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Abstract

In this paper, we continue the research on groups whose subgroups are normal or have finite conjugacy classes already considered in a previous one (see [8]). Module theoretical methods are used in our study.

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1 Introduction

One of the most important topic in group theory is the study of the influence of given families of subgroups of a group on the structure of the group itself. The first result in this direction was obtained by R. Dedekind; in its classical paper

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[4] he described finite groups all whose subgroups are normal. Dedekind's description was extended to arbitrary groups by R. Baer [1]. Groups with only normal subgroups are known as *Dedekind group*.

We focus our analysis on the family S_G of all non-normal subgroups of a group G. Thus, Dedekind groups are exactly the groups G for which $S_G = \emptyset$. Since Dedekind, many authors have studied groups G for which the members of the family S_G satisfy some finiteness condition as the minimal (S. N. Chernikov [3]), the maximal (L. A. Kurdachenko, N. F. Kuzenny and N. N. Semko [7]) and many others or in which the members of S_G satisfy some property like being finite (Chernikov [2]) or being abelian (G. M. Romalis and N. F. Sesekin [12, 13, 14]). The fairest extension both of the class of all finite groups and of the class of abelian groups is the class of the FC–groups or groups with finite conjugacy classes. Nowadays, the theory of FC–groups forms a well developed topic and represents one of the main branches of the theory of groups with finiteness conditions (see [5, 16]).

Recently, the authors have considered the class of all groups in which every non-normal subgroup have finite conjugacy classes or is an FC–group (see [8]). They prove, among other results, that a soluble–by–finite group whose non-normal subgroups have finite conjugacy classes is a locally FC-group and its commutator subgroup is an FC-group. Moreover, the groups in this class whose commutator subgroup is Chernikov are investigated.

The arguments used in [8] are group-theoretical in their whole. The aim of this paper is to go on with this research but making use of ideas coming from the Theory of Modules. Roughly speaking, we proceed as follows. Suppose we have a group G which contains an abelian normal subgroup A. Then A can be made into a $\mathbb{Z}G$–module, where G acts on A by conjugation. If A is further an elementary abelian p–subgroup (which frequently happens) and $g \in G \backslash C_G(A)$, then we think of A as an $\mathbb{F}_p\langle y\rangle(G)$–module, where F_p is the prime field of p elements and $\langle y\rangle$ is an infinite cyclic group acting on A by $a \cdot y = a^g, a \in A$. As the rings $\mathbb{Z}$ and $\mathbb{F}_p\langle y\rangle$ are principal ideal domains, they share many module-theoretical properties and we may make use of the powerful of the theory of modules to deduce deeper properties on the structure of G. We develop first the general ideas from the Theory of Modules we need and then we prepare the more specific properties we are going to apply. We call this study as the research of *the local structure of a group whose non-normal subgroups have finite conjugacy classes*. Our main results are:

Theorem A *Let G be a metabelian group whose non-normal subgroups have finite conjugacy classes. Then either*

(1) G is locally nilpotent and in this case

(1.1) G is a Fitting group, or

(1.2) the commutator subgroup of G is Chernikov.

or

(2) G is an FC–group, or

(3) the commutator subgroup of G is Chernikov.

Theorem B *Let G be a soluble–by–finite group whose non-normal subgroups have finite conjugacy classes in which $G/[G,G]$ finitely generated. Then either*

(1) G contains an FC–subgroup of finite index, or

(2) the commutator subgroup $[G,G]$ of G is a central–by–finite Chernikov group.

Most of our notation is standard and can be found in [10]. We shall use the monograph [15] as a general reference for module-theoretical methods.

2 Some technical results

Let G be a group including an abelian normal subgroup A. Then A can be made into a $\mathbb{Z}G$–module, where G acts on A by conjugation, that is $a{\cdot}g = a^g$ for every $a \in A$ and $g \in G$. Further, if A is an elementary abelian p–subgroup for some prime p, then we think of A as an $\mathbb{F}_pG$–module. In this case, if $g \in G \setminus A$, A is seen as an $\mathbb{F}_p\langle y\rangle$–module, where $\langle y\rangle$ is an infinite cyclic and y acts on A by $ay = a^g,\ a \in A$. Since both rings $\mathbb{Z}$ and $\mathbb{F}_p\langle y\rangle$ are principal ideal domains, they share many module-theoretical arguments, which we are developing now. A minor remark concerning notation. When A is looked as a module, we write additively its inner law and by juxtaposition its outer one. However, when A is viewed as a subgroup of G, we keep its inner multiplicative law. Though this can seem confusing, it is widely employed as it is clear enough in any case.

Let J be a principal ideal domain, and put $\mathrm{Spec}(J)$ to denote the set of maximal ideals of J. Let A be a J–module. Then the set

$$T_J(A) = \{a \in A |\ \mathrm{Ann}_J(a) \neq \{0\}\}$$

is a submodule of A called the *J–periodic part of A*. The module A is called *J–periodic* if $A = T_J(A)$ and *J–torsion-free* if $T_J(A) = \{0\}$. If $a \in T_J(A)$, then $\mathrm{Ann}_J(A) = P_1^{k(1)} \cdots P_s^{k(s)}$ for some $P_1, \cdots, P_s \in \mathrm{Spec}(J)$ and $k(1), \cdots, k(s) \in \mathbb{N}$. Then we put

$$\Pi_J(a) = \{P_1, \cdots, P_s\} \text{ and } \Pi_J(A) = \bigcup_{a \in T_J(A)} \Pi_J(a).$$

If I is an ideal of J, then $AI = \{a \in A \mid aI^n = \{0\}$ for some $n \in \mathbb{N}\}$ is a J–submodule of A called *the I–component of A*. Also $\Omega_{I,n}(A) = \{a \in A \mid aI^n = \{0\}\}$ is another J–submodule of A such that $\Omega_{I,n}(A) \leq \Omega_{I,n+1}(A)$, for every $n \in \mathbb{N}$, and $A_I = \bigcup_{n\in\mathbb{N}} \Omega_{I,n}(A)$. As for abelian groups, we may prove that

$$T_J(A) = \bigoplus_{P \in \Pi_J(A)} A_P.$$

Let $P \in \mathrm{Spec}(J)$. Choose an element $y_{(P)}$ such that $P = y_{(P)}J$. The mapping $x \mapsto xy_{(P)},\ x \in J$ is an injective J–homomorphism and so J and $y_{(P)}J = P$ are isomorphic J–modules. It follows that $J/P^n = J/y_{(P)}^n J \cong y_{(P)}J/y_{(P)}^{n+1}J = P/P^{n+1}$, what means that $\{J/P^n \mid n \in \mathbb{N}\}$ is an injective family of J–modules. The corresponding injective limit

$$C_{P^\infty} = \varinjlim_{n\in\mathbb{N}} J/P^n$$

is a module called *the Prüfer P–module*. It is easy to see that $C_{P^\infty} = \sum_{n\in\mathbb{N}} a_n J$, where $a_1 y_{(P)} = 0$ and $a_{n+1}y_{(P)} = a_n$ for every $n > 1$. It follows that every proper J–submodule of C_{P^∞} coincides with some $a_n J$.

