interval $0 < k < k_c$.

The Hamiltonian of the problem is obtained as the Legendre transformation of the Lagrangian function (4):

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) - (xY - yX) + U(x, y) \tag{12}$$

with U the potential function already obtained in equation (1).

In the phase space (x, y, X, Y), the isosceles equilibria are

$$E_2 = (0, y_0, -y_0, 0), \quad E_4 = (0, -y_0, y_0, 0),$$

where $y_0 = \sqrt{r^2 - 1/4}$, with r the distance of the point to the end-points of the segment. Because of the symmetry of the problem, it is sufficient the analysis of one of these points, say E_2.

By means of the translation

$$x = \xi, \quad y = \eta + y_0, \qquad X = P_\xi - y_0, \quad Y = P_\eta$$

we put the origin at E_2, and the Hamiltonian in this set of variables is now

$$\mathcal{H} = \frac{1}{2}(P_\xi^2 + P_\eta^2) - (\xi P_\eta - \eta P_\xi) - \frac{y_0^2}{2} - \eta y_0 + U(\xi, \eta),$$

with

$$U(\xi, \eta) = -k \log\left(\frac{s(\xi, \eta) + 1}{s(\xi, \eta) - 1}\right),$$
$$s(\xi, \eta) = \sqrt{(\xi + 1/2)^2 + (\eta + y_0)^2} + \sqrt{(\xi - 1/2)^2 + (\eta + y_0)^2}.$$

Next, we have to expand the Hamiltonian in power series of the new variables in a neighborhood of the origin. With the help of an algebraic manipulator, we easily obtain

$$\mathcal{H} = \sum_{2 \le j} \mathcal{H}_j,$$

where each term $\mathcal{H}_j$ is an homogeneous polynomial of degree j.

In our case, we obtain

$$\mathcal{H}_2 = \frac{1}{2}(P_\xi^2 + P_\eta^2) - (\xi P_\eta - \eta P_\xi) + \frac{1}{2}(\alpha_{20}\xi^2 + \alpha_{02}\eta^2), \tag{13}$$

and the rest, till order 6,

$$\begin{aligned}
\mathcal{H}_3 &= \alpha_{12}\xi\eta^2 + \alpha_{03}\eta^3, \\
\mathcal{H}_4 &= \alpha_{40}\xi^4 + \alpha_{22}\xi^2\eta^2 + \alpha_{04}\eta^4, \\
\mathcal{H}_5 &= \alpha_{41}\xi^4\eta + \alpha_{23}\xi^2\eta^3 + \alpha_{05}\eta^5, \\
\mathcal{H}_6 &= \alpha_{60}\xi^6 + \alpha_{42}\xi^4\eta^2 + \alpha_{24}\xi^2\eta^4 + \alpha_{06}\eta^6.
\end{aligned} \tag{14}$$

$\mathcal{H}_2$	$\alpha_{20} = 1 - \dfrac{1}{4r^2}, \quad \alpha_{02} = -2 + \dfrac{1}{4r^2}$
$\mathcal{H}_3$	$\alpha_{21} = -\dfrac{3\sqrt{(4r^2-1)^3}}{16r^4}, \quad \alpha_{03} = \dfrac{96r^4 - 28r^2 + 3}{48r^4\sqrt{4r^2-1}}$
$\mathcal{H}_4$	$\alpha_{40} = \dfrac{-48r^4 + 32r^2 - 5}{128r^6}, \quad \alpha_{22} = \dfrac{3(64r^4 - 36r^2 + 5)}{64r^6},$ $\alpha_{04} = \dfrac{5 - 60r^2 + 256r^4 - 512r^6}{128r^6(4r^2-1)}.$
$\mathcal{H}_5$	$\alpha_{41} = \dfrac{175 - 2400r^2 + 12000r^4 - 25600r^6 + 19200r^8}{1280r^8\sqrt{(4r^2-1)^3}}$ $\alpha_{23} = \dfrac{-350 + 5000r^2 - 26400r^4 + 60800r^6 - 51200r^8}{1280r^8\sqrt{(4r^2-1)^3}}$ $\alpha_{05} = \dfrac{35 - 520r^2 + 2928r^4 - 7680r^6 + 10240r^8}{1280r^8\sqrt{(4r^2-1)^3}}$
$\mathcal{H}_6$	$\alpha_{60} = \dfrac{-63 + 532r^2 - 1360r^4 + 960r^6}{3072r^{10}}$ $\alpha_{42} = -\dfrac{5(-63 + 560r^2 - 1520r^4 + 1152r^6)}{1024r^{10}}$ $\alpha_{24} = \dfrac{15(-21 + 196r^2 - 576r^4 + 512r^6)}{1024r^{10}}$ $\alpha_{06} = \dfrac{63 - 1120r^2 + 7920r^4 - 28032r^6 + 51200r^8 - 49152r^{10})}{3072r^{10}(-1+2r)^2(1+2r)^2}$

