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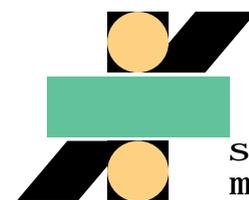
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Have Finite Conjugacy Classes

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Groups whose Non-Normal Subgroups Have Finite Conjugacy Classes*

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Abstract

The class \mathfrak{X} of all groups whose non-normal subgroups have finite conjugacy classes has been investigated. In particular, it will be proved that a soluble-by-finite \mathfrak{X} -group is locally (central-by-finite) and its commutator subgroup is an FC -group. The structure of periodic \mathfrak{X} -groups with Chernikov commutator subgroup is described.

1 Introduction

The structure of groups for which the set of non-normal subgroups has prescribed properties has been investigated by several authors. The first step was of course the description of groups in which all subgroups are normal; it is well known that such groups either are abelian or can be decomposed as a direct product of Q_8 (the quaternion group of order 8) and a periodic abelian group with no elements of order 4.

Romalis and Sesekin ([12] [13] [14]) investigated soluble groups whose non-normal subgroups are abelian (*metahamiltonian groups*), proving in particular that every metahamiltonian group has derived length at most 3 and its commutator subgroup is a finite group with prime-power order. The final description of metahamiltonian groups was obtained by Kuzenny and Semko (see [6], [7] and [8]).

On the other hand, S.N. Chernikov ([1]) studied the class of soluble groups whose non-normal subgroups are finite. He showed that such groups either are Dedekind groups or extensions of a Prüfer group by a finite Dedekind group.

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A group G is called an FC -group if every element of G has finitely many conjugates, or equivalently if for each $x \in G$ the centralizer $C_G(x)$ of x has finite index in G . Groups with finite conjugacy classes can be considered as the most natural tool in order to study properties which are common both to finite groups and abelian groups. The theory of FC -groups had a strong development in the second half of last century, and also in recent years many authors have investigated properties of such groups (for details we refer in particular to the survey [16]).

The aim of this article is to investigate the class \mathfrak{X} of all groups whose non- FC -subgroups are normal. This class, also contains the class of *minimal-non-FC* groups, i.e. those groups which are not FC -groups while all its proper subgroups have the property FC , studied by several authors (see, for instance [2], [4]). We will prove, among other results, that a soluble-by-finite \mathfrak{X} -group is locally (central-by-finite) and its commutator subgroup is an FC -group. Moreover, it will be investigated the structure of \mathfrak{X} -groups with Chernikov commutator subgroup and will be found some classes of \mathfrak{X} -groups whose commutator subgroup is a Chernikov group.

Most of our notation is standard and can be found in [9] and [10]. We shall also use the monograph [15] as a general reference for properties of FC -groups.

2 Some preliminar results

Clearly, the class \mathfrak{X} is closed with respect to subgroups and homomorphic images. Moreover, if G is an \mathfrak{X} -group and S is a non- FC -subgroup of G , then S is normal and G/S is a Dedekind group.

Lemma 2.1 *Let G be an \mathfrak{X} -group and A a torsion-free finitely generated abelian normal subgroup of G . Then A centralizes every element of finite order of G .*

PROOF – Suppose by contradiction that there exists a periodic element g of G which is not contained in $C_G(A)$, and let p be a prime number such that $\langle g, A^{p^n} \rangle$ is not abelian, for every positive integer n . Since the commutator subgroup of an FC -group is periodic (see, for instance, [9], Theorem 4.32), then $\langle g, A^{p^n} \rangle$ is not an FC -subgroup, for each $n \in \mathbb{N}$, and hence $\langle g \rangle = \bigcap_{n \in \mathbb{N}} \langle g, A^{p^n} \rangle$ is normal. It follows that $g \in C_G(A)$. This contradiction proves the lemma. \square

A group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Obviously, all locally (soluble-by-finite) groups are locally graded. Recall that a minimal-non- FC group having proper commutator subgroup is a Chernikov group, and that every perfect locally graded group is countable locally finite p -group for some prime number p (see [15], pp.156-160 and [16], pp. 281-282).