Let $0 \neq x \in J$. The module A is said to be an *x– divisible* if $A = Ax$. If A is x–divisible for each $0 \neq x \in J$, then A is called *J–divisible.*

Lemma 2.1 *Let G be a group which includes an abelian normal p-subgroup A, for some prime p, and let g be an element of infinite order of G such that $\langle g\rangle \cap C_G(A) = \langle 1\rangle$. If the $\mathbb{F}_{(P)}\langle g\rangle$-module $A_1 = \Omega_1(A)$ is periodic, then the $\mathbb{Z}\langle g\rangle$-submodule $C := c\mathbb{Z}\langle g\rangle = \langle c\rangle^{\langle g\rangle}$ is finite for each element $c \in A$.*

Proof – Put $J = \mathbb{F}_p\langle g\rangle$ and $R = \mathbb{Z}\langle g\rangle$. We have $\mid c \mid = p^k$, for some $k \in \mathbb{N}$. If $k = 1$ then $C = cJ \cong J/I$, where $I = \mathrm{Ann}_J(c) \neq \{0\}$. Since every ideal of J has finite index, C is finite, as required. Let $k > 1$ and proceed by induction on k. The map $\phi : u \to pu,\ u \in C$, is an R-endomorphism of C such that $\mathrm{Im}\ \phi = pC$ and $\mathrm{Ker}\ \phi = \Omega_1(C)$. Moreover, $pC = pCR$ is a cyclic R-submodule and $\mid pc \mid = p^{k-1}$, so that, by induction, pC is finite. Furthermore, $pC = \mathrm{Im}\ \phi \cong C/\mathrm{Ker}\ \phi = C/\Omega_1(C)$, and in particular $C/\Omega_1(C)$ is finite. On the other hand, R is noetherian, so that every finitely generated R-module is likewise noetherian. It follows that the submodule $\Omega_1(C)$ is finitely generated. We have already noted that every cyclic J-submodule of A is finite, so that $\Omega_1(C)$ is finite. Thus C is finite, and the proof is completed. □

Let G be an abelian group and A a $\mathbb{Z}G$-module. Put

$$Z_{\mathbb{Z}G}(A) = \{a \in A \mid ax = a, \forall x \in G\}.$$

It is easy to see that $Z_{\mathbb{Z}G}(A)$ is a $\mathbb{Z}G$-submodule of A, called the *$\mathbb{Z}G$-centre* of A. We construct the *upper $\mathbb{Z}G$-central series* of A,

$$\{0\} = C_0 \leq C_1 \ldots C_\alpha \leq C_{\alpha+1} \leq \ldots,$$

by the following rules:

$$C_1 = Z_{\mathbb{Z}G}(A),\ C_{\alpha+1}/C_\alpha = Z_{\mathbb{Z}G}(A/C_\alpha),$$

if $\alpha < \gamma$, and

$$C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha,$$

if λ is a limit ordinal. The last term C_γ of this series is called the *upper $\mathbb{Z}G$–hypercentre* of A, and it is clear that $Z_{\mathbb{Z}G}(A/C_\gamma) = \{0\}$.

Let B and C be $\mathbb{Z}G$-submodules of A such that $B \leq C$. The factor module C/B is said to be *G-central* (respectively *G-eccentric*) if $C_G(C/B) = G$ (respectively $C_G(C/B) \neq G$).

Lemma 2.2 *Let G be an abelian group, and suppose that A is a $\mathbb{Z}G$-module such that $C_G(A) = \langle 1 \rangle$. Let B be a $\mathbb{Z}G$-submodule satisfying the following conditions:*

(1) B and A/B are simple $\mathbb{Z}G$-modules; and

(2) B is G-central and A/B is G-eccentric.

Then there exists a $\mathbb{Z}G$-submodule C such that $A = B \oplus C$.

Proof – Let $1 \neq g \in G$. Since G is abelian, the map $\phi : a \to a(g-1)$, $a \in A$, is a $\mathbb{Z}G$-endomorphism of A. In particular, Im ϕ and Ker ϕ are $\mathbb{Z}G$-submodules of A. Moreover, Ker $\phi \geq B$. If Ker $\phi \neq B$, then Ker $\phi = A$, because A/B is simple. Hence Ker $\phi = B$, that is $A/B \cong$ Im $\phi := C$ is a $\mathbb{Z}G$-submodule of A such that $C_G(C)$ is properly contained in G. Since C is also simple, $C \cap B = \{0\}$. The simplicity of A/B again yields $C + B = A$, that is $A = B \oplus C$. □

Lemma 2.3 *Let G be an abelian group, and let A be a $\mathbb{Z}G$-module such that $C_G(A) = \langle 1 \rangle$ and B a $\mathbb{Z}G$-submodule satisfying the following conditions:*

(1) B and A/B are simple $\mathbb{Z}G$-modules;

(2) A/B is G-central and B is G-eccentric.

Then there exists a $\mathbb{Z}G$-submodule C such that $A = B \oplus C$.

Proof – Let $1 \neq g \in G$. Again the map $\phi : a \to a(g-1)$, $a \in A$, is a $\mathbb{Z}G$-endomorphism of A, so that Im ϕ and Ker ϕ are $\mathbb{Z}G$-submodules of A. Since $C_G(A/B) = G$, then Im $\phi \leq B$. If Im $\phi = \{0\}$, then $g \in C_G(A) = \langle 1 \rangle$, a contradiction. Hence Im $\phi \neq \{0\}$ and so Im $\phi = B$. The factor $A(g-1)/B(g-1)$ is a $\mathbb{Z}G$-homomorphic image of A/B and $C_G(A(g-1)/B(g-1)) = G$. As B is simple, $A(g-1) =$ Im $\phi = B$ and, as $C_G(B) \neq G$, we obtain $A(g-1) = B(g-1)$. Hence for every $a \in A$ we have $a(g-1) = b(g-1)$, for some $b \in B$. It follows that $(a-b)(g-1) = 0$, that is $a - b \in C_G(g)$. Then $A = B + C_A(g)$. Since $C_A(g) =$ Ker ϕ, $C := C_A(g)$ is a $\mathbb{Z}G$-submodule of A. As B is simple, either $B \cap C = \{0\}$ or $B \leq C$. In the latter, $A = C$ and therefore $g \in C_G(A) = \langle 1 \rangle$. This contradiction shows that $B \cap C = \{0\}$ and so $A = B \oplus C$, as required. □

Let G be an abelian group and A a $\mathbb{Z}G$-module. A submodule C is said to be *G-hypereccentric* if C has an ascending series

$$\{0\} = C_0 \leq C_1 \leq \ldots C_\alpha \leq C_{\alpha+1} \leq \ldots \leq C_\gamma = C$$

of $\mathbb{Z}G$-submodule of A such that $C_{\alpha+1}/C_\alpha$ is a simple $\mathbb{Z}G$-module and a G-eccentric factor for every $\alpha < \gamma$.

Corollary 2.4 *Let G be an abelian group. Suppose that A is a $\mathbb{Z}G$-module such that every finitely generated $\mathbb{Z}G$-submodule of A has finite composition series. If C is the upper $\mathbb{Z}G$-hypercentre of A, then*

(1) $A = C \oplus E$, where E is a maximal G-hypereccentric $\mathbb{Z}G$-submodule of A;and

(2) E includes every G-hypereccentric $\mathbb{Z}G$-submodule of A.

