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Asymptotic properties of Laguerre–Sobolev orthogonal polynomials

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Abstract

Let S_n be polynomials orthogonal with respect to the inner product

$$(f,g)_S = \int_0^\infty fg\,d\mu_0 + \lambda \int_0^\infty f'g'\,d\mu_1$$

where $d\mu_0 = x^\alpha e^{-x}dx$, $d\mu_1 = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\,\delta_\xi$ with $\alpha > -1$, $\xi \le 0$, $M \ge 0$, and $\lambda > 0$. A strong asymptotic on $(0,\infty)$, a Mehler-Heine type formula, a Plancherel-Rotach type exterior asymptotic as well as an upper estimate for S_n are obtained. As consequence, we give asymptotic results for the zeros and critical points of S_n and the distribution of contracted zeros. Some numerical examples are shown.

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1 Introduction

The asymptotic behaviour of the polynomials and their zeros is one of the central problems of the theory of orthogonal polynomials.

In this paper we are concerned with the asymptotic properties of Sobolev orthogonal polynomials, that is, polynomials orthogonal with respect to an inner product involving derivatives. More precisely, we consider the Sobolev inner product:

$$(f,g)_S = \int_0^{\infty} f g \, d\mu_0 + \lambda \int_0^{\infty} f' g' \, d\mu_1 \tag{1.1}$$

where

$$d\mu_0 = x^{\alpha} e^{-x} dx, \quad d\mu_1 = \frac{x^{\alpha+1} e^{-x}}{x-\xi} dx + M \, \delta_{\xi}$$

with $\alpha > -1$, $\xi \le 0$, $M \ge 0$, and $\lambda > 0$. The pair of measures (μ_0, μ_1) constitutes one of the so-called coherent pairs.

The goal of coherence is the fact we can establish a relation between two consecutive Sobolev orthogonal polynomials and two consecutive orthogonal polynomials associated with the first measure μ_0. This relation plays an important role in the study of Sobolev polynomials and was one of the properties that Iserles et al. looked for the new polynomials that they introduced in [4] as the solution to an isoperimetric problem. Moreover, the existence of this kind of relation was the reason for the introduction of the concept of coherence. Although this finite relation between Sobolev polynomials and standard orthogonal polynomials is an important feature of coherence, it is *not* exclusive of coherent pairs.

Of course, it is possible to study Sobolev orthogonal polynomials without these algebraic relations (see, for example, [7], [8]) and very interesting analytic results can be obtained, but from the point of view of numerical analysis, it still remains a lot to be known about generating, computing and experimenting with Sobolev polynomials orthogonal with respect to an inner product of the form

$$(f,g)_S = \int f \, g \, d\mu_0 + \int f' \, g' \, d\mu_1,$$

having both measures absolutely continuous part non zero. However, in this sense there has been an important first step in [3].

The complete characterization of all coherent pairs of measures was done in [9]. In the case of unbounded support measures, there are two general families of polynomials both related with Laguerre polynomials. The first one, usually named of type I, corresponds to the pair (μ_0, μ_1) where either $d\mu_0(x) = (x - \xi)x^{\alpha-1} e^{-x} \, dx$, $d\mu_1(x) = x^{\alpha} e^{-x} \, dx$ with $\xi \le 0$ and $\alpha > 0$ or $d\mu_0(x) = e^{-x} \, dx + M \, \delta_0(x)$ with $M \ge 0$ and $d\mu_1(x) = e^{-x} \, dx$. The second one (type II) is the pair described in (1.1).

The asymptotic behaviour of Sobolev polynomials for coherent pairs of type I has been widely studied (see, for instance, [6], [11], [12]) while, with respect to type II, only the comparative asymptotic has been treated (see [11]). The aim of this paper is to complete the study of asymptotic properties for polynomials of type II.

The paper is organized as follows. Some properties of classical Laguerre polynomials are exposed in this Section. In Section 2, polynomials orthogonal with respect to the measure μ_1 are analyzed. The interest of these polynomials becames from the fact that the absolutely continuous part of μ_1 is a rational perturbation of the Laguerre weight. Section 3 is dedicated to asymptotics of Sobolev polynomials: a strong asymptotic on $(0,+\infty)$, a Mehler-Heine type formula and Plancherel-Rotach type exterior asymptotics are derived. Moreover, as consequence, asymptotics of zeros and critical points of Sobolev polynomials as well as the distribution of contracted zeros and the nth root asymptotic are obtained. Also, some numerical examples are presented. Finally, in the last Section an upper estimate for the Sobolev polynomials is given.

Consider the Sobolev inner product (1.1). Denote $\{S_n\}_n$ and $\{T_n\}_n$ the sequences of polynomials orthogonal with respect to (1.1) and the measure μ_1, respectively, normalized by the condition that S_n and T_n have the same leading coefficient as the classical Laguerre polynomial $L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\,x^n + \ldots$. Observe that $T_0 = S_0 = L_0^{(\alpha)}$, and $S_1 = L_1^{(\alpha)}$.

Throughout this paper the following notation will be used:

$$\|L_n^{(\alpha)}\|^2_{\mu_0} = \int_0^\infty (L_n^{(\alpha)}(x))^2\, d\mu_0(x)\,, \quad \|T_n\|^2_{\mu_1} = \int_0^\infty (T_n(x))^2\, d\mu_1(x)$$
$$\text{and} \quad \|S_n\|^2_S = (S_n, S_n)_S.$$

Many of the properties of classical Laguerre polynomials can be seen, for example, in the classical book [13] of Szegő. Next, in the following Proposition we summarize those of them who play an important role in this paper giving a reference for the reader:

Proposition 1.1 *The following properties hold for Laguerre polynomials:*

(a) [13, formula (5.1.1)]:

$$\|L_n^{(\alpha)}\|^2_{\mu_0} = \int_0^\infty (L_n^{(\alpha)}(x))^2\, x^\alpha\, e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}, \quad \alpha > -1. \tag{1.2}$$