Table 1: Coefficients of the Taylor series of the Hamiltonian in a neighborhood of the equilibrium E_2.

The coefficients α_{ij} depend on the parameters of the problem, either k or r, since both are related through $4r^3 - r - 4k = 0$. In Table 1 we present them in terms of r.

But the term $\mathcal{H}_2$ such as appears in Equation (13) still is not in the form required by Arnold's theorem. Thus, an extra work is required. Following Deprit and Deprit-Bartholomé's paper [4], we will construct a symplectic transformation to achieve our goal.

In the case here considered, that is, the isosceles point E_2 with $k_c < k$, the eigenvalues of the linearized equations are pure imaginary numbers (Eq. (10)), thus, we can consider the frequencies

$$\omega_1^2 = \frac{1}{2}\left(1 + \sqrt{4\delta^2 - 12\delta + 1}\right), \qquad \omega_2^2 = \frac{1}{2}\left(1 - \sqrt{4\delta^2 - 12\delta + 1}\right).$$

With these frequencies, we can define a set of parameters

$$a_i^2 = \frac{\omega_i^2 + 1 - \alpha_{02}}{\omega_i(\omega_1^2 - \omega_2^2)}, \qquad b_i^2 = \frac{\omega_i^2 + 1 - \alpha_{20}}{\omega_i(\omega_1^2 - \omega_2^2)}, \quad i = 1, 2,$$

such that the transformation $(\xi, \eta, P_\xi, P_\eta) \longrightarrow (\psi_1, \psi_2, \Psi_1, \Psi_2)$ given by

$$\begin{aligned}
\xi &= a_1\sqrt{2\Psi_1}\sin\psi_1 + a_2\sqrt{2\Psi_2}\sin\psi_2,\\
\eta &= b_1\sqrt{2\Psi_1}\cos\psi_1 - b_2\sqrt{2\Psi_2}\cos\psi_2,\\
P_\xi &= (a_1\omega_1 - b_1)\sqrt{2\Psi_1}\cos\psi_1 + (b_2 - a_2\omega_2)\sqrt{2\Psi_2}\cos\psi_2,\\
P_\eta &= (a_1 - b_1\omega_1)\sqrt{2\Psi_1}\sin\psi_1 + (a_2 - b_2\omega_2)\sqrt{2\Psi_2}\sin\psi_2,
\end{aligned} \tag{15}$$

is symplectic and rends the quadratic part (13) into the sought expression, namely,

$$\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2. \tag{16}$$

Details about how this transformation is obtained are given in [20].

The above transformation makes simpler the quadratic part $\mathcal{H}_2$, but there is a penalty to pay for; indeed, the rest of the Taylor terms (14) become rather complicated. However its expression can be easily obtained with the help of an algebraic manipulator.

