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R. Celorrio
V. Domínguez
F.J. Sayas

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Universidad de Zaragoza

Overlapped BEM–FEM and some Schwarz iterations *

Ricardo Celorrio[1], Víctor Domínguez[2] & Francisco–Javier Sayas[3]

[1] Dep. Matemática Aplicada, Universidad de Zaragoza, EUITIZ, 50018 Zaragoza, Spain

e–mail: celorrio@posta.unizar.es

[2] Dep. Matemática e Informática, Universidad Pública de Navarra

Campus de Arrosadía, 31006 Pamplona, Spain

e–mail: victor.dominguez@unavarra.es

[3] Dep. Matemática Aplicada, Universidad de Zaragoza, C.P.S., 50018 Zaragoza, Spain

e–mail: jsayas@posta.unizar.es

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Abstract

In this work we consider the numerical solution of the Laplace equation in a domain with holes by means of the overlapping of finite and boundary elements. The essence of the method is considering a finite element solution of the Laplace equation in the domain without holes and an exterior single–layer solution on the unbounded domain around these holes. This solution can be viewed as a limit of a discretized interior–exterior Schwarz–type iteration. A convergence analysis of both the iteration and the discrete solution is carried out, taking full generality in the BEM scheme. Some numerical experiments are also given.

AMS subject classification. 65N30, 65N38, 65F10

Key Words. Finite elements, boundary elements, Schwarz method

1 Introduction

The present work is concerned with the numerical solution of a model problem, namely Laplace's equation with Dirichlet boundary conditions, on a domain with holes (henceforth referred to as obstacles), in two or three dimensions. The basic idea is to take out the interior obstacles and make a triangulation of the simpler remaining domain using finite elements on it. At the same time we will consider the exterior problems to the set of boundaries of the obstacles and use boundary elements, based on an indirect single layer potential formulation, for this kind of problem. Then we iterate between both problems, interchanging traces, i.e., the finite element solutions generate traces on the boundaries of the obstacles and the boundary element solution is plugged into the single layer potential

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to obtain values on the outer boundary. The iteration is then carried out till the process becomes stationary.

The method thus proposed is a discrete version of the interior–exterior Schwarz iteration studied in [3]. The kind of ideas of this method is heavily related to multiple scattering techniques and somewhat resembles Chimera type methods [2] . Just for this model situation, the method provides a simple discretization of a problem in a domain with a complicated geometry by taking out holes (in fact we do not have to eliminate all the obstacles, just the part of them we choose).

For the discretization of the interior problem we consider $\mathbb{P}_k$ finite elements on a simplicial triangulation. The reason for this restriction is more on the analysis side of the problem than on practical issues, basically since we want to use Scott–Zhang's operator [18]. All what is used of these finite elements is stated in the Appendix, so all the results are valid as long as we can prove similar bounds for other finite element spaces. For the boundary element part, we consider any convergent Galerkin method, be it based on finite elements or not. Part of the analysis requires a somewhat stronger approximation property of the discrete space. This property is however satisfied by all relevant boundary element spaces.

In fact the work develops two different approaches to the problem. First we study the iteration and we prove that in some situations it has a limit. On the other hand we can ignore the iterative process and study its possible fixed point even when this is not reached as the limit. This second approach allows to think not only of better iterations (based on GMRES for instance), but also of more general elliptic problems.

The paper is structured as follows. In Section 2 we state the model problem, describe the Schwarz BEM–FEM iteration and write down the discrete equations satisfied by its possible limit. In Section 3 we write the problem and the iteration as an equivalent set of operator equations, traces of the uncoupled problems becoming now the unknowns. This allows for a study of the problem in relation with the continuous interior–exterior Schwarz iteration of [3]. By only assuming convergence of the Galerkin BEM and refinement of the interior triangulation, we show in Section 4 that the discrete iteration converges in all cases when the continuous iteration converges.

In Section 5, we prove uniform boundedness of the inverse of the discrete operator obtained in Section 3. This gives an alternative proof of the same fact for the three dimensional case (where everything works before discretization) and in the good two dimensional cases. It also extends the result for some two dimensional cases. The proof does not use the fact that the spectral radius of a certain operator obtained in Section 3 is less that one, but relies on Fredholm properties of this operator. This opens some expectations on the possibility of extending the results to more general situations.

In Section 6 we give an asymptotic analysis of the error as the discretization (both finite and boundary element) becomes finer. We comment on possible choices for the BEM spaces. Section 7 is devoted to illustrating some of the results in a simple two dimensional example, by using a non–conforming Dirac delta scheme (a quadrature method) for the boundary integral part.

An appendix gathers some results concerning mainly convergence and stability of the finite element method in Sobolev spaces of fractional order.