(b) [13, formula (5.1.13)]:

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x)\,, \quad \alpha \in \mathbb{R}. \tag{1.3}$$

(c) Three term recurrence relation [13, formula (5.1.10)]:

$$xL_n^{(\alpha)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \tag{1.4}$$

$$L_{-1}^{(\alpha)}(x) = 0 \quad \textit{and} \quad L_0^{(\alpha)}(x) = 1.$$

(d) *The sequence* $\left\{\frac{L_n^{(\alpha)}(x)}{n^{\alpha/2-1/4}}\right\}_n$ *is uniformly bounded on compact subsets of* $(0,+\infty)$ *([13, Th. (8.22.1)]).*

(e) *It holds*

$$\frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}\,J_\alpha(2\sqrt{nx}) + O(n^{-3/4}), \tag{1.5}$$

uniformly on compact subsets of $(0,+\infty)$ *where* J_α *is the Bessel function ([13, Section 8.22 and formula (1.71.7)]).*

(f) *Mehler–Heine formula [13, Th.8.1.3]:*

$$\lim_{n\to\infty}\frac{L_n^{(\alpha)}(x/n)}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}), \tag{1.6}$$

uniformly on compact subsets of $\mathbb{C}$.

(g) *Ratio asymptotics for scaled Laguerre polynomials ([14, p.117]:*

$$\lim_{n\to\infty}\frac{L_{n-1}^{(\alpha)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = -\frac{1}{\varphi\left((x-2)/2\right)}\,, \tag{1.7}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$ *and uniformly on* $j\in\mathbb{N}\cup\{0\}$ *where* φ *is the conformal mapping of* $\mathbb{C}\setminus[-1,1]$ *onto the exterior of the unit circle given by*

$$\varphi(x) = x+\sqrt{x^2-1}\,, \quad x\in\mathbb{C}\setminus[-1,1], \tag{1.8}$$

with $\sqrt{x^2-1}>0$ *when* $x>1$.

We want to remark that from (1.6), it can be shown that

$$\lim_{n\to\infty}\frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}), \tag{1.9}$$

uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j\in\mathbb{N}\cup\{0\}$.

2 The orthogonal polynomials T_n and the Sobolev orthogonal polynomials S_n

Really, the polynomials T_n orthogonal with respect to the measure μ_1 have a particular interest by themselves since they are orthogonal with respect to a measure which has as absolutely continuous part a rational modification of the Laguerre weight function $x^{\alpha+1}e^{-x}$ on $[0,\infty)$ and possibly a mass point (a Dirac delta) at $\xi\le 0$. In fact, we use the following results established in [11].

Lemma 2.1 *(a) [11, Lemma 4.1]. The polynomials T_n satisfy the relation*

$$T_n(x) = L_n^{(\alpha+1)}(x) - c_n L_{n-1}^{(\alpha+1)}(x), \quad n \geq 0\,, \tag{2.1}$$

where

$$c_n = \frac{\|T_n\|_{\mu_1}^2}{\|L_n^{(\alpha)}\|_{\mu_0}^2},\ n \geq 0. \tag{2.2}$$

(b) Relation (2.1) can be expressed as

$$T_n(x) = L_n^{(\alpha)}(x) - d_n L_{n-1}^{(\alpha+1)}(x), \quad n \geq 0, \tag{2.3}$$

where $d_n = c_n - 1,\ n \geq 0\,.$

(c) [11, Lemma 4.4]. It holds

$$\lim_n \sqrt{n}\, d_n = d(\xi) = \begin{cases} -\sqrt{-\xi} & \text{if} \quad M = 0, \\ \sqrt{-\xi} & \text{if} \quad M > 0, \end{cases} \tag{2.4}$$

and therefore $\lim_n c_n = 1$*. In particular,*

- *If* $\xi = 0$ *and* $M > 0$*, we get*

$$\lim_{n\to\infty} n\, d_n = \alpha + 1. \tag{2.5}$$

- *If* $\xi = M = 0$*, then* $d_n = 0$ *and therefore* $c_n = 1$*, for all* n*.*

We have the following explicitly relation between Sobolev orthogonal polynomials and Laguerre polynomials (see [11, Lemma 4.7] and [4] in a more general framework):

Lemma 2.2 *It holds*

$$L_n^{(\alpha)}(x) - c_{n-1} L_{n-1}^{(\alpha)}(x) = S_n(x) - a_{n-1} S_{n-1}(x), \quad n \geq 1, \tag{2.6}$$

where $a_n = c_n \,\frac{\|L_n^{(\alpha)}\|_{\mu_0}^2}{\|S_n\|_S^2}$*. Moreover (see [11, Lemma 4.10]),*

$$\lim_{n\to\infty} a_n = a = \frac{1}{\varphi\left((\lambda+2)/2\right)}\,, \tag{2.7}$$

where φ *is defined by (1.8).*

It is clear from (2.6) that we can compute S_n in a recursive way, and even, we can give an explicit expression for S_n in terms of Laguerre polynomials and the sequences $\{c_n\}$ and $\{a_n\}$. Thus, if we want to compute the polynomials S_n, calculate its zeros or realize any numerical experiment with these polynomials, we have to compute effectively the sequence $\{c_n\}$ that appears in relation (2.1) and the sequence $\{a_n\}$.

First, we obtain a nonlinear recurrence relation for $\{c_n\}$.