In particular, this decomposition is unique.

Proof – There is no loss if we assume $C_G(A) = \langle 1 \rangle$.

(1) Clearly the $\mathbb{Z}G$-module A has a local system of finitely generated $\mathbb{Z}G$-submodules $\{A_\lambda \mid \lambda \in \Lambda\}$. Every A_λ has finite $\mathbb{Z}G$-composition series and, by Lemma 2.2 and Lemma 2.3, $A_\lambda = C_\lambda \oplus E_\lambda$, where C_λ is the upper $\mathbb{Z}G$-hypercentre of A_λ (so that $C_\lambda = C \cap A_\lambda$) and E_λ is a maximal G-hypereccentric $\mathbb{Z}G$-submodule of A_λ. Suppose that $A_\lambda \leq A_\mu$. Since

$$A_\lambda/(A_\lambda \cap E_\mu) \cong_{\mathbb{Z}G} A_\lambda E_\mu/E_\mu \leq A_\mu/E_\mu \cong_{\mathbb{Z}G} C_\mu,$$

we obtain that $A_\lambda E_\mu$ is a maximal G-hypereccentric $\mathbb{Z}G$-submodule of A_λ. Therefore $A_\lambda \cap E_\mu = E_\lambda$. Put $E = \bigcup_{\lambda \in \Lambda} E_\lambda$. Then E is a $\mathbb{Z}G$-submodule of A such that $E \cap A_\lambda = E_\lambda$, for each $\lambda \in \Lambda$. By construction, E is a G-hypereccentric $\mathbb{Z}G$-submodule of A. On the other hand, since $A_\lambda = C_\lambda \oplus E_\lambda = (C \cap A_\lambda) \oplus (E \cap A_\lambda)$, it follows that $A = C \oplus E$. In particular, E is a maximal G-hypereccentric $\mathbb{Z}G$-submodule of A.

(2) Let now B be a G-hypereccentric $\mathbb{Z}G$-submodule of A. If $(B + E)/E$ is non-zero, it includes a non-zero simple $\mathbb{Z}G$-submodule D/E. Since $(B + E)/E \cong B/(B + E)$, then D/E is isomorphic with some simple $\mathbb{Z}G$-factor of B. It follows that $C_G(D/E) \neq G$. On the other hand, $(B+E)/E \leq A/E \cong C$, that is $C_G(D/E) = G$. This contradiction shows that $B \leq E$. □

Corollary 2.5 *Let G be a group and A be a normal abelian p-subgroup, for some prime p, such that $G/C_G(A)$ is abelian, Moreover, let g be an element of G which does not belong to $C_G(A)$. If the $\mathbb{F}_p\langle g \rangle$-module $\Omega_1(A)$ is periodic, then A includes G-invariant subgroups C and E such that $A = C \oplus E$, where C is the upper $\mathbb{Z}\langle g \rangle$-hypercentre of A and E is the unique maximal $\langle g \rangle$-hypereccentric $\mathbb{Z}G$-submodule of A.*

Proof – If g has infinite order and $\langle g \rangle \cap C_G(A) = \langle 1 \rangle$, then Lemma 2.1 yields that $\langle a \rangle^{\langle g \rangle}$ is finite, for every element a of A. If $g^n \in C_G(A)$, for some $n \in \mathbb{N}$, then it is clear that $\langle a \rangle^{\langle g \rangle}$ is also finite. In particular, every finitely generated $\mathbb{Z}G$-submodule of A has finite composition series. Hence we may apply Corollary 2.4 to get the decomposition. Since $G/C_G(A)$ is abelian, C is a $\mathbb{Z}G$-submodule. Applying Corollary 2.4 again, every $\langle g \rangle$-hypereccentric $\mathbb{Z}\langle g \rangle$-submodule of A lies in E and so E is also G-invariant. □

Lemma 2.6 *Let G be a group, and assume that G includes a normal elementary abelian p-group A, where p is a prime. Let g be an element of infinite order of G such that $\langle g\rangle \cap C_G(A) = \langle 1\rangle$. If the $\mathbb{F}_p\langle g\rangle$-module A is periodic and contains a non-zero divisible $\mathbb{F}_p\langle g\rangle$-submodule, then there exist an element $1 \neq x \in \langle g\rangle$ and an $\langle x\rangle$-invariant subgroup U of A, such that $U\langle x\rangle$ is not normal and $C_{\langle x\rangle}(U) = \langle 1\rangle$.*

Proof – Put $J = \mathbb{F}_p\langle g\rangle$, and let D be the J-divisible part of A. Since A is J-periodic, then $D = \bigoplus_{\lambda\in\Lambda} D_\lambda$, where D_λ is a Prüfer P_λ-submodule, for some $P_\lambda \in \mathrm{Spec}(J)$, $\lambda \in \Lambda$. Suppose that $\mid \Lambda \mid > 1$, and consider two indexes λ and μ, such that $\lambda \neq \mu$. Since D_λ and D_μ are J-divisible, it follows that $\mathrm{Ann}_J(D_\lambda) = \{0\} = \mathrm{Ann}_J(D_\mu)$. In particular, we have $C_{\langle g\rangle}(D_\lambda) = C_{\langle g\rangle}(D_\mu) = \langle 1\rangle$. Therefore we can choose an element $d \in D_\lambda$ such that $d^g = d_1 \neq d$. Thus $[d, g] = d^{-1}d_1 \neq 1$ and $d^{-1}gd = d^{-1}d_1g^{-1}$. On the other hand, since $D_\lambda \cap D_\mu = \langle 1\rangle$, then $d^{-1}d_1g^{-1} \notin D_\mu\langle g\rangle$. It follows that $D_\mu\langle g\rangle$ is not normal, so that we may put $U = D_\mu$ and $x = g$.

Suppose now that D is a Prüfer P-module, for some $P \in Spec(J)$. Let $L := \Omega_{p,1}(D)$. Then $L \cong J/P$. Since every non-zero ideal of J has finite index, L is finite. Put $\langle z\rangle = C_{\langle g\rangle}(L)$, and choose an element $0 \neq a_1 \in L$. Then $a_1(z-1) = 0$. Since D is J-divisible, there exists an element $a_2 \in D$ such that $a_2(z-1) = a_1$. In a similar way, we choose a set of elements $\{a_n \mid n \geq 3\}$ satisfying the following conditions: $a_3(z-1) = a_2$ and $a_{n+1}(z-1) = a_n$, for any $n \in \mathbb{N}$. In other words, $Q = \langle a_n \mid n \in \mathbb{N}\rangle$ is a Prüfer S-module, where $S = \mathbb{F}_p\langle z\rangle$. Let $u_1 = a_1$ and $v_1 = a_2$. Again we can choose sets $\{u_n \mid n \in \mathbb{N}\}$ and $\{v_n \mid n \in \mathbb{N}\}$, such that $u_1(x-1) = 0 = v_1(x-1)$, $u_{n+1}(x-1) = u_n$ and $v_{n+1}(x-1) = v_n$, for any $n \in \mathbb{N}$. Then both subgroups U and V are $\langle x\rangle$-invariant and $C_{\langle x\rangle}(U) = C_{\langle x\rangle}(V) = \langle 1\rangle$. Pick an element $w \in V$ such that $w^x = w_1 \neq 1$. We have $[w, x] = w^{-1}w_1 \neq 1$ and $w^{-1}x^{-1}w = w^{-1}w_1x^{-1}$. On the other hand, since $U \cap V = \langle 1\rangle$, it follows that $w^{-1}w_1x^{-1} \notin U\langle x\rangle$. In particular, the subgroup $U\langle x\rangle$ is not normal. □