Proposition 2.2 *Let G be a locally graded \mathfrak{X} -group. Then the set of all elements of finite order is a (characteristic) subgroup of G .*

PROOF – Let x and y periodic elements of G , and show that even xy is periodic. Clearly we may assume that $G = \langle x, y \rangle$. Suppose first that every subgroup of finite index in G is not an FC -group. Hence, every finite homomorphic image of G is a Dedekind group, and G' is properly contained in G . Let K be a subgroup of G' . If K is not an FC -group, then the index $|G' : K|$ is at most 2, and K is a minimal non FC -group. It follows that K and even G are periodic. Thus we may suppose that G' is an FC -group. Put T the torsion subgroup of G' . Clearly G/T is a finitely generated metabelian group whose finite homomorphic images are Dedekind groups, so that G/T is nilpotent (see [11],15.5.3) and G is periodic.

Therefore we may assume that G has a (normal) subgroup H of finite index which is an FC -group. Clearly H is finitely generated, and hence $G/Z(H)$ is finite. Let L be a torsion free subgroup of finite index in $Z(H)$. The factor group $Z(H)/Core_G(L)$ has finite exponent, and so it is finite. It follows by Lemma 2.1 that $Core_G(L)$ is contained in the centre $Z(G)$, and so $G = \langle x, y \rangle$ is periodic. \square

Proposition 2.3 *Let G be a locally graded \mathfrak{X} -group. If G is torsion-free, then G is abelian.*

PROOF – Since a torsion-free FC -group is abelian, then G is metahamiltonian. Thus the statement follows from Corollary 3 of [3]. \square

Corollary 2.4 *Let G be a locally graded \mathfrak{X} -group. Then the commutator subgroup of G is periodic.*

Corollary 2.5 *Let G be a locally (soluble-by-finite) \mathfrak{X} -group. Then G is locally (central-by-finite).*

PROOF – Clearly we may assume that G is finitely generated. If G is not nilpotent, then G contains a subgroup H of finite index which is not subnormal (see [11],15.5.3). Therefore H is an FC -group, so that the factor group $G/Z(\text{Core}_G(H))$ is finite. It follows that G is polycyclic, and Corollary 2.4 yields that G' is finite. In particular, $Z(G)$ has finite index in G . \square

Lemma 2.6 *Let G be an \mathfrak{X} -group. If G is a non- FC -group, then the commutator subgroup of G is countable.*

PROOF – Let $K = [G, G]$. Since G is a non- FC -group, there is an element $g \in G$ such that the index $|G : C_G(g)|$ is infinite. Then G includes a countable subgroup H such that $g \in H$ and the index $|H : C_H(g)|$ is infinite. It follows that H is not an FC -group, and G/H is a Dedekind group. It turns out that $K/(K \cap H) (\simeq KH/H)$ has order at most 2. Thus K is countable. \square

The last result of this section is a technical lemma that will be useful in the following.

Lemma 2.7 *Let G be an \mathfrak{X} -group, and let g be an element of G and A and B subgroups of G satisfying the following conditions:*

- (1) A and B are $\langle g \rangle$ -invariant;
- (2) $\langle A, B \rangle \cap \langle g \rangle = \langle 1 \rangle$;
- (3) $A \cap B = \langle 1 \rangle$;
- (4) $C_A(g) \neq A$.

Then $\langle B, g \rangle$ is an FC -group, and in particular the index $|B : C_B(g)|$ is finite.

PROOF – Clearly the subgroup A contains an element a such that $[a, g] = d \neq 1$. Then $a^{-1}ga = gd \neq g$. Suppose first that gd belongs to $B\langle g \rangle$, and let $gd = g^m b$ for some $m \in \mathbb{Z}$ and $b \in B$. Then $g^{m-1} = db^{-1}$, and, by hypothesis (2), $db^{-1} = 1$. It follows, by (3), that $d = b = 1$. This contradiction shows that $gd \notin B\langle g \rangle$, and in particular, $B\langle g \rangle$ is not normal. Hence $B\langle g \rangle$ is an FC -subgroup. \square

3 Periodic \mathfrak{X} -groups with commutator Chernikov subgroup

A very special class of FC -groups is the class BFC , of all groups with boundedly finite conjugacy classes. A well known result of B. H. Neumann ensures that a group satisfies BFC if and only if its commutator subgroup is finite (see [9], Theorem 4.35).

The natural semidirect product $G = Q \rtimes P$, where P is a Prüfer p -group, (p is a prime number), and Q is a subgroup of $Aut P$, is an example of group whose non-normal subgroups are BFC . In fact, if H is an arbitrary subgroup of G , which is not normal in G , then H does not contain P , so that the intersection $P \cap H$ is finite. On the other hand, G/P is abelian, and hence H' is contained in $P \cap H$. In particular, H is a BFC -group.