Proposition 2.3 *It holds, for $n \geq 0$,*

$$c_{n+1} = \frac{2n+2+\alpha-\xi}{n+1} - \frac{n+1+\alpha}{(n+1)\,c_n}\,, \tag{2.8}$$

with

$$c_0 = \frac{\int_0^\infty \frac{x^{\alpha+1}e^{-x}}{x-\xi}\,dx + M}{\Gamma(\alpha+1)}\,.$$

Proof. We express the polynomial $-\frac{x-\xi}{n+1}\,L_n^{(\alpha+1)}(x)$ in terms of the basis $\{T_i\}_{i=0}^{n+1}$ and we obtain:

$$-\frac{x-\xi}{n+1}\,L_n^{(\alpha+1)}(x) = T_{n+1}(x) - \frac{n+1+\alpha}{(n+1)\,c_n}\,T_n(x)\,, \quad n \geq 0. \tag{2.9}$$

Then, multiplying (2.9) by $L_n^{(\alpha+1)}(x)$ and integrating with respect to the measure $x^{\alpha+1}e^{-x}dx$ on $[0,\infty)$, we can derived the result using the formulas (1.4) and (2.1). □

The sequence $\{c_n\}$ also plays an important role for the polynomials $\{T_n\}$ from computational point of view as well as to obtain asymptotic properties.

It is well known (see [2]) that if we want to compute zeros of polynomials T_n they are calculated as the eigenvalues of the symmetric tridiagonal Jacobi matrix, whose entries are the coefficients of the three term recurrence relation for the orthonormal polynomials t_n with positive leader coefficient:

$$xt_n(x) = \beta_{n+1}t_{n+1}(x) + \gamma_n t_n(x) + \beta_n t_{n-1}(x), \quad n \geq 0,$$

with $t_{-1}(x) = 0,\ t_0(x) = \|T_0\|_{\mu_1}^{-1}$. Straightforward computations show that

$$\beta_n = \sqrt{n(n+\alpha)\,\frac{c_n}{c_{n-1}}} \quad \text{and} \quad \gamma_n = nc_n + \frac{n+\alpha+1}{c_n} + \xi.$$

Now, we present several analytic properties of the polynomials T_n.

Proposition 2.4 *For $\alpha > -1$, the following properties hold:*

(a) The sequence $\left\{\frac{T_n(x)}{n^{\alpha/2-1/4}}\right\}_n$ is uniformly bounded on compact subsets of $(0,+\infty)$.

(b) Asymptotics on $(0,+\infty)$ for T_n : if $\xi < 0$,

$$\frac{T_n(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}\,J_\alpha(2\sqrt{nx}) + O(n^{-1/4}),$$

and, if $\xi = 0$,

$$\frac{T_n(x)}{n^{\alpha/2}} = e^{x/2}x^{-\alpha/2}\,J_\alpha(2\sqrt{nx}) + O(n^{-3/4}).$$

Both uniformly on compact subsets of $(0,+\infty)$.

(c) Mehler–Heine type formula for T_n : If $\xi < 0$,

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha+1/2}} = -d(\xi)\, x^{-(\alpha+1)/2} J_{\alpha+1}(2\sqrt{x}),$$

if $\xi = 0$ and $M > 0$,

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha}} = -\, x^{-\alpha/2} J_{\alpha+2}(2\sqrt{x}),$$

and, if $\xi = M = 0$,

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha}} = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}).$$

All the limits hold uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N}\cup\{0\}$, where $d(\xi)$ is given by (2.4).

(d) Plancherel–Rotach type exterior asymptotics for T_n :

$$\lim_{n\to\infty} \frac{T_n((n+j)x)}{L_n^{(\alpha+1)}((n+j)x)} = 1 + \varphi\left(\frac{x-2}{2}\right)^{-1}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$ and uniformly on $j \in \mathbb{N}\cup\{0\}$.

Proof. If $\xi = M = 0$ all the results are true because of $T_n(x) = L_n^{(\alpha)}(x)$, for all n.
(a) We divide (2.3) by $n^{\alpha/2-1/4}$. Then, using (2.4) and Proposition 1.1 (d) the result follows.
(b) If $\xi < 0$ we divide (2.3) by $n^{\alpha/2}$ and using again Proposition 1.1 (d) and (2.4) we get

$$\begin{aligned}\frac{T_n(x)}{n^{\alpha/2}} &= \frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} - \frac{1}{(n-1)^{1/4}}\sqrt{n}\, d_n \left(\frac{n-1}{n}\right)^{(\alpha+1)/2} \frac{L_{n-1}^{(\alpha+1)}(x)}{(n-1)^{(\alpha+1)/2-1/4}} \\ &= \frac{L_n^{(\alpha)}(x)}{n^{\alpha/2}} + O(n^{-1/4}).\end{aligned}$$

Thus, the result follows from (1.5). On the other hand, if $\xi = 0$ and $M > 0$, we can proceed in the same way using now (2.5).
(c) Whenever $\xi < 0$, scaling the variable as $x \to x/(n+j)$ in relation (2.3) we get

$$\frac{T_n(x/(n+j))}{n^{\alpha+1/2}} = \frac{L_n^{(\alpha)}(x/(n+j))}{n^{\alpha+1/2}} - \sqrt{n}\, d_n \frac{L_{n-1}^{(\alpha+1)}(x/(n+j))}{n^{\alpha+1}}.$$

It only remains to use (1.9) and (2.4) to reach the result.

If $\xi = 0$ and $M > 0$, proceeding as above and using (2.5) it follows that

$$\lim_{n\to\infty} \frac{T_n(x/(n+j))}{n^{\alpha}} = x^{-\alpha/2}\, J_\alpha(2\sqrt{x}) - (\alpha+1)\, x^{-(\alpha+1)/2}\, J_{\alpha+1}(2\sqrt{x}).$$

Now, using

$$2\alpha\, z^{-1} J_\alpha(z) = J_{\alpha-1}(z) + J_{\alpha+1}(z) \tag{2.10}$$

(see, [13, formula (1.71.5)]), we have the result.
(d) In the same way as in (c), scaling the variable as $x \to (n+j)x$ in relation (2.1), dividing by $L_n^{(\alpha+1)}((n+j)x)$ and using (1.7) and $\lim_n c_n = 1$, the result arises. □

3 Asymptotics of Sobolev orthogonal polynomials S_n

In this section, first of all, we will obtain the strong asymptotic of S_n on the positive semiaxis and analogues of the Mehler-Heine and Plancherel-Rotach type asymptotic formulas for the Sobolev polynomials.