Proposition 2.7 *Let G be a group which includes a periodic normal subgroup A, and take $g \in G\backslash C_G(A)$. If B is a $\langle g\rangle$-invariant $\langle g\rangle$- hypereccentric subgroup of A, then $[g, B] = B$.*

Proof – Since B is $\langle g\rangle$–hypereccentric, it has an ascending series

$$\{1\} = B_0 \leq B_1 \leq \ \cdots \ \leq B_\alpha \pounds B_{\alpha+1} \leq \ \cdots B_\gamma = B$$

of $\langle g\rangle$-invariant subgroups satisfying the following conditions for every $\alpha < \gamma$:

(i) $B_{\alpha+}1/B_\alpha$ is a chief $\langle g\rangle$–factor, and

(ii) $B_{\alpha+1}/B_\alpha$ is a $\langle g\rangle$–eccentric factor.

The subgroup $[g, B_1]$ is $\langle g\rangle$-invariant and therefore either $[g, B_1] = \langle 1\rangle$ or $[g, B_1] = B_1$. But $[g, B1] = \{1\}$ implies $\langle g\rangle = C_{\langle g\rangle}(B_1)$, what is impossible. Hence $[g, B_1] = B_1$. In the same way $[g, B_2/B_1] = B_2/B_1$ and it follows that $[g, B_2] = B_2$. Applying transfinite induction, we obtain that $[g, B] = B$ as required. □

3 The local structure of an $\mathfrak{X}$–group

Throughout this section, G is an group whose non-normal subgroups have finite conjugacy classes having a periodic abelian normal subgroup A and $g \in G \setminus C_G(A)$. Our aim is to study the structure of A as a $\mathbb{Z}\langle g\rangle$–module, following the approach set out and developed in the above. We note that in the majority of our results A has prime exponent p (that is, A is an elementary abelian p–subgroup) so that A can be further viewed as an $\mathbb{F}_p\langle g\rangle$–module.

Lemma 3.1 *Let G be a group whose non-normal subgrous have finite conjugacy classes and suppose that A is an elementary abelian p–subgroup of G. Suppose that $g \in G \setminus C_G(A)$ has infinite order and $\langle g\rangle \cap C_G(A) = \langle 1\rangle$. Then A is $\mathbb{F}_p\langle g\rangle$–periodic.*

Proof – Put $J = \mathbb{F}_p\langle g\rangle$. Suppose that A contains an element c such that $\mathrm{Ann}_J(a) = \{0\}$. We have $C = cJ \cong J$. If we put $x = g^2$, then $J = \mathbb{F}_p\langle x\rangle \oplus \mathbb{F}_p\langle g\rangle$ and therefore $C = C_1 \oplus C_1 g$, where $C_1 = \langle c\rangle\mathbb{F}_p\langle x\rangle$. Clearly the subgroups C_1 and $C_2 = C_1 g$ are both $\langle x\rangle$-invariant, and we have $c^x = c_1 \neq c$, that is $c^{-1}x^{-1}cx = c^{-1}c_1 \neq 1$ and $c^{-1}x^{-1}c = c^{-1}c_1x^{-1}$. Since $C_1 \cap C_2 = \{1\}$, $c^{-1}c_1x^{-1} \notin C_2\langle x\rangle$. It follows that $C_2\langle x\rangle$ is an FC–subgroup. Therefore $x^n \in C_G(C_2)$ for some $n \in \mathbb{N}$. This is a contradiction because C_2 is $\mathbb{F}_p\langle x\rangle$-torsion-free. The lemma is proved. □

Lemma 3.2 *Let G be a group whose non-normal subgroups have finite conjugacy classes that includes an elementary abelian p–subgroup A and let $g \in G \setminus C_G(A)$. Suppose that there is a decomposition $A = Dr_{\lambda\in\Lambda}A_\lambda$, where every A_λ is a minimal $\langle g\rangle$–invariant subgroup, $\lambda \in \Lambda$. Then there exists a finite subset $\Gamma \subseteq \Lambda$ such that $A_\lambda \leq C_{\langle g\rangle}(A)$, for every $\lambda \in \Lambda \setminus \Gamma$.*

Proof – If the order of g is infinite, then $\langle g\rangle \cap A = \{1\}$. If $|g|$ is finite, then there is a finite subset $\Delta \leq \Lambda$ such that $\langle g\rangle \cap A \leq \mathrm{Dr}_{\lambda\in\Delta}A_\lambda$. In the latter, $A_\lambda \leq C_{\langle g\rangle}(A)$ for every $\lambda \in \Lambda \setminus \Delta$ and we are done. Therefore we may assume that there is an index $\lambda \in \Lambda \setminus \Delta = \Sigma$ such that $C_{\langle g\rangle}(A_\lambda) \neq \langle g\rangle$. Put $B = \mathrm{Dr}_{\mu\in\Sigma\setminus\{\lambda\}}A_\mu$. By [8, Lemma 2.8], $|B : C_B(\langle g\rangle)|$ is finite. Hence there is a finite subset $\Gamma \supseteq \Delta$ such that $A_\lambda \leq C_{\langle g\rangle}(A)$ for each $\lambda \in \Lambda \setminus \Gamma$. □

Theorem 3.3 *Let G be a group whose non-normal subgroups have finite conjugacy classes that includes a periodic abelian normal subgroup A and let $g \in G \setminus C_C(A)$.*

(1) If A has prime exponent p, then the maximal $\langle g\rangle$–hypereccentric $\mathbb{F}_p\langle g\rangle$-submodules of A are finite and the maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$–submodules of A are Chernikov.

(2) The maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$–submodules of A are Chernikov.