In the following we will describe the structure of periodic \mathfrak{X} -groups whose commutator subgroup is a Chernikov group and is not contained in the centre.

Let A be an infinite abelian group, $H \leq Aut(A)$. We will say that H is an *infinitely irreducible* automorphism group of A , if A contains no proper infinite H -invariant subgroup. If $H = \langle x \rangle$, then we say that the element x acts infinitely irreducible on A .

Lemma 3.1 *Let G be any group, and let D be an abelian divisible subgroup of G satisfying the minimal condition on subgroups. If g is an element of G of finite order such that $g \in N_G(D) \setminus C_G(D)$, then D contains a divisible $\langle g \rangle$ -invariant subgroup A such that $g \notin C_G(A)$ and g acts infinitely irreducible on A .*

PROOF – Let R be a divisible $\langle g \rangle$ -invariant subgroup of D such that $g \notin C_G(R)$, and assume that the subgroup $[R, g]$ is centralized by g . Then the subgroup $\langle R, g \rangle$ is nilpotent of class 2, and clearly satisfies the minimal condition on subgroups. It follows that $\langle R, g \rangle$ is central-by-finite (see [9], Theorem 3.14), and Schur's Theorem yields that it is also an FC -group. Thus $g \in C_G(R)$ (see [15], Theorem 1.9). This last contradiction shows that $g \notin C_G([R, g])$.

It follows that

$$D \geq [D, g] \geq [D, {}_2g] \geq \dots \geq [D, {}_ng] \geq \dots \quad (n \in \mathbb{N})$$

is a descending chain of infinite $\langle g \rangle$ -invariant divisible subgroups of D . Since D verifies the minimal condition on subgroups, there exists a positive integer n such that $X = [D, {}_ng] = [X, g]$, and clearly $C_X(g)$ is finite.

Let now A be a subgroup of X which is minimal with respect to the properties to be infinite, $\langle g \rangle$ -invariant and such that $g \notin C_G(A)$. If the finite residual J of A is properly contained in A , then by minimality of A , we have that g centralizes J , which is a contradiction, since $C_X(g)$ is finite. Therefore A is divisible, and clearly the action of g on A is infinitely irreducible. The lemma is proved. \square

The previous lemma is not true when the element g has infinite order. For, let p be a prime number, and denote by D the direct product $A \times B$, where A and B are Prüfer p -groups with the natural generators $a_1, a_2, \dots, a_n, \dots$ and $b_1, b_2, \dots, b_n, \dots$, respectively. If g is the automorphism of D of infinite order defined by the rule $(a_n, b_n)^x = (a_n, a_n b_n)$, then the semidirect product $G = \langle g \rangle \rtimes D$, has the subgroup $\langle b_1, A \rangle$ which is minimal with respect to the conditions to be infinite $\langle g \rangle$ -invariant and not centralized by g . We note also that G is an \mathfrak{X} -group.

Lemma 3.2 *Let G be an \mathfrak{X} -group, and let D be a divisible subgroup of G satisfying the minimal condition on subgroups. If $C_G(D)$ is properly contained in $N_G(D)$, then the commutator subgroup of G is a Chernikov group.*

PROOF – Let g be an element of G such that $g \in N_G(D) \setminus C_G(D)$, and suppose first that g has infinite order. Since the centre of an FC -group contains every \mathfrak{F} -perfect subgroup (see[15], Theorem 1.9), the subgroup $\langle g, D \rangle$ is not an FC -group. It follows that $\langle g, D \rangle$ is a normal subgroup of G , and the factor group $G/\langle g, D \rangle$ is a Dedekind group. Therefore, if we denote by K the commutator subgroup of G , then the index

$$|KD : KD \cap \langle g, D \rangle| = |K\langle g, D \rangle : \langle g, D \rangle|$$

is finite. On the other hand, by Corollary 2.4, KD is periodic, and the modular law yields that the factor group KD/D is finite, and so K is a Chernikov group.