If we look for analytic properties of the Sobolev orthogonal polynomials S_n, we have to pay attention to the polynomials in the left hand side of (2.6), that is

$$V_n(x) := L_n^{(\alpha)}(x) - c_{n-1} L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x) - d_{n-1} L_{n-1}^{(\alpha)}(x), \quad n \geq 0, \tag{3.1}$$

where the last equality is a consequence of (1.3) and the relation between the coefficients c_n and d_n. We can observe that the polynomials V_n are, in some sense, close to the polynomials T_n, exactly V_n is a primitive of $-T_{n-1}$, i.e., $V_n'(x) = -T_{n-1}(x)$.

First, we give the strong asymptotic of S_n on $(0, +\infty)$. In order to do this, we will use several analytic properties of these polynomials V_n. Notice that, to establish Proposition 2.4 it is only necessary to know the asymptotic behaviour of the sequence $\{d_n\}$ and of the corresponding Laguerre polynomials involved in the algebraic relation: in the case of T_n they are the Laguerre polynomials with parameter $\alpha+1$ and in the case of V_n the Laguerre polynomials with parameter α.

Theorem 3.1 *For $\alpha > -1$, we have*

$$\frac{S_n(x)}{n^{(\alpha-1)/2}} = e^{x/2} x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{nx}) + O(n^{-1/4}),$$

uniformly on compact subsets of $(0, +\infty)$.

Proof. From (2.6) and (3.1),

$$S_n(x) = V_n(x) + a_{n-1} S_{n-1}(x) \tag{3.2}$$

so,

$$\frac{S_n(x)}{n^{\alpha/2-3/4}} = \frac{V_n(x)}{n^{\alpha/2-3/4}} + a_{n-1} \left(\frac{n-1}{n}\right)^{\alpha/2-3/4} \frac{S_{n-1}(x)}{(n-1)^{\alpha/2-3/4}}.$$

Dividing in (3.1) by $n^{\alpha/2-3/4}$ and taking into account Proposition 1.1 (d) and (2.4), we have that $\left\{V_n(x)/n^{\alpha/2-3/4}\right\}_n$ is uniformly bounded on compact sets of $(0,+\infty)$. Since $a_{n-1}\left(\frac{n-1}{n}\right)^{\alpha/2-3/4} \longrightarrow a \in (0,1)$, standard arguments yield $\left\{S_n(x)/n^{\alpha/2-3/4}\right\}_n$ is also uniformly bounded.

On the other hand, using (1.5) and Lemma 2.1 (c), it can be deduced that if $\xi < 0$,

$$\frac{V_n(x)}{n^{(\alpha-1)/2}} = \frac{L_n^{(\alpha-1)}(x)}{n^{(\alpha-1)/2}} + O(n^{-1/4})$$

and if $\xi = 0$

$$\frac{V_n(x)}{n^{(\alpha-1)/2}} = \frac{L_n^{(\alpha-1)}(x)}{n^{(\alpha-1)/2}} + O(n^{-3/4})$$

where the bound for the remainder holds uniformly on compact sets of $(0,+\infty)$.

Finally, observe that

$$\begin{aligned}\frac{S_n(x)}{n^{(\alpha-1)/2}} &= \frac{V_n(x)}{n^{(\alpha-1)/2}} + \frac{a_{n-1}}{(n-1)^{1/4}}\left(\frac{n-1}{n}\right)^{(\alpha-1)/2}\frac{S_{n-1}(x)}{(n-1)^{\alpha/2-3/4}} \\ &= \frac{V_n(x)}{n^{(\alpha-1)/2}} + O(n^{-1/4}) = \frac{L_n^{(\alpha-1)}(x)}{n^{(\alpha-1)/2}} + O(n^{-1/4}),\end{aligned}$$

uniformly on compact subsets of $(0,+\infty)$.

Using (1.5), the theorem follows. □

Such as we mention in Section 2, we can express the polynomials S_n in terms of the Laguerre polynomials with parameter α, that is, using (3.2) in a recursive way and taking into account (3.1) we obtain

$$S_n(x) = \sum_{i=0}^{n} b_i^{(n)} V_{n-i}(x) = \sum_{i=0}^{n} b_i^{(n)}\left(L_{n-i}^{(\alpha)}(x) - c_{n-i-1}L_{n-i-1}^{(\alpha)}(x)\right), \quad n \geq 0, \tag{3.3}$$

$$\text{where} \quad b_i^{(n)} = \prod_{j=1}^{i} a_{n-j} \quad \textit{and} \quad b_0^{(n)} = 1.$$

Moreover, from (2.7) we have

$$\lim_{n\to\infty} b_i^{(n)} = \varphi\left(\frac{\lambda+2}{2}\right)^{-i} = a^i, \quad \text{for all } i. \tag{3.4}$$

Next, we obtain asymptotic results for the Sobolev orthogonal polynomials S_n. Before, we want to remark that for the case corresponding to $\xi = M = 0$, that is, $d\mu_0 = d\mu_1 = x^\alpha e^{-x}dx$, $\alpha > -1$, Mehler–Heine type formula and Plancherel–Rotach type exterior asymptotics were obtained in Theorem 5 of [6], in other framework. Here, we include this case for a more comprehensive reading.