Proof – (1) Suppose first that the order of g is infinite so that $\langle g\rangle \cap A = \langle 1\rangle$. By Lemma 3.1, the J–module A is periodic, where $J = \mathbb{Z}_p\langle g\rangle$. Let $P \in \mathrm{Spec}(J)$ such that $P \neq (g-1)J$ and let B be a P–component of A. Put $C = \Omega_{P,1}(B)$. Then $C = \bigoplus_{\lambda\in\Lambda} C_\lambda$, where every C_λ is a simple J–submodule such that $\mathrm{Ann}_J(C_\lambda) = P$. In particular $C_{\langle g\rangle}(C_\lambda) \neq \langle g\rangle$ for every $\lambda \in \Lambda$. By Lemma 3.2, the set Λ is finite. It follows that $B = \bigoplus_{\lambda\in\Lambda} E_\lambda$ where E_λ is either a cyclic J–submodule or a Prüfer P–submodule ([9, Lemma 5.6 and Theorem 5.7]). Suppose that some E_λ is divisible. By Lemma 2.6, the subgroup $\langle g\rangle$ contains an element x such that A includes an $\langle x\rangle$–invariant subgroup U such that $U\langle x\rangle$ is not normal and $C_{\langle x\rangle}(U) = \langle 1\rangle$. Then $U\langle x\rangle$ is an FC–subgroup and so there is some $m \in \mathbb{N}$ such that $x^m \in C_{\langle x\rangle}(U)$. This contradiction shows that no E_λ is divisible. In other words, every submodule E_λ has to be cyclic and hence finite. Since Λ is finite, B is finite too. If $\Pi = \Pi_J(A)$, we have $A = \bigoplus_{P\in\Pi} A_P$, where A_P is the P–component of A. If $P \neq \langle g-1\rangle J$, then $C = \Omega_{P,1}(A_P) = \bigoplus_{\lambda\in\Lambda} C_\lambda$, where C_λ is a simple J–submodule such that $\mathrm{Ann}_J(C_\lambda) = P$. In particular $C_C(C_\lambda) = \langle 1\rangle$ for every $\lambda \in \Lambda$. It follows that $C_{A_P}(g) = \langle 1\rangle$ whenever $P \neq \langle g-1\rangle J$. Let $\Sigma = \Pi \setminus \{(g-1)J\}$. If $\Sigma = \{P\}$, by the arguments given in the above paragraph, we are done. Therefore suppose that $\Sigma \neq \{P\}$ and let $\Delta = \Sigma \setminus \{P\}$ and $L = \bigoplus_{Q\in\Delta} A_Q$. Since g has infinite order, $\langle g\rangle \cap A = \langle 1\rangle$. By [8, Lemma 2.8], $|L : CL(g)|$ is finite and, since $C_L(g) = \langle 1\rangle$, L is finite. It follows that $A_1 := \bigoplus_{P\in\Sigma} A_P$ is finite. But A_1 is exactly a maximal $\langle g\rangle$–hypereccentric $\mathbb{F}_p\langle g\rangle$-submodule of A.

Consider now the case in which $|g|$ is finite or $|g|$ is infinite but $g^k \in C_G(A)$ for some $k \in \mathbb{N}$. Put $J = \mathbb{F}_p\langle y\rangle$, where $\langle y\rangle$ is an infinite cyclic group and think of A as a J–module as explained above. We have $\mathrm{Ann}_J(A) = I \neq \{0\}$. Again we have that $A = \bigoplus_{P\in\Pi} A_P$ where the set $\Pi = \Pi_J(A)$ is finite. Let $B = A_P$. Then $B = \bigoplus_{\lambda in\Lambda} B_\lambda$, where B_λ is a cyclic J–submodule for every $\lambda \in \Lambda$([15, Theorem 6.14]). As above, we may show that $\Omega_{P,1}(A_P)$ is finite and then the set Λ is likewise finite. Thus it follows that A_P is finite. Since $\Sigma = \Pi \setminus \{(y-1)J\}$ is finite, it follows that $A_2 := \bigoplus_{\lambda\in\Lambda} A_P$ is finite. But A_2 is exactly a maximal $\langle g\rangle$–hypereccentric $\mathbb{F}_p\langle g\rangle$–submodule of A.

Let now E be a maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$–submodule of A. Applying the result shown above, $\Omega_1(E)$ is finite and therefore E is Chernikov.

(2) If E is a maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$-submodule of A, then $E = \mathrm{Dr}_{P\in\Pi(E)} E_p$, where E_p is a Sylow p-subgroup of E. By (1), E_p is Chernikov and, by [8, Lemma 2.8], $\Pi(E)$ is finite. Thus E is Chernikov. □

Lemma 3.4 *Let G be a group whose non-normal subgroups have finite conjugacy classes that includes an elementary abelian p–subgroup A and let $g \in G\setminus C_G(A)$. If C is the upper $\mathbb{F}_p\langle g\rangle$–hypercentre of A, then $|C : C_C(g)|$ is finite.*

Proof – Suppose first that the order of g is infinite and $\langle g\rangle \cap A = \langle 1\rangle$. If $J = \mathbb{F}_p\langle g\rangle$, then C is the J_1–component of A, where $J_1 := (g-1)J$. As for abelian groups, C includes a pure J–submodule $B = \bigoplus_{\lambda\in\Lambda} B_\lambda$, where B_λ is a cyclic J–submodule for every $\lambda \in \Lambda$, such that C/B is divisible ([6, Section

33]). If $B \neq C_B(g)$, then there is an index $\nu \in \Lambda$ such that $C_B(g)$ does not include B_ν. Put $E = \bigoplus_{\lambda \neq \nu} B_\lambda$. By [8, Lemma 2.8], $|E : C_E(g)|$ is finite and so is $|B : C_B(g)|$. In other words, $\mathrm{Ann}_J(B) \neq \{0\}$. As in [6, Theorem 27.5], we may decompose $C = B \oplus D$. Since $D \cong C/B$, D is divisible and, as other times, we show that $D = \{0\}$, that is $C = B$. Thus $|C : C_C(g)|$ is finite.

Consider now the case in which $|g|$ is finite or $|g|$ is infinite but $g^k \in C_G(A)$ for some $k \in \mathbb{N}$. Put $J = \mathbb{F}_p\langle y\rangle$, where $\langle y\rangle$ is an infinite cyclic group, so that, as indicated, A is a J–module and $\mathrm{Ann}_J(A) \neq \{0\}$. In particular, $\mathrm{Ann}_J(C) \neq \{0\}$. Then $C = \bigoplus_{\lambda \in \Lambda} B_\lambda$, where B_λ is a cyclic J–submodule for every $\lambda \in \Lambda$ (see [15, Theorem 6.14]). As above we obtain that $|C : C_C(g)|$ is finite. $\square$

Theorem 3.5 *Let G be a group whose non-normal subgroups have finite conjugacy classes that includes a periodic abelian normal subgroup A and let $g \in G \setminus C_G(A)$. If C is the upper $\mathbb{Z}\langle g\rangle$–hypercentre of A, then $[g, C]$ is Chernikov.*

Proof – If the order of g is finite, then the order of $S := A \cap \langle g\rangle$ is finite too. Otherwise $S = A \cap \langle g\rangle = \langle 1\rangle$. Since S is $\langle g\rangle$–invariant, we may replace A by A/S to assume that $A \cap \langle g\rangle = \langle 1\rangle$.