Assume now that the element g has finite order. By Lemma 3.1 there exists a divisible $\langle g \rangle$ -invariant subgroup A of D such that $g \notin C_G(A)$ and g acts infinitely irreducibly on A . Clearly, the index $|\langle g, A \rangle : A|$ is finite, so that, since $\langle A, g \rangle$ is not an FC -group, the factor group $K\langle A, g \rangle/\langle A, g \rangle$ is finite. If $K \cap \langle A, g \rangle = \langle A, g \rangle$, then $\langle A, g \rangle \leq K$, and hence K is a Chernikov group. Thus we may suppose that the $\langle g \rangle$ -invariant subgroup $K \cap \langle A, g \rangle$ is properly contained in $\langle A, g \rangle$, and so it is finite. It follows that also K is finite, a contradiction. This completes the proof. \square

Lemma 3.3 *Let G be an \mathfrak{X} -group, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K and g is an element of G of finite order such that $g \notin C_G(D)$, then g acts infinitely irreducible on D . In particular, D is a p -subgroup for some prime p .*

PROOF – By Lemma 3.1 there exists a divisible $\langle g \rangle$ -invariant subgroup A of D such that $g \notin C_G(A)$ and g acts infinitely irreducible on A . Using the same arguments as in the last previous lemma, we can obtain that A is the divisible part of the commutator subgroup K . This means that $A = D$, and so g acts infinitely irreducible on D . \square

Corollary 3.4 *Let G be an \mathfrak{X} -group, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K and g is an element of G of finite order such that $g \notin C_G(D)$, then $C_D(g)$ is finite and $D = [g, D]$.*

PROOF – Clearly the map $\phi : d \in D \rightarrow [g, d] \in D$, is a $\mathbb{Z}\langle g \rangle$ -endomorphism of D . Therefore, $Im\phi = [g, D]$ and $Ker\phi = C_D(g)$ are $\langle g \rangle$ -invariant subgroups of D . If we assume that $C_D(g)$ is infinite, then the Lemma 3.3 yields that $C_D(g) = D$, a contradiction. It follows that $C_D(g)$ is finite. On the other hand, since $[g, D]$ is isomorphic with $D/C_G(g)$, then $[g, D]$ is infinite, and again by Lemma 3.3, $D = [g, D]$. \square

Corollary 3.5 *Let G be an \mathfrak{X} -group, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K and $P/C_G(D)$ is a periodic subgroup of $G/C_G(D)$, then $P/C_G(D)$ is a finite cyclic group. Moreover, $\pi(D) = \{p\}$ and either $|P/C_G(D)| = p$, or $P/C_G(D)$ is a cyclic p' -subgroup.*

PROOF – Clearly, we may assume that G is not an FC -group. Since the periodic subgroups of an automorphism group of a divisible Chernikov group are finite (see, for example [9], Theorem 3.29.2), then $P/C_G(D)$ is likewise finite. On the other hand, the group $P/C_G(D)$ is an infinitely irreducible automorphism group of D by Lemma 3.3. Thus the statement follows easily from Theorem 6.1 of [5]. \square

Corollary 3.6 *Let G be an \mathfrak{X} -group, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K , then $G/C_G(D)$ is abelian. In particular, K is central-by-finite.*

PROOF – We may suppose that the factor group $G/C_G(D)$ is not periodic, by Corollary 3.5. Clearly, every subgroup of G properly containing $C_G(D)$ is not an FC-group. It follows that $G/C_G(D)$ is abelian. \square

Corollary 3.7 *Let G be a soluble-by-finite \mathfrak{X} -group. Then the commutator subgroup of G is an FC-group.*

PROOF – Assume, by contradiction, that $K = [G, G]$ is not an FC-group. If every proper subgroup of K is an FC-group, then K is a Chernikov group (see [15], pp.156-158). This is a contradiction by Corollary 3.6. Therefore K contains a proper subgroup H which is not an FC-group. Then H is normal in G , and the factor group G/H is a Dedekind group. Thus $|K/H| \leq 2$, and clearly H is a minimal non-FC-group. It follows that H , and hence even K , is a Chernikov group, which contradicts again to Corollary 3.6. \square

Theorem 3.8 *Let G be a periodic \mathfrak{X} -group with Chernikov commutator subgroup K , and let D be the divisible part of K . If $C_G(D) \neq G$, then $G = DL$, where $D \cap L$ is a finite G -invariant subgroup and L is a BFC-group. Moreover, every non-normal subgroup of G is a BFC-group.*

PROOF – Let g be an element of G such that $g \notin C_G(D)$. Then the subgroup $\langle g, D \rangle$ is not an FC-subgroup, so that by hypothesis, $\langle gD \rangle$ is a normal subgroup of the factor group G/D . If x is an arbitrary element of G , then $(gD)^{xD} = g^k D$ for some positive integer k . It follows that $g^x = g^k d$ for some $d \in D$. If $d = 1$, then $x \in N_G(\langle g \rangle)$. Let $d \neq 1$. Since $g^k \notin C_G(D)$, Corollary 3.4 yields that $D = [g^k, D]$, and in particular, $d = [g^k, d_1]$, for some element $d_1 \in D$. It follows

$$g^x = g^k [g^k, d_1] = d_1^{-1} g^k d_1,$$

so that $xd_1^{-1} \in N_G(\langle g \rangle) = L$. Thus, in any case $G = DL$.