Previously, we give the following technical result:

Lemma 3.2 *There exist constants C and r with $C > 1$ and $0 < r < 1$ such that the coefficients $b_i^{(n)}$ in (3.3) verify $b_i^{(n)} < C\, r^i$ for all $n \geq 0$ and $0 \leq i \leq n$.*

Proof. From Lemma 2.2 we know that $a_n > 0$ and $\lim_n a_n = a < 1$, then there exists $r \in (a, 1)$ such that $0 < a_n < r < 1$ for all $n \geq n_0$. Therefore, whenever $1 \leq i \leq n - n_0$, $\;b_i^{(n)} < r^i$ and for the remainder values of i, taking $M = \max\{1, a_0, a_1, \ldots, a_{n_0-1}\}$, we have

$$b_i^{(n)} = \prod_{j=1}^{n-n_0} a_{n-j} \prod_{j=n-n_0+1}^{i} a_{n-j} < r^{n-n_0} M^{i-n+n_0} \leq r^{n-n_0} M^{n_0} \leq r^i \left(\frac{M}{r}\right)^{n_0}.$$

The result follows with $C = \left(\frac{M}{r}\right)^{n_0}$. $\square$

Theorem 3.3 *Let $\alpha > -1$, the polynomials S_n orthogonal with respect to the inner product (1.1) satisfy*

(a) A Mehler–Heine type formula. It holds: if $\xi < 0$,

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^{\alpha-1/2}} = -\frac{d(\xi)}{1-a}\, x^{-\alpha/2}\, J_\alpha(2\sqrt{x}),$$

if $\xi = 0$ and $M > 0$,

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^{\alpha-1}} = \frac{1}{1-a}\, s(x),$$

and, if $\xi = M = 0$,

$$\lim_{n\to\infty} \frac{S_n(x/n)}{n^{\alpha-1}} = \frac{1}{1-a}\, x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}),$$

where a and $d(\xi)$ are given by (2.7) and (2.4), respectively, and

$$s(x) = x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) - (\alpha+1)\, x^{-\alpha/2} J_\alpha(2\sqrt{x}). \qquad (3.5)$$

All the limits hold uniformly on compact subsets of $\mathbb{C}$.

(b) Plancherel–Rotach type exterior asymptotics. It holds

$$\lim_{n\to\infty} \frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = \frac{\varphi(\frac{x-2}{2}) + 1}{\varphi(\frac{x-2}{2}) + a},$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, 4]$ where φ and a are given by (1.8) and (2.7), respectively.

Proof. (a) From (3.3), we have

$$\frac{S_n(x/n)}{n^{\alpha-1/2}} = \sum_{i=0}^{n} b_i^{(n)}\, \frac{V_{n-i}(x/n)}{n^{\alpha-1/2}} =: \sum_{i=0}^{n} v_{n,i}(x/n). \qquad (3.6)$$

Whenever $\xi < 0$, dividing by $n^{\alpha-1/2}$ in formula (3.1) evaluated at $x/(n+j)$, and using (1.9) and (2.4), we deduce that

$$\lim_{n\to\infty} \frac{V_n(x/(n+j))}{n^{\alpha-1/2}} = -d(\xi)\, x^{-\alpha/2} J_\alpha(2\sqrt{x}), \tag{3.7}$$

holds uniformly on compact sets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$.

Given a compact set $K \subset \mathbb{C}$, because of this last result and Lemma 3.2, there exists a constant D, depending only on K, such that $|v_{n,i}(x/n)| < D\, r^i$ for $i = 0, \dots, n$ and $x \in K$. Therefore, by Lebesgue's dominated convergence theorem, (3.7) and (3.4), we have

$$\lim_{n\to\infty} \sum_{i=0}^{n} v_{n,i}(x/n) = \sum_{i=0}^{\infty} \lim_{n\to\infty} v_{n,i}(x/n) = -d(\xi)\, x^{-\alpha/2}\, J_\alpha(2\sqrt{x}) \sum_{i=0}^{\infty} a^i$$

uniformly on compact subsets of $\mathbb{C}$ and the result follows.

For the remainder cases, formula (3.7) adopts the form:

$$\lim_{n\to\infty} \frac{V_n(x/(n+j))}{n^{\alpha-1}} = x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) - (\alpha+1)\, x^{-\alpha/2} J_\alpha(2\sqrt{x}), \quad \xi = 0,\ M > 0,$$
$$\lim_{n\to\infty} \frac{V_n(x/(n+j))}{n^{\alpha-1}} = x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}), \quad \xi = 0 = M.$$

Now it suffices to handle as above.
(b) From (3.3) we can write

$$\frac{S_n(nx)}{L_n^{(\alpha)}(nx)} = \sum_{i=0}^{n} b_i^{(n)}\, \frac{V_{n-i}(nx)}{L_n^{(\alpha)}(nx)}, \quad x \in \mathbb{C} \setminus [0,4].$$

Polynomials V_n satisfy the following Plancherel–Rotach type exterior asymptotics

$$\lim_{n\to\infty} \frac{V_n((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = 1 + \varphi\left(\frac{x-2}{2}\right)^{-1} \tag{3.8}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,4]$ and uniformly on $j \in \mathbb{N} \cup \{0\}$. This is a simple consequence of (1.7) and (3.1).

Now, handling in the same way as in (a) and using again (1.7), we can deduce

$$\lim_{n\to\infty} \sum_{i=0}^{n} b_i^{(n)}\, \frac{V_{n-i}(nx)}{L_n^{(\alpha)}(nx)} = \lim_{n\to\infty} \sum_{i=0}^{n} b_i^{(n)}\, \frac{V_{n-i}(nx)}{L_{n-i}^{(\alpha)}(nx)}\, \frac{L_{n-i}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}$$
$$= \sum_{i=0}^{\infty} \lim_{n\to\infty} \left(b_i^{(n)}\, \frac{V_{n-i}(nx)}{L_{n-i}^{(\alpha)}(nx)}\, \frac{L_{n-i}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} \right) = \left(1 + \varphi\left(\frac{x-2}{2}\right)^{-1}\right) \sum_{i=0}^{\infty} \left(\frac{-a}{\varphi\left(\frac{x-2}{2}\right)} \right)^i,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,4]$, and thus, the result follows. □

The above theorem allows us to obtain additional results about asymptotic properties of zeros and critical points of Sobolev polynomials S_n. First, recall that S_n has n different, real zeros, and at most one of them is outside $(0,+\infty)$, they interlace with those of $L_n^{(\alpha)}$ and the zeros of S'_n with those of T_{n-1} (for more information about location of these zeros, see [10]). Moreover, from Theorem 4.11 in [11], it follows that they accumulate on $\{\xi\}\cup[0,+\infty)$ (in particular on $[0,+\infty)$, when $M=0$).