Assume first that A has exponent prime p. If $[g, C] = \langle 1\rangle$, then we are done. Therefore we suppose that there is some $c \in C$ such that $[g, c] \neq 1$. Put $C_1 = \langle c\rangle^{\langle g\rangle}$ so that C_1 is finite and $[g, C_1] \neq \langle 1\rangle$. Let $L = \Omega_1(C)$ be the lower layer of C. Put $J = \mathbb{F}_p\langle y\rangle$, where $\langle y\rangle$ is an infinite cyclic group, so that A is a J–module. By Lemma 3.4, $\mathrm{Ann}_J(C) \neq \{0\}$. Then $C = \bigoplus_{\lambda \in \Lambda} B_\lambda$, where B_λ is a cyclic J–submodule for every $\lambda \in \Lambda$ (see [15, Theorem 6.14]). By Lemma 3.4, there exists a finite subset $\Phi \subseteq \Lambda$ such that $B_\lambda(y - 1) = \{0\}$ for every $\lambda \in \Lambda \setminus \Phi$. In other words, $[g, B_\lambda] = \langle 1\rangle$ for every $\lambda \in \Lambda \setminus \Phi$. Since C_1 is finite, $L \cap C_1$ is finite too and so there is a finite subset $\Psi \subseteq \Lambda$ such that $L \cap C_1 \in \bigoplus_{\lambda \in \Psi} B_\lambda$. Put $\Delta = \Phi \cup \Psi$, $Q = \bigoplus_{\lambda \in \Delta} B_\lambda$. Thus Q is finite. Choose a maximal $\langle g\rangle$–invariant subgroup E of C such that $\bigoplus_{\lambda \in \Lambda \setminus \Delta} B_\lambda \leq E$ and $Q \cap E = \langle 1\rangle$. Put $R/E = \Omega_1(C/E)$. By Lemma 3.4, $|R/E : C_{R/E}(g)|$ is finite and hence R/E is finite by the election of E. It follows that C/E is Chernikov. By [8, Lemma 2.8], $|E : C_E(g)|$ is finite and thus $C/C_C(g)$ is Chernikov. The mapping $\phi : a \mapsto [a, c],\ c \in C$ is a $\langle g\rangle$–endomorphism of A. In particular, Im $\phi = [g, C]$ and Ker $\phi = C_C(g)$ are $\langle g\rangle$–invariant subgroups of A. Now our result follows in this case since $[g, C] \cong C/C_C(g)$ is Chernikov.

In the general case, we decompose $C = \mathrm{Dr}_{\mathrm{p}\in\pi}\mathrm{C_p}$ where $\pi = \pi(C)$ and C_p is the Sylow p–subgroup of C for every $p \in \Pi$. If there is some $q \in \Pi$ such that $g, C_q \neq \langle 1\rangle$,then, as we showed above, $[g, C_q]$ is Chernikov. Let $L = \mathrm{Dr}_{p\neq q} C_p$. By [8, Lemma 2.8], $|L : C_L(g)|$ is finite and it follows that $[g, L]$ is finite too. Since $[g, C] = [g, Cq] \times [g, L]$, $[g, C]$ is likewise Chernikov. $\square$

4 Some applications

Lemma 4.1 *Let G be a group and assume that G contains a normal periodic abelian subgroup A such that G/A is abelian. If $\langle A, g\rangle$ is locally nilpotent for every $g \in G$, then G is likewise locally nilpotent.*

Proof – Let p be a prime and P be the Sylow p–subgroup of A. Put $B := \Omega_1(P)$. If $g \in G \setminus A$, given an infinite cyclic group $\langle y\rangle$, we consider B as an $\mathbb{F}_p\langle y\rangle$-module in the usual way. We claim that B is J–periodic. Otherwise, there is some $a \in B$ such that $\mathrm{Ann}_J(a) = \{0\}$ and $\langle a, g\rangle$ is isomorphic to the wreath product of $\langle a\rangle$ and $\langle g\rangle$, which is not nilpotent.

If $b \in B$, then $\langle b\rangle^{\langle g\rangle} = bJ \cong J/\mathrm{Ann}_J(b)$. Since every non-zero ideal of the ring J has finite index, $\langle b\rangle^{\langle g\rangle}$ is finite. If E is a minimal $\langle g\rangle$- invariant subgroup of $\langle b\rangle^{\langle g\rangle}$, then $Q = \mathrm{Ann}_J(E) \in Spec(J)$. Since $\langle E, g\rangle$ is nilpotent, $[E, g] = E(x-1) = \langle 1\rangle$. It follows that $Q = (x-1)J$. Since this holds for any simple J–submodule of B, it follows that $\Pi_J(B) = \{(x-1)J\}$. In other words, B has an ascending series

$$\langle 1\rangle = C_0 \leq C_1 \leq \ldots \leq C_n \leq C_{n+1} \leq \ldots C_\omega = \bigcup_{n\in\mathbb{N}} C_n = B$$

such that

$$C_1 = C_B(g) = \mathrm{Ann}_B(x-1),\ C_{n+1}/C_n = C_{B/C_n}(g),\ n \in \mathbb{N}$$

(the upper $Z\langle g\rangle$–central series of B). Since $G/C_G(B)$ is abelian, every subgroup C_n is G–invariant, $n \in \mathbb{N}$. Put now $B_1 = B,\ B_n = \Omega_n(P),\ n \in \mathbb{N}$. The mapping $\varphi : b \to pb$, $b \in B_2$, is an $\mathbb{Z}\langle g\rangle$–homomorphism. Then Im $\varphi \leq B_1$ and Ker $\varphi = B_1$ and so B_2/B_1 is isomorphic to some $\mathbb{Z}\langle g\rangle$–submodule of B_1. It follows that B_2/B_1 is $\mathbb{Z}\langle g\rangle$–hypercentre. By induction, B_{n+1}/B_n is $\mathbb{Z}\langle g\rangle$–hypercentre foe every $n \in \mathbb{N}$. It turns out that P is likewise $\mathbb{Z}\langle g\rangle$–hypercentral. Moreover, the length of the upper $\mathbb{Z}\langle g\rangle$-central series of P is at most ω. For, $\langle u\rangle^{\langle g\rangle}$ $(u \in P)$ is finite by Lemma 2.1. Since the $\mathbb{Z}\langle g\rangle$–center of a $\mathbb{Z}\langle g\rangle$–hypercentral module has a non-zero intersection with every non-zero $\mathbb{Z}\langle g\rangle$–submodule, there is some $n \in \mathbb{N}$ such that $C_n \geq \langle u\rangle^{\langle g\rangle}$. Since this is true for each Sylow subgroup of A, the $\mathbb{Z}\langle g\rangle$–module A is also $\mathbb{Z}\langle g\rangle$–hypercentral with $\mathbb{Z}\langle g\rangle$–hypercentral length at most ω. Thus, A has an ascending series

$$\langle 1\rangle = Z_0 \leq Z_1 \leq \ldots \leq Z_n \leq \ldots Z_\omega = \bigcup_{n\in\mathbb{N}} Z_n = A$$

such that $Z_1 = C_A(g),\ Z_{n+1}/Z_n = C_{A/Z_n}(g),\ n \in \mathbb{N}$.