Since D is abelian, $D \cap L$ is a normal subgroup of G . In particular, $D \cap L$ is $\langle g \rangle$ -invariant. If we suppose that $D \cap L$ is infinite, Lemma 3.3 yields that $D \cap L = D$, that is $D \leq N_G(\langle g \rangle)$. As g has finite order, then $\langle g, D \rangle$ has finite commutator subgroup, so that $g \in C_G(D)$. This contradiction shows that $D \cap L$ is finite.

Clearly, the factor group G/D is finite-by-abelian. Thus $L/(L \cap D)$ has finite commutator subgroup, so that, since $L \cap D$ is finite, also $[L, L]$ is finite.

Finally, let H be an arbitrary subgroup of G . Assume first that $H \leq C_G(D) = D(L \cap C_G(D))$. Then H has finite commutator subgroup.

Suppose now that $H \not\leq C_G(D)$. If the intersection $H \cap D$ is infinite, then $H \cap D = D$, because an element $y \in H \setminus C_G(D)$ acts infinitely irreducibly on D by Lemma 3.3. Moreover, $\langle y, D \rangle$ is not an FC-group, so that it is normal in G and $G/\langle y, D \rangle$ is a Dedekind group. It follows that H is normal in G . So we may suppose that $H \cap D$ is finite. Since $H/(H \cap D)$ is isomorphic with a subgroup of G/D , then $H/(H \cap D)$ has finite commutator subgroup, and H is a BFC-group. The theorem is proved. \square

4 The structure of periodic \mathfrak{X} -groups having an FC-subgroup of finite index

A group G is said to be an *almost FC-group* if G contains a subgroup which is an FC-group and has finite index in G . Examples of almost FC-groups are soluble metahamiltonian groups and soluble groups whose non-normal subgroups are finite. In this section we will concern with periodic \mathfrak{X} -groups which are almost FC. It will turn out that such groups either are FC-groups or have Chernikov commutator subgroup.

Lemma 4.1 *Let G be an \mathfrak{X} -group. If G is not an FC-group and contains an abelian subgroup of finite index, then the commutator subgroup G' is Chernikov group.*

PROOF – Suppose first that G is periodic. Denote by A a normal abelian subgroup having finite index in G . By hypothesis G is not an FC-group, so that it contains an element g which does not centralizes A and such that $|G : C_G(g)|$ is infinite. In particular, $|A : C_A(g)|$ is infinite, and the subgroup $H = \langle g, A \rangle$ is not an FC-group.

Clearly the intersection $B = \langle g \rangle \cap A$ is finite and it is also contained in $Z(H)$. If we suppose that the index $|A/B : C_{A/B}(gB)|$ is finite, then H/B is central-by-finite. In particular, by Schur's theorem (see, for example, [9], Theorem 4.12) H/B and hence H has finite commutator subgroup. This contradiction shows that the index $|A/B : C_{A/B}(gB)|$ is infinite. Assume that $B = \langle 1 \rangle$. Let a be an element of A such that $[g, a] = a_1 \neq 1$. Put $C = \langle a \rangle^{\langle g \rangle}$. Then C is a finite $\langle g \rangle$ -invariant subgroup. Clearly a_1 belongs to C and there exists a subgroup E of A such that $E \cap C = \langle 1 \rangle$ which is maximal with respect to this condition. Then A/E is a Chernikov group. Since $\langle g \rangle$ is finite, the set

$$\{E^x \mid x \in \langle g \rangle\} = \{E_1, \dots, E_m\}$$

is finite, and the factor group $A/Core_{\langle g \rangle}(E)$ is isomorphic with a subgroup of $A/E_1 \times \dots \times A/E_m$, and so it is also a Chernikov group. Put $Q = Core_{\langle g \rangle}(E)$. Then Q is a $\langle g \rangle$ -invariant subgroup and $C \cap Q = \langle 1 \rangle$. By Lemma 2.7 $Q\langle g \rangle$ is an FC -subgroup, and in particular $C_Q(g)$ has finite index in Q . Since A/Q is a Chernikov group, $A/C_A(g)$ is also a Chernikov group.