Corollary 3.4 *For $\alpha>-1$, denote with $j_{\alpha,i}$ the ith positive zero of Bessel function $J_\alpha(x)$. Let $\{x_{n,i}\}_{i=1}^{n}$ be the zeros in increasing order of the polynomial S_n orthogonal with respect to the inner product (1.1) and $\{\tilde{x}_{n,i}\}_{i=1}^{n-1}$ be the critical points of S_n. Then,*

(a) If $\xi<0$, we have

$$\lim_{n\to\infty} n\,x_{n,i}=\frac{j_{\alpha,i}^2}{4} \quad \text{and} \quad \lim_{n\to\infty} n\,\tilde{x}_{n,i}=\frac{j_{\alpha+1,i}^2}{4}.$$

(b) If $\xi=0$ and $M>0$, we have

$$\lim_{n\to\infty} n\,x_{n,i}=s_{\alpha,i}, \quad \lim_{n\to\infty} n\,\tilde{x}_{n,1}=0 \quad \text{and} \quad \lim_{n\to\infty} n\,\tilde{x}_{n,i}=\frac{j_{\alpha+2,i-1}^2}{4},\ i\geq 2,$$

where $s_{\alpha,i}$ denotes the ith real zero of function $s(x)$ defined in (3.5).

(c) If $\xi=M=0$, we have

$$\lim_{n\to\infty} n\,x_{n,i}=\frac{j_{\alpha-1,i}^2}{4} \quad \text{and} \quad \lim_{n\to\infty} n\,\tilde{x}_{n,i}=\frac{j_{\alpha,i}^2}{4},$$

where three cases are possible:

- *If $-1<\alpha<0$, $j_{\alpha-1,1}$ is any of the two purely imaginary zeros of $J_{\alpha-1}(x)$ and, for $i\geq 2$, $j_{\alpha-1,i}$ is the $(i-1)$th positive real zero of $J_{\alpha-1}(x)$.*
- *If $\alpha=0$, $j_{\alpha-1,1}=0$ and, for $i\geq 2$, $j_{\alpha-1,i}$ is the $(i-1)$th positive real zero of $J_{\alpha-1}(x)$.*
- *If $\alpha>0$, $j_{\alpha-1,i}$ is the ith positive real zero of $J_{\alpha-1}(x)$.*

Proof. (a) The result for the zeros is a consequence of Theorem 3.3 (a) and Hurwitz's theorem. Concerning the critical points, since we have uniform convergence in the Mehler–Heine type formula (Th. 3.3 (a)), taking derivatives and using properties of Bessel functions ([13, Section 1.7]) we get

$$\lim_{n\to\infty}\frac{S'_n(x/n)}{n^{\alpha+1/2}}=\frac{d(\xi)}{1-a}\,x^{-(\alpha+1)/2}J_{\alpha+1}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$, which yields the result.

(b) Denote $g_\alpha(x) = x^{-\alpha/2} J_\alpha(2\sqrt{x}) = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)}, \quad x \in \mathbb{C}$. From definition of $s(x)$ (see (3.5) and (2.10)), we can write

$$s(x) = -g_\alpha(x) - x\, g_{\alpha+1}(x) = \sum_{i=0}^{\infty} \frac{(i-1)}{i!} \frac{(-x)^i}{\Gamma(i+\alpha+1)},$$

for $\alpha > -1$ and $x \in \mathbb{C}$.

Observe that, if $x \in (-\infty, 0)$, then $g_\alpha(x) > 0$, $\lim_{x\to-\infty} g_\alpha(x) = +\infty$ and $\lim_{x\to-\infty} s(x) = +\infty$.

Using formula (1.71.5) in [13] we have $s'(x) = x g_{\alpha+2}(x)$, $x \in \mathbb{C}$ and therefore $s(x)$ is a decreasing function on $(-\infty, 0)$. Since $s(0) < 0$, we have that $s(x)$ has only one negative zero. Moreover, because of the positive zeros of $J_\alpha(x)$ interlace with those of $J_{\alpha+1}(x)$, we can deduce that there is precisely one zero of $s(x)$ between two consecutive positive zeros of $J_{\alpha+1}(2\sqrt{x})$.

Now, again by Hurwitz's theorem the result for the zeros follows. Finally, we have

$$\lim_{n\to\infty} \frac{S'(x/n)}{n^\alpha} = \frac{1}{1-a} s'(x) = \frac{1}{1-a} x^{-\alpha/2}\, J_{\alpha+2}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$, which implies the result.
(c) It can be also obtained in a similar way (see Proposition 4 and Remark 2 in [6]). □

Remark. The existence of a negative zero of S_n is an interesting problem (see, for example, Section 5 of [10] and [6]). Here, we have found the range of values of the parameters α, ξ, and M for which the polynomials S_n have a negative zero for n sufficiently large i.e.,

- The polynomials S_n have one negative zero for n sufficiently large if and only if either $\alpha > -1$, $\xi = 0$, and $M > 0$ or $-1 < \alpha < 0$ and $\xi = M = 0$.
- Moreover, the critical points of S_n for n sufficiently large lie on $[0, +\infty)$.

Finally, observe that, fixed i positive integer, the zeros of S_n satisfy $\lim_n x_{ni} = 0$; more precisely $x_{ni} = O(1/n)$. Even, whenever S_n has a negative zero x_{n1}, $\lim_n x_{n1} = 0$.

In order to illustrate these anaytic results, we show numerically the behaviour of the first zero, x_{n1}, of S_n in the cases of Corollary 3.4 where the nonlinear recurrence relation satisfied by c_n (formula (2.8)) and a_n (formula (4.7) in [11]) have been used.