Let h be another element of G. Consider the $\mathbb{Z}\langle h\rangle$-module Z_1. As above, Z_1 is also $\mathbb{Z}\langle h\rangle$-hypercentral. Let

$$\langle 1\rangle = H_0 \leq H_1 \leq \ldots \leq H_n \leq H_{n+1} \leq \ldots H_\omega = \bigcup_{n\in\mathbb{N}} H_n = Z_1$$

be the upper $\mathbb{Z}\langle h\rangle$-central series of Z_1. By the election of Z_1, this series is $\mathbb{Z}\langle g,h\rangle$-central. Since G/A is abelian, the mapping $\psi : z \mapsto z(g-1),\ z \in Z_2$ is a $\mathbb{Z}G$-homomorphism such that Ker $\psi = Z_1$, Im $\psi \leq Z_1$. Hence Z_2/Z_1 is isomorphic to some submodule of Z_1. In particular, Z_2/Z_1 is $\mathbb{Z}\langle g,h\rangle$-hypercentral. By induction, Z_{n+1}/Z_n is likewise$\mathbb{Z}\langle g,h\rangle$-hypercentral, $n \in \mathbb{N}$. It follows that A is $\mathbb{Z}\langle g,h\rangle$-hypercentral. In other words $\langle A,g,h\rangle$ is hypercentral.

We proceed terating the argument of the above paragraphs. In fact, reasoning in the same way, we prove that $\langle A, g_1, g_2, \ldots, g_n\rangle$ is hypercentral for every finite subset $\{g_1, g_2, \ldots, g_n\}$ of elements. The result is proved □

We are now in a position to deduce the parts (2) and (3) of **Theorem A**

Theorem 4.2 *Let G be a metabelian group whose non-normal subgroups have finite conjugacy classes and suppose that G is not locally nilpotent. Then either G is an FC–group or the commutator subgroup of G is Chernikov.*

Proof – By Lemma 4.1, there is some $g \in G \setminus C_G(A)$ such that $\langle A,g\rangle$ is not locally nilpotent. If C is the upper $\mathbb{Z}\langle g\rangle$–hypercentre of A and E is the unique maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$–submodule of A, by Lemma 3.1 and Corollary 2.5, $A = C \times E$. Since $\langle A,g\rangle$ is not hypercentral, $E \neq \langle 1\rangle$. By Theorem 3.3, E is Chernikov. If the divisible part D of E is non-trivial, by the election of g, then $g \notin C_G(D)$. By [8, Lemma 3.3], the commutator subgroup of G is Chernikov. Suppose now that E is finite. Put $Q = G/C$, $U = A/C$ and $z = gC$. Then $U = A/C = CE/C \cong_{\mathbb{Z}\langle g\rangle} E$. By Lemma 4.1, $[g,E] = E$ and hence $[z,U] = U$. Let h be an arbitrary element of Q. Then $z^h = zu$, for some $u \in U$. Since $U = [z,U]$, we may write $u = [z,v]$, for some $v \in U$ and then $z^h = zu = z[z,v] = z^v$. It follows that $h^{v-1} \in C_Q(z)$ and thus $Q = UV$, where $V = C_Q(z)$. Since the $\mathbb{Z}\langle g\rangle$–modules U and E are isomorphic, $U \cap V = \langle 1\rangle$. If we suppose that V is normal in Q, then $[U,V] = \langle 1\rangle$ and then $U \leq C_Q(z)$. This contradiction shows that V cannot be normal. Thus the preimage W of V in G neither is normal and then W is an FC–subgroup. Moreover, $G = AW = ECW = EW$. Since E is finite, the index $|G : W|$ is likewise finite and it follows that $W \leq FC(G)$. But $E \leq FC(G)$, hence $G = FC(G)$, that is G is an FC–group. □

Corollary 4.3 *Let G be a group whose non-normal subgroups have finite conjugacy classes. If H is a periodic normal FC–subgroup of G such that G/H is finitely generated, then $H \leq FC(G)$.*

Corollary 4.4 *Let G be a locally graded group whose non-normal subgroups have finite conjugacy classes. Then G is locally central-by-finite.*

Proof – As the hypotheses are inherited by subgroups, we may suppose that G is finitely generated. Then G' is the normal closure of a finite subset (see [10, Lemma 1.43]), and it is periodic by [8, Corollary 2.4]. It follows that G' is contained in the FC–centre of G by Corollary 4.3. An easy application of Dietzman's Lemma shows that G' is finite, and G is central-by-finite. □

The next result will show the part (1) of **Theorem A** so that the proof of this result will be completed.

Theorem 4.5 *Let G be a locally nilpotent metabelian group whose non-normal subgroups have finite conjugacy classes. Then either G is a Fitting group or the commutator subgroup of G is Chernikov.*

Proof – If $\langle A, g\rangle$ is nilpotent for every $g \in G \setminus C_G(A)$, then G is a Fitting group. Otherwise there is some $g \in G\setminus C_G(A)$ such that $\langle g, A\rangle$ is not nilpotent. By Lemma 3.1 and Corollary 2.5, A decomposes as $A = C \times E$, where C is the upper $\mathbb{Z}\langle g\rangle$–hypercentre of A and E is the unique maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$–submodule of A. Since G is locally nilpotent, $E = \langle 1\rangle$ and so $A = C$. By Theorem 3.5, $K := [g, A]$ is Chernikov. If the divisible part D of K is trivial, then the commutator of the locally nilpotent subgroup $\langle g, A\rangle$ is finite and so $\langle g, A\rangle$ is nilpotent. This contradiction shows that $D \neq \langle 1\rangle$. If $g \in C_G(D)$, then $\langle g, A\rangle$ is again nilpotent. Hence $g \notin C_G(D)$ and, by [8, Lemma 3.3], the commutator subgroup of G is Chernikov. □

Actually, the local analysis says us much more. Let G be an group whose non-normal subgroups have finite conjugacy classes that includes a periodic abelian subgroup A and suppose that there exists some $g \in N_G(A) \setminus C_G(A)$ such that $\langle g, A\rangle$ is not an FC–group. By Lemma 3.1 and Corollary 2.5, A decomposes as $A = C \times E$, where C is the upper $\mathbb{Z}\langle g\rangle$–hypercentre of A and E is the unique maximal $\langle g\rangle$–hypereccentric $\mathbb{Z}\langle g\rangle$–submodule of A. In fact, E is Chernikov by Theorem 3.3. If the divisible part D_0 of E is non-trivial, by the election of g, then $g \notin C_G(D_0)$ and, by [8, Lemma 3.3], the commutator subgroup of G is Chernikov. Suppose now that E is finite and consider C. By Theorem 3.5, $K = [g, C]$ is Chernikov. If the divisible part D of K is trivial, then the finiteness of E implies that the commutator subgroup of $\langle g, A\rangle$ is finite, what contradicts the election of g. Thus $D \neq \langle 1\rangle$. If $g \notin C_G(D)$, by [8, Lemma 3.3], the commutator subgroup of G is Chernikov. Suppose now that $[g, D] = \langle 1\rangle$. Then the commutator subgroup of $\langle g, C\rangle/D$ is finite so that $L/D = C_{C/D}(gD)$ has finite index in C/D. Put $T = \langle g, L\rangle.ThenD \leq Z(T)$ and so T/D is abelian. Hence the mapping $\phi : xmapsto[g, x],\ x \in T$ is an endomorphism of T and it follows that Im $\phi = [g, T] \cong T/C_T(g) = T/$Ker ϕ. If $[g, T]$ is finite, then $T/C_T(g)$ is finite and consequently $A/C_A(g)$ is also finite, what implies that $\langle g, A\rangle$ is an FC–subgroup. This contradiction shows that $[g, T]$ is a non-finite Chernikov group, which only can appear when the order of g is infinite. Then T is not an FC–group and then T is normal and G/T is a Dedekind group. Put $x = g^8$. It is not hard to see that $[x, T]$ is also infinite and hence $\langle x, L\rangle$ neither is an FC–subgroup. The factor-group $G/\langle x, L\rangle$ contains elements of order 8 so that it must be abelian. Furthermore, by [8, Corollary 2.5], the commutator subgroup of G is periodic provided G is locally (soluble–by–finite). This implies that G/L is abelian and so G is metabelian.