The mapping $\phi : a \rightarrow [g, a]$, $a \in A$, is a $\mathbb{Z}\langle g \rangle$ -endomorphism of A . Therefore $Im\phi = [g, A]$ and $Ker\phi = C_A(g)$ are the $\langle g \rangle$ -invariant subgroup and $[g, A] \cong A/C_A(g)$. It follows that $R = [g, A]$ is a Chernikov subgroup. If $B \neq 1$ the argument above shows that $[g, A]B/B$ is a Chernikov group, and hence R is likewise a Chernikov group. Let S be a finite G -invariant subgroup of R such that R/S is divisible. It is enough to show that the commutator subgroup of G/S is a Chernikov group, and hence we may suppose that $S = \{1\}$.

Assume that $g \in C_G(R)$. Therefore, H is a nilpotent subgroup of nilpotency class at most 2. Let $|g| = k$, then $[g, h]^k = [g^k, h] = 1$ for each element $h \in H$. It follows that $[g, H] = [g, A]$ is a bounded Chernikov subgroup, and hence it is finite. In particular H is an FC -group. This contradiction shows that $g \notin C_G(R)$, and Lemma 3.2 yields that $[G, G]$ is a Chernikov subgroup when G is periodic.

In the general case, we can consider a maximal torsion-free subgroup U of A . Since G/A is finite, the set $\{U_1, \dots, U_m\}$ of all conjugates of U in G is finite. Put $V = Core_G(U)$. Clearly A/U is periodic, and the natural embedding of $A/Core_G(U)$ in $A/U_1 \times \dots \times A/U_m$ shows that also A/V is periodic. The first part of the proof shows that $G'V/V \cong G'/(G' \cap V)$ is a Chernikov group. On the other hand, G' is periodic by Corollary 2.4, so that $G' \cap V = \{1\}$ and G' is a Chernikov group. \square

Lemma 4.2 *Let G be a periodic \mathfrak{X} -group, and let H be a normal residually finite subgroup of G such that G/H is cyclic. If H is an FC -group, then G is likewise an FC -group.*

PROOF – Put $G/H = \langle gH \rangle$. Since H is residually finite and the element g has finite order, then H contains a G -invariant subgroup R such that the factor group G/R is finite and $R \cap \langle g \rangle = \langle 1 \rangle$.

Assume first that $[g, R] = \{1\}$. Therefore $C_H(g)$ has finite index in H . On the other hand, G/H is finite, so that g is an FC -element, and G is an FC -group.

Thus, we may suppose that R contains an element x which does not centralize g . Put $A = \langle g, x \rangle = \langle g \rangle \rtimes B$, where $B = \langle x \rangle^G$. Clearly A is finite. It follows that H contains a G -invariant subgroup V of finite index

such that $V \cap A = \{1\}$, and Lemma 2.7 yields that $V\langle g \rangle$ is an FC -group. In particular, $|V : C_V(g)|$ is finite, and G is an FC -group. \square

Lemma 4.3 *Let G be a periodic \mathfrak{X} -group, and let H be a normal subgroup of G such that G/H is cyclic of prime order. If H is an FC -group, then G either is an FC -group or its commutator subgroup is a Chernikov group.*

PROOF – Let g be an element of G such that $G/H = \langle gH \rangle$, and assume that G is not an FC -group. If N is a normal finite subgroup of G , then G/N is not an FC -group. Therefore, in order to show that G' is a Chernikov group, we may replace G with G/N , and assume $N = \{1\}$. In particular, we may suppose that the normal closure of $\langle g \rangle \cap H$ in G is trivial by Dietzmann Lemma.

Denote by Z the centre of H . If the factor group H/Z is finite, then G is abelian-by-finite, and Lemma 4.1 yields that G' is a Chernikov group. So we may suppose that H/Z is infinite. Moreover G/Z is an FC -group by Lemma 4.2. Let now x be an element of G which is not contained in the FC -centre of G . Then every subgroup of finite index in G which contains x is not FC -group, and hence it is normal. It follows that the cyclic subgroup $\langle xZ \rangle$ is normal in G/Z .