(a) $\alpha = -0.5,\ \xi = -10,\ \lambda = 1.$

	$M = 0$		$M = 2$	
	nx_{n1}	$x_{n,1}$	$nx_{n,1}$	$x_{n,1}$
$n = 50$	0.5985025263	0.0119700505	0.6560458759	0.0131209175
$n = 100$	0.6022670343	0.0060226703	0.6421427906	0.0064214279
$n = 150$	0.6042891938	0.0040285946	0.6366175985	0.0042441173
$n = 200$	0.6056173146	0.0030280866	0.6335134224	0.0031675671
$n = 250$	0.6065803700	0.0024263219	0.6314766976	0.0025259068
	0.6168502751		0.6168502751	
	$\frac{j^2_{\alpha,1}}{4}$		$\frac{j^2_{\alpha,1}}{4}$	

(b) $\xi = 0,\ M = 2,\ \lambda = 1.$

	$\alpha = -0.5$		$\alpha = 2.5$	
	$nx_{n,1}$	$x_{n,1}$	$nx_{n,1}$	$x_{n,1}$
$n = 50$	−0.9995290524	−0.0199905810	−4.4617547547	−0.0892350951
$n = 100$	−1.0118720710	−0.0101187207	−4.3961517653	−0.0439615177
$n = 150$	−1.0191664985	−0.0067944433	−4.3745120007	−0.0291634133
$n = 200$	−1.0240655338	−0.0051203277	−4.3637453016	−0.0218187265
$n = 250$	−1.0276502314	−0.0041106009	−4.3573033494	−0.0174292134
	−1.066582516		−4.3325842295	
	$s_{\alpha,1}$		$s_{\alpha,1}$	

(c) $\xi = M = 0,\ \lambda = 1.$

	$\alpha = -0.5$		$\alpha = 0$	$\alpha = 2.5$	
	$nx_{n,1}$	$x_{n,1}$	$nx_{n,1}$	$nx_{n,1}$	$x_{n,1}$
$n = 50$	−0.366308	−0.007326	$1.41153\,10^{-19}$	4.98876	0.09978
$n = 100$	−0.362992	−0.003630	$3.56416\,10^{-40}$	5.01701	0.05017
$n = 150$	−0.361917	−0.002413	$6.74969\,10^{-61}$	5.02696	0.03351
$n = 200$	−0.361384	−0.001807	$1.13621\,10^{-81}$	5.03204	0.02516
$n = 250$	−0.361066	−0.001444	$1.79310\,10^{-102}$	5.03512	0.02014
	−0.359807		0	5.04768	
	$\frac{j^2_{\alpha-1,1}}{4}$		$\frac{j^2_{\alpha-1,1}}{4}$	$\frac{j^2_{\alpha-1,1}}{4}$	

For a better reading we have rounded to six digits the numerical results in the case (c) and we also eliminated the column $x_{n,1}$ for $\alpha = 0$.

From Theorem 3.3 and the zero distribution of the orthonormal Laguerre polynomials $l_n^{(\alpha)}$ (see [14, Section 4.3]), the asymptotic distribution of the contracted zeros and the nth root asymptotics for the scaled Sobolev polynomials can be derived:

Corollary 3.5 *(a) The contracted zeros of S_n, $\frac{x_{ni}}{n}$, accumulate on $[0,4]$ and they have the same asymptotic distribution as the contracted zeros of the orthonormal Laguerre polynomials $l_n^{(\alpha)}$, that is, it has density $d\nu(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,dx$.*

(b) The formula

$$\lim_n |S_n(nx)|^{1/n} = exp\,\{1 + \int_0^4 \log|x-y|\,d\nu(y)\}$$

is true uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$.

Remark. For monic Sobolev polynomials $\widehat{S_n}$ we have

$$\lim_n \frac{1}{n}\,|\widehat{S_n}(nx)|^{1/n} = exp\,\{\frac{1}{2\pi}\int_0^4 \log|x-y|\,\sqrt{\frac{4-y}{y}}\,dy\}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$ or, equivalently,

$$\lim_n \frac{1}{2n}\,|\widehat{S_n}(2nx)|^{1/n} = exp\,\{\frac{1}{\pi}\int_0^2 \log|x-y|\,\sqrt{\frac{2-y}{y}}\,dy\}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,2]$.

Observe that this is exactly the result for monic Laguerre–Sobolev polynomials of type I obtained in [12, Th. 2.2], using potential theory. (In all the results in [12] concerned with nth root asymptotic, the locally uniformly convergence holds in $\mathbb{C}\setminus[0,2]$ instead of in $[0,2]$).

4 Upper bound for Sobolev orthogonal polynomials S_n

To obtain an upper bound for Sobolev orthogonal polynomials our starting-point will be formula (3.3). A global estimates for classical Laguerre polynomials with respect to n, x, and α is known (see formulas (22.14.13) and (22.14.14) in [1]): For $x \geq 0$, $n \geq 0$ and $\alpha > -1$, the inequality

$$|L_n^{(\alpha)}(x)| \leq A(n,\alpha)\,e^{x/2} \tag{4.1}$$

where

$$A(n,\alpha) = \begin{cases} \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)} & \text{if} \quad \alpha \geq 0, \\ 2 - \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)} & \text{if} \quad -1 < \alpha \leq 0, \end{cases} \tag{4.2}$$

holds.

Therefore, we need upper estimates for the coefficients $b_i^{(n)}$ (that is, for a_n) and c_n. This is done in the next Lemma.

Lemma 4.1 *For $n \geq 1$, the coefficients c_n and a_n in Lemma 2.2 satisfy*

$$\frac{n+1+\alpha}{2(n+1)+\alpha-\xi} < c_n < 2 + \frac{\alpha-\xi}{n}, \quad n \geq 1, \tag{4.3}$$

and

$$a_n < \left(2 + \frac{\alpha-\xi}{n}\right) \frac{2n+\alpha-\xi}{(2+\lambda)n+\alpha-\xi}, \quad n \geq 1. \tag{4.4}$$

Proof. From the recurrence relation (2.8) for the parameters c_n, since $c_n > 0$ for every n, we get the inequalities (4.3).