Another requirement of the Arnold theorem is to have the Hamiltonian in normal form up to a certain order. At this point, let us recall that a given Hamiltonian

$$\mathcal{H} = \sum_{i\geq 2}\mathcal{H}_i$$

is in normal form up to a degree k, if $\mathcal{H}_j \in \ker L_2$, $\forall 2 \leq j \leq k$, where the operator $L_2(-)$ is the Lie derivative with respect to $\mathcal{H}_2$.

We carried out the normalization by means of the Lie-Deprit method [3], and we found for the first order normalized Hamiltonian the expression

$$\mathcal{H}' = \mathcal{H}'_2 + \mathcal{H}'_4 + \ldots,$$

with

$$\mathcal{H}'_2 = \omega_1 \Psi'_1 - \omega_2 \Psi'_2,$$

$$\mathcal{H}'_4 = A{\Psi'}_1^2 - 2B{\Psi'}_1{\Psi'}_2 + C{\Psi'}_2^2,$$

where the coefficients A, B and C are

$$\begin{aligned}
A &= \Big(-216\delta^2 + 3102\delta^3 - 7418\delta^4 + 7354\delta^5 - 2942\delta^6 - 404\delta^7 \\
&\quad +696\delta^8 - 144\delta^9 + \Delta^{1/2}\left(-54\delta^2 - 1344\delta^3 + 3460\delta^4\right. \\
&\quad \left.-4036\delta^5 + 2426\delta^6 - 696\delta^7 + 72\delta^8\right)\Big) / \\
&\quad \Big(18 - 456\delta + 3318\delta^2 - 5520\delta^3 + 3504\delta^4 - 960\delta^5 + 96\delta^6 \\
&\quad +\Delta^{1/2}\left(18 - 375\delta + 2211\delta^2 - 3513\delta^3 + 2199\delta^4 - 600\delta^5 + 60\delta^6\right)\Big), \\
B &= \Big(-17280\delta^4 + 432000\delta^5 - 2047680\delta^6 + 3609856\delta^7 - 3275776\delta^8 \\
&\quad +1678976\delta^9 - 480960\delta^{10} + 69888\delta^{11} - 3840\delta^{12}\Big) / \\
&\quad \Big(\left(-4 + 127\delta - 1064\delta^2 + 1541\delta^3 - 700\delta^4 + 100\delta^5\right) \\
&\quad \sqrt{1 - \Delta^{1/2}}(1 + \Delta^{1/2})^{\frac{5}{2}}(-1 + 6\delta - 2\delta^2 + \Delta^{1/2})\Big), \\
C &= \Big(216\delta^2 - 3102\delta^3 + 7418\delta^4 - 7354\delta^5 + 2942\delta^6 + 404\delta^7 \\
&\quad -696\delta^8 + 144\delta^9 + \Delta^{1/2}\left(-54\delta^2 - 1344\delta^3 + 3460\delta^4\right. \\
&\quad \left.-4036\delta^5 + 2426\delta^6 - 696\delta^7 + 72\delta^8\right)\Big) / \\
&\quad \Big(-18 + 456\delta - 3318\delta^2 + 5520\delta^3 - 3504\delta^4 + 960\delta^5 - 96\delta^6 \\
&\quad +\Delta^{1/2}\left(18 - 375\delta + 2211\delta^2 - 3513\delta^3 + 2199\delta^4 - 600\delta^5 + 60\delta^6\right)\Big),
\end{aligned}$$

with $\Delta = 1 - 12\delta + 4\delta^2$ (see Eq. (10)). Thus, we can now apply Arnold's theorem.

(N.B. For sake of simplifying the notation, we will omit the primes from here on.)