Suppose first that $g \in C_G(Z)$. Then Z is contained in the centre of G . Put $C/Z = C_{G/Z}(\langle gZ \rangle)$. Clearly C has finite index in G and $g \notin FC(C)$. Moreover the map $\phi : x \in C \rightarrow [g, x] \in C$ is an endomorphism of C , so that $[g, C]$ is isomorphic with $C/C_C(g)$. Let $p = |G/H|$. Since $g^p = 1$, the commutator $[g, C]$ is an infinite elementary abelian p -group.

Let now a_1 an element of $H \cap C$ such that $[g, a_1] = b_1 \neq 1$. Then $|C : C_C(a_1)|$ is finite. Since $g \notin FC(C)$, there is an element $a_2 \in C_C(a_1)$ such that $[g, a_2] = b_2 \neq b_1$. By induction we can define a sequence $\{a_n\}_n \in \mathbb{N}$ of elements of C such that $a_{n+1} \in C_C(a_1, \dots, a_n)$ and $b_{n+1} \notin \langle b_1, \dots, b_n \rangle$ where $\{b_n = [g, a_n]\}_n \in \mathbb{N}$. Put $E = \langle b, a_n : n \in \mathbb{N} \rangle$ and $Z_1 = \langle a_n, b_n : n \in \mathbb{N} \rangle$. Then $E = \langle g \rangle Z_1$ is abelian-by-finite, and by Lemma 4.1 either E is an FC -group or $[E, E]$ is a Chernikov group. On the other hand, $[E, E] = \text{Dr}_{n \in \mathbb{N}} b_n$, and this is a contradiction.

So we may suppose that g does not centralize Z . If Z is a Chernikov group, it contains a finite G -invariant subgroup S such that Z/S is divisible. Thus, by above argument, it can be assumed that Z is divisible, and Lemma 3.2 yields that G' is a Chernikov group.

Suppose now that Z is not a Chernikov group. Let z be an element of Z such that $[g, z] \neq \{1\}$, and put $Y = \langle z \rangle^{\langle g \rangle}$. Moreover, let U be a subgroup of Z which is maximal with respect to the condition $U \cap Y = \{1\}$. Since Y is finite, then Z/U is a Chernikov group.

On the other hand, if $\{U_1, \dots, U_m\}$ is the conjugacy class of U in G , then the natural embedding of $Z/\text{Core}_G(U)$ in $Z/U_1 \times \dots \times Z/U_m$ shows that there exists a $\langle g \rangle$ -invariant subgroup V of Z such that Z/V is a Chernikov group. Therefore, by Lemma 2.7 the index $|V : C_V(g)|$ is finite, and so $Z/C_Z(g)$ is likewise a Chernikov group. The endomorphism of Z defined by the rule $z \rightarrow [g, z]$ shows that also $W = [g, Z]$ is a Chernikov group. Clearly $[g, Z]$ is a G -invariant subgroup of G and as above, we may suppose that W is a non-trivial divisible group. In particular W has not finite exponent, and hence $g \notin C_G(W)$. A new application of Lemma 3.2 completes the proof. \square

Theorem 4.4 *Let G be a periodic \mathfrak{X} -group. If G is an almost FC -group, then G either is an FC -group or its commutator subgroup is a Chernikov group.*

PROOF – Let H be a normal subgroup of G which is an FC -group and has finite index in G , and assume that G is a counterexample such that the order of G/H is minimal. Clearly, by Lemma 4.3 $|G/H|$ is not a prime number.

If X is a proper subgroup of G which contains H and is not an FC -group, then X is normal in G , X' is a Chernikov group and the factor group G/X' is abelian-by-finite. It follows, by Lemma 4.1 that G/X' either is central-by-finite or its commutator subgroup is a Chernikov group. Therefore G' is likewise a Chernikov group.

This contradiction shows that every proper subgroup of G which contains H is an FC -group. Put now $G = g_1H \cup \dots \cup g_tH$, where g_1, \dots, g_t are element of G and $t \in \mathbb{N}$, and assume that every subgroup $\langle g_i \rangle H$ is proper. Then, by above G is an FC -group. Hence there exists an element $g \in G$ such that $G/H = \langle gH \rangle$.

If L/H is a proper subgroup of G/H , then L is a proper normal FC -subgroup of G , so that by minimality of counterexample, G either is an FC -group or its commutator subgroup is a Chernikov group. This last contradiction proves the theorem. \square

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