On the other hand, remind that the coefficients a_n in formula (2.6) are defined by $a_n = c_n \frac{\|L_n^{(\alpha)}\|^2_{\mu_0}}{\|S_n\|^2_S}$. As a consequence of the extremal property of the norms of the monic orthogonal polynomials, we have

$$\|S_n\|^2_S \geq \|L_n^{(\alpha)}\|^2_{\mu_0} + \lambda \|T_{n-1}\|^2_{\mu_1}, \quad n \geq 1,$$

which, by the definition of c_n, (see (2.2)), and (1.2) leads to

$$\frac{\|S_n\|^2_S}{\|L_n^{(\alpha)}\|^2_{\mu_0}} \geq 1 + \lambda \frac{\|T_{n-1}\|^2_{\mu_1}}{\|L_n^{(\alpha)}\|^2_{\mu_0}} = 1 + \lambda \frac{n}{n+\alpha} c_{n-1}. \tag{4.5}$$

Thus, from (4.3) and (4.5), we obtain $\frac{\|L_n^{(\alpha)}\|^2_{\mu_0}}{\|S_n\|^2_S} \leq \left(1 + \frac{\lambda n}{2n+\alpha-\xi}\right)^{-1}$ and so (4.4) holds. $\square$

A global estimate for Sobolev orthogonal polynomials is now deduced:

Theorem 4.2 *For $x \geq 0$, $\alpha > -1$ and $n \geq 1$ we have*

$$|S_n(x)| \leq C \frac{1-r^n}{1-r} A(n,\alpha)\, e^{x/2},$$

where $C = \begin{cases} 3+\alpha-\xi & \text{if} \quad \alpha \geq \xi, \\ 3 & \text{if} \quad \alpha \leq \xi, \end{cases}$ *and* $r = \begin{cases} \frac{(2+\alpha-\xi)^2}{2+\lambda+\alpha-\xi} & \text{if} \quad \alpha \geq \xi, \\ \frac{4}{2+\lambda} & \text{if} \quad \alpha \leq \xi. \end{cases}$

Proof. Observe that, using $b_n^{(n)} = a_0\, b_{n-1}^{(n)}$, formula (3.3) can be writen in the form

$$S_n(x) = \sum_{i=0}^{n-2} b_i^{(n)} \left(L_{n-i}^{(\alpha)}(x) - c_{n-i-1}\, L_{n-i-1}^{(\alpha)}(x)\right) + b_{n-1}^{(n)} \left(L_1^{(\alpha)}(x) - c_0 + a_0\right).$$

Then, as $a_0 = c_0$,

$$|S_n(x)| \leq \sum_{i=0}^{n-2} b_i^{(n)} \left(|L_{n-i}^{((\alpha)}(x)| + c_{n-i-1}\, |L_{n-i-1}^{(\alpha)}(x)|\right) + b_{n-1}^{(n)}\, |L_1^{(\alpha)}(x)|. \tag{4.6}$$

It is easy to prove that, for $\alpha > -1$ and $i = 0, 1, \ldots, n$, $A(n-i,\alpha) \leq A(n,\alpha)$ and therefore, by (4.1), $|L_{n-i}^{(\alpha)}(x)| \leq A(n,\alpha)\, e^{x/2}$ which leads to

$$|S_n(x)| \leq \left[\sum_{i=0}^{n-2} b_i^{(n)}(1 + c_{n-i-1}) + b_{n-1}^{(n)}\right] A(n,\alpha)\, e^{x/2}.$$

From (4.3), analizing separately the cases $\alpha - \xi < 0$ (that is, $-1 < \alpha < \xi \leq 0$) and $\alpha - \xi \geq 0$, we get $c_n < c = \begin{cases} 2 + \alpha - \xi & \text{if} \quad \alpha \geq \xi, \\ 2 & \text{if} \quad \alpha \leq \xi \end{cases}$. In a similar way, from (4.4) we deduce that $a_n < r = \begin{cases} \frac{(2+\alpha-\xi)^2}{2+\lambda+\alpha-\xi} & \text{if} \quad \alpha \geq \xi, \\ \frac{4}{2+\lambda} & \text{if} \quad \alpha \leq \xi \end{cases}$. It suffices to write $C = 1 + c$ and the result follows. □

In some particular cases the upper estimate for the Sobolev polynomials S_n can be improved. One of them occurs when $M = 0$ in the inner product (1.1), that is $d\,\mu_1 = \frac{x^{\alpha+1}\, e^{-x}}{x-\xi}\, dx$. In this situation, integrating in formula (2.1) with respect to the measure μ_1, we have

$$c_n \int_0^\infty L_{n-1}^{(\alpha+1)}(x)\, d\mu_1(x) = \int_0^\infty L_n^{(\alpha+1)}(x)\, d\mu_1(x), \quad n \geq 1.$$

Using Rodrigues' formula for Laguerre polynomials and after integration by parts $n-1$ times, it can be derived, see ([11]),

$$\int_0^\infty L_{n-1}^{(\alpha+1)}(x)\, d\mu_1(x) = \int_0^\infty \frac{x^{n+\alpha}\, e^{-x}}{(x-\xi)^n}\, dx.$$

This implies that, for every $n \geq 1$, $c_n \leq 1$. (Observe that $c_n = 1$ only if $\xi = 0$).

As consequence, we have $a_n \leq 1$ and $b_i^n \leq 1$ for every $n \geq 1$ and $i = 0, \ldots, n-1$. Thus, the upper estimate for S_n in Theorem 4.2 becames:

$$|S_n(x)| \leq 2\, n\, A(n,\alpha)\, e^{x/2}.$$

Improvements of the estimates for $|L_n^\alpha(x)|$ lead to improvements of the ones for $|S_n(x)|$, according formula (4.6) (see, for instance [5]).

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