The quantity $D_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1)$ appearing in Arnold's theorem, usually named determinant, for $k = 2$ is

$$D_4 = A\omega_2^2 - 2B\omega_1\omega_2 + C\omega_1^2,$$

and for the values of A, B and C above obtained is

$$\begin{aligned}
D_4 &= \delta(-324 + 6219\delta + 9582\delta^2 - 221564\delta^3 + 491220\delta^4 \\
&\quad -499633\delta^5 + 282390\delta^6 - 90770\delta^7 + 15456\delta^8 - 1080\delta^9) / \\
&\quad 3(\delta - 3)^2(\delta - 1)(1 - 12\delta + 4\delta^2)(4 - 75\delta + 25\delta^2).
\end{aligned}$$

Let us remember that in our analysis $\delta \in (0, \delta_c) = (0, (3 - 2\sqrt{2})/2)$, then the function $D_4(\delta)$ presents two vertical asymptotes at the values $\delta_1 = (15 - \sqrt{209})/10$ and $\delta_c = 3/2 - \sqrt{2}$ (see Figure 2), and it has only one zero at $\delta_0 = 0.05243194999951975$.

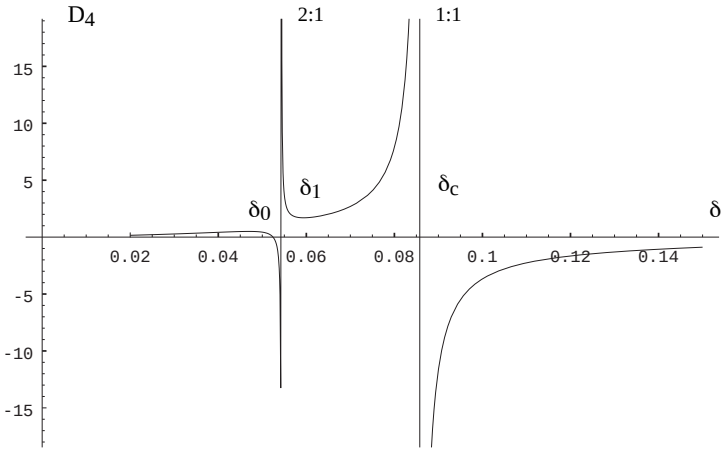

Figure 2: D_4 versus δ. We can see that in the interval $(0, \delta_c)$ of linear stability for the isosceles points, there are two singularities, δ_1 and δ_c, and only one zero δ_0.

According with Arnold's theorem of stability, we conclude that

Proposition 4 *The isosceles points are nonlinearly stable for*

$$\delta \in (0, \delta_0) \cup (\delta_0, \delta_1) \cup (\delta_1, \delta_c).$$

Since $D_4(\delta_0) = 0$, the stability for this value is not decided yet. We have to increase the order of the normalization until the obtaining of $\mathcal{H}_6(\Psi_1, \Psi_2)$, and then, see whether $D_6 = \mathcal{H}_6(\omega_2, \omega_1)$ is null or not. If $D_6 \neq 0$ we apply Arnold's theorem and conclude that the equilibrium is unstable; if $D_6 = 0$, we have to push the normalization two orderes more, and so on.

But this high order normalization is hard to obtain with symbolic coefficients. However, we are interested in the normalization not for whatever value of the parameter δ, but only for one specific value, namely δ_0. Hence, the coefficients are real numbers and we perform our Lie transformations in floating point arithmetic. By doing so, we obtain

$$\begin{aligned}
\mathcal{H}' = {} & 0.89941931767562\,\Psi_1 - 0.4370868231735232\,\Psi_2) \\
& - \tfrac{1}{2!}(0.0813546084234034\,\Psi_1^2 + 0.4845199550607618\,\Psi_1\Psi_2 \\
& \quad + 0.2162471063670814\,\Psi_2^2) \\
& + \tfrac{1}{4!}\,(1.18372785058136\Psi_1^3 + 255.91791007170106\Psi_1^2\Psi_2 \\
& \quad - 1212.23356486903499\Psi_1\Psi_2^2 + 179.37037131960825\Psi_2^3).
\end{aligned}$$

Thus, the frequencies for this Hamiltonian are

$$\omega_1 = 0.89941931767562, \qquad \omega_2 = 0.4370868231735232.$$

and the determinants are

$$D_4(\omega_2, \omega_1) = 9.5\ 10^{-15},$$

$$D_6(\omega_2, \omega_1) = 25.125375 \neq 0.$$

Proposition 5 *The isosceles points are nonlinearly stable for $\delta = \delta_0$.*