The next example shows that a case as the described above can appear.

Example. Let p be a prime. If $j \in \mathbb{N}$, let H_j be the metabelian group $\langle c_j \rangle \ltimes (\langle a_j \rangle \times \langle b_j \rangle)$, where $|a_j| = |b_j| = |c_j| = p^j$, $[a_j, c_j] = 1$ and $[c_j, b_j] = a_j$ so that H_j is a p–group. Put $C = \Pi_{j \in \mathbb{N}} H_j$, $g = (c_j)_{j \in \mathbb{N}}$. Consider $H = \langle g, P \rangle$, where $P = \mathrm{Dr}_{j \in \mathbb{N}} H_j$. Then $Z(H)$ includes $A := \mathrm{Dr}_{j \in \mathbb{N}} \langle a_j \rangle$. Put $E := \langle a_j a_{j+1}^{-p} \mid j \in \mathbb{N} \rangle$ so that $E \leq Z(H)$ and $D := A/E$ is a Prüfer p–group. Define $G := H/E$, $T := P/E$, $d_j := a_j E$, $u_j := b_j E$, $v_j := c_j E$ and $x := gE$. Then $D \leq Z(G)$, $D = [G, G]$ and $[x, G] = D$. In particular, $G/C_G(x) \cong D$ is a Prüfer p–group. It follows that G is not an FC–group. Let S be a subgroup of G. If $S \geq D$, then S is normal in G. If $S \not\leq D$, then $D \cap S$ is finite and $S/(D \cap S) \cong SD/D$ is abelian. Hence the commutator subgroup of S is finite.

Theorem 4.6 *Let G be an group whose non-normal subgroups have finite conjugacy classes and put $F = FC(G)$. Then either G includes an FC–subgroup of finite index or F is nilpotent of nilpotency class at most 3 having a normal subgroup H such that $Z(F) \leq H \leq F$, $H/Z(F)$ is abelian and F/H is a quaternion group.*

Proof – If $x \in F$, then $X = \langle x \rangle^G$ is finitely generated and $G/C_G(X)$ is finite. Let $\mathcal{L}$ be the family of finitely generated G–invariant subgroups of F, $\mathcal{C} = \{C_G(L) \mid L \in \mathcal{L}\}$ and Z the intersection of the members of $\mathcal{C}$. Then $Z = C_G(F)$ so that $Z \cap F \leq Z(F)$ and $F/(Z \cap F) \cong FZ/Z \leq G/Z$. Therefore if G is not an almost FC–group, we may replace G by G/Z to assume that G is residually finite. Since no subgroup of finite index can be an FC–group, if H is a subgroup of finite index, then H is normal and G/H is a Dedekind group. Since G is residually finite, if

$$\mathcal{L} = \{H \leq G \mid |G : H| < \infty\},$$

by Remak's theorem, $G \hookrightarrow \Pi_{H \in \mathcal{L}} G/H$. If G is not abelian, then G includes a normal subgroup U of finite index such that G/U is a non-abelian Dedekind group so that $G/U = (Q/U) \times (A/U)$, where Q/U is a quaternion group and A/U is the direct product of an elemenraty abelian 2–subgroup and a periodic abelian $2'$–subgroup (see [11, 5.3.7]). Let S be a subgroup of A having finite index. Then S is normal in G and G/S is a non-abelian Dedekind group. It follows that A/S is abelian. Put now

$$\mathcal{M} = \{Y \leq A \mid |A : Y| < \infty\}$$

and let M be the intersection of the members of $\mathcal{M}$. We have $A/M \hookrightarrow \Pi_{Y \in \mathcal{M}} A/Y$ and so A/M is abelian. Since G/A is finite, $M = \langle 1 \rangle$. □

Lemma 4.7 *Let G be an group whose non-normal subgroups have finite conjugacy classes which includes a periodic residually finite abelian normal subgroup A such that G/A is finitely generated abelian. Then G is residually finite central–by–finite.*

Proof – Suppose that $G/A = \langle g_1A, \cdots, g_mA\rangle$. By Theorem 3.3 and Theorem 3.5, $[g_j, A]$ is Chernikov for every $1 \leq j \leq m$. Actually, each $[g_j, A]$ is finite since A is residually finite. Put $K := [g_1, A] \cdots [g_m, A]$ so that K is finite and $A/K \leq Z(G/K)$. By [8, Corollary 2.5], the subgroup $L/K := \langle g_1K, \cdots, g_mK\rangle$ is central–by–finite. It follows that $B/K := A/K \cap L/K$ is finite. Since $G/B = A/B \times L/B$, G/B is abelian. Hence $[G, G]$ is finite. Let U be a maximal torsion-free subgroup of $Z(G)$. Then G/U is periodic and further G/U includes the residually finite abelian normal subgroup $AU/U \cong A/(A \cap U) \cong A$ that has finite index. It follows that G/U is residually finite. By Remak's theorem $G \hookrightarrow G/U \times G/A$ and then G is residually finite. Then there is a normal subgroup C of finite index such that $C \cap [G, G] = \langle 1\rangle$. Clearly $C \leq Z(G)$. □

PROOF of **Theorem B**.

Suppose that G is not an FC–group and put $K := [G, G]$. By [8, Corollary 2.5], K is periodic and, by [8, Corollary 3.10], K is an FC–subgroup.. Therefore $K \leq FC(G)$ by Corollary 4.3. Let $Z = Z(K)$. By Theorem 4.6, $K/Z(K)$ is abelian–by–finite. Since this is an FC–group, the commutator subgroup $D/Z(K)$ of $K/Z(K)$ is finite. Then K/D is periodic residually finite abelian. By Lemma 4.7, G/D is central–by–finite. By Schur's theorem, the commutator subgroup of G/D is finite. It follows that $K/Z(K)$ is finite.

To finish, it suffices to see that K is Chernikov. If D_1 is the commutator subgroup of K, then D_1 is finite and so we may replace G by G/D_1 to assume that $D_1 = \langle 1\rangle$. As above, K is a periodic FC–group. If $G/K = \langle g_1K, \cdots, g_mK\rangle$, by Theorem 3.3 and Theorem 3.5, $[g_j, K]$ is Chernikov for every $1 \leq j \leq m$. Put $R := [g_1, K] \cdots [g_m, K]$ so that R is Chernikov and $K/R \leq Z(G/R)$. By [8, Corollary 2.5], $L/R := \langle g_1R, \cdots, g_mR\rangle$ is central–by–finite. It follows that $B/R := K/R \cap L/R$ is finite. Since $G/B = A/B \times L/B$, we have that G/B is abelian. Hence $[G, G]$ is Chernikov. □

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