Still remains to analyze the values δ_1 and δ_c where $D_4(\delta)$ was singular. But the point δ_c was already proved to be linearly unstable, and hence it is unstable. For the case δ_1, we need another procedure, since for this value, the frequencies happen to be $\omega_1 = \sqrt{0.8}$ and $\omega_2 = \sqrt{0.2}$, that is to say, $\omega_1 = 2\omega_2$. Thus, we are in presence of the resonance 2:1, and in principle, we will meet small divisors in the normalization for these specific values. However, there are some results that allows the stability analysis for resonant cases. In this context, the authors [8] gave a geometrical way to investigate the nonlinear stability in the case of general resonances. Besides, there are several papers dealing with specific resonances, thus, we find in [15] the following result for the third order resonance:

Theorem 4 (Markeev) *If the real normal form of the Hamiltonian contains resonant terms in the form*

$$\mathcal{H} = 2\omega\Psi_1 - \omega\Psi_2 - \sqrt{\omega(A_{12}^2 + B_{12}^2)}\ \Psi_2\sqrt{\Psi_1}\cos(\psi_1 + 2\psi_2) + \mathcal{O}(\Psi_1^2 + \Psi_2^2),$$

if $A_{12}^2 + B_{12}^2 \neq 0$, then the equilibrium position is unstable.

By putting $\delta = \delta_1$, $\omega_1 = 2\sqrt{0.2}$ and $\omega_2 = \sqrt{0.2}$ into the transformation (15), the Hamiltonian becomes

$$\mathcal{H}_2 = \sqrt{0.2}(2\Psi_1 - \Psi_2) = \omega(2\Psi_1 - \Psi_2),$$

$$\begin{aligned}\mathcal{H}_3 = &(-2.08807\Psi_1 - 11.169\Psi_2)\sqrt{\Psi_1}\cos\psi_1 + 4.64257\Psi_1^{3/2}\cos 3\psi_1 \\ &+15.8737\sqrt{\Psi_1}\Psi_2\cos(\psi_1 - 2\psi_2) - 14.946\Psi_1\sqrt{\Psi_2}\cos(2\psi_1 - \psi_2) \\ &+(3.22232\Psi_1 + 11.4.30901\Psi_2)\sqrt{\Psi_2}\cos\psi_2 - 5.48251\Psi_2^{3/2}\cos 3\psi_2 \\ &+5.81053\Psi_1\sqrt{\Psi_2}\cos(2\psi_1 + \psi_2) - 0.142089\sqrt{\Psi_1}\Psi_2\cos(\psi_1 + 2\psi_2).\end{aligned}$$

By means of a Lie transformation, we can eliminate every term in $\mathcal{H}_3$, except the one in $\cos(\psi_1 + 2\psi_2)$, since this resonant term belongs to the kernel of L_2, and therefore, it must be in the normalized Hamiltonian. Hence, the normal form of the 2:1 resonant Hamiltonian is

$$\mathcal{H}' = \omega(2\Psi_1 - \Psi_2) - 0.142089\Psi_2\sqrt{\Psi_1}\cos(\psi_1 + 2\psi_2), \tag{17}$$

that is precisely in the form of the Markeev's theorem. Consequently, we conclude that, for the resonant case, the isosceles equilibrium is unstable.

With the precedent analysis, we can conclude

Theorem 5 *The dynamics determined by the gravitational action of a finite straight segment, rotating uniformly on a fixed plane about the perpendicular axis passing through its center of mass, has two pairs of equilibria, the colinear ones that are unstable everywhere, and the isosceles, that are stable for the parameter* $\delta \in (0, \delta_1) \cup (\delta_1, \delta_c)$*, and unstable for* $\delta \in \{\delta_1\} \cup [\delta_c, 1)$*.*

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