Notational foreword. We will make extensive use of the Sobolev spaces $H^s(\mathcal{O})$, where $\mathcal{O}$ is an open set in $\mathbb{R}^d$ and $\mathbb{R} \ni s \geq 0$. Their norms will be denoted by $\| \cdot \|_{s,\mathcal{O}}$. For Lipschitz curves or surfaces Θ, we also will consider the Sobolev spaces of real order $H^s(\Theta)$ with $s \in [-1, 1]$, with similar notations for their norms. Definitions and properties of these spaces, in the form that are needed here, can be found in [7, 12].

With C (with possible sub and superscripts) we will denote a positive constant independent of the discretization parameter h, and of all quantities it is multiplied by, possibly different in each occurrence.

2 Description of the method

Let $Q \subset \mathbb{R}^d$ $(d = 2, 3)$ be a domain with polygonal (polyhedral) boundary Σ, strictly containing a possibly non connected open set with Lipschitz boundary Γ. The corresponding anular domain, exterior to Γ and interior to Σ will be denoted by Ω.

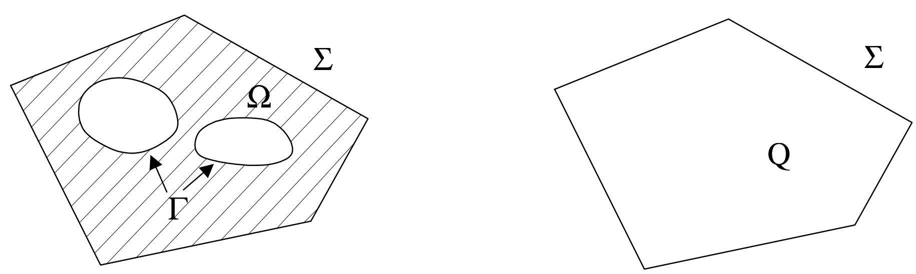

Figure 1: The domain with and without holes

The method we propose here is a Schwarz–type iteration for the boundary value problem: find $w \in H^1(\Omega)$ such that

$$\left| \begin{array}{l} \Delta w = 0, \quad \text{in } \Omega, \\ \\ \gamma_\Sigma w = g_\Sigma, \quad \gamma_\Gamma w = g_\Gamma \end{array} \right. \tag{1}$$

for given $g_\Sigma \in H^{1/2}(\Sigma)$, $g_\Gamma \in H^{1/2}(\Gamma)$, γ_Σ and γ_Γ being the corresponding trace operators.

Associated to the boundary Γ we consider the single–layer potential

$$\mathcal{S}_\Gamma \psi := \int_\Gamma \Phi(\,\cdot\,, y)\psi(y)\mathrm{d}\sigma(y) \;:\; \mathbb{R}^d \to \mathbb{R} \tag{2}$$

for an arbitrary density $\psi \in H^{-1/2}(\Gamma)$, being $\Phi(x, y) := 1/|x - y|$ in three dimensions and $\Phi(x, y) := -\log |r(x - y)|$ in two dimensions. The choice of the parameter $r > 0$ has some consequences on properties of layer potentials (see Proposition 1 below).

We also consider the single–layer operator $V_\Gamma : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$

$$V_\Gamma \psi := \gamma_\Gamma \mathcal{S}_\Gamma \psi = \int_\Gamma \Phi(\,\cdot\,, y)\psi(y)\mathrm{d}\sigma(y) \;:\; \Gamma \to \mathbb{R}.$$

For properties of this operator and restrictions–extensions to other Sobolev spaces of real order, see [5, 12]. In two dimensions, the concept of logarithmic capacity of a curve or set of curves is relevant in properties of inversibility of V_Γ. For its definition, see [12, 19]. We gather in the following proposition some well–known properties of V_Γ (see [12] Chapter 8, for instance).

Proposition 1 *In the three dimensional case, V_Γ is elliptic and therefore inversible in $H^{-1/2}(\Gamma)$. In two dimensions, V_Γ is inversible if and only if the logarithmic capacity of Γ differs from $1/r$ and elliptic if and only if the capacity is less than $1/r$.*

The method we propose needs two families of discrete spaces, one defined in Q (the domain without the holes) and the other on the boundary Γ. For simplicity we will consider a single discretization parameter $h := (h_Q, h_\Gamma) \to 0$, grouping the refinement possibilities of both families. The parameter h_Γ does not necessarily have a geometrical meaning. In the last part of this work (section 6) we will separate the effects of discretizations in Q and Γ.

Let $\mathcal{T}_h$ be a regular family of simplicial triangulations in the usual sense (see [1, 6]) and let $P_h^Q \subset H^1(Q)$ be a polynomial finite element space associated to this triangulation

$$P_h^Q := \{u_h \in \mathcal{C}(\overline{Q}) \mid u_h|_K \in \mathbb{P}_k, \quad \forall K \in \mathcal{T}_h\}.$$

We will not make apparent the degree of the polynomials ($k \geq 1$) in the definitions, since it does not appear in the analysis until the moment when we consider convergence orders. The finite element space restricted to the boundary

$$X_h^\Sigma := \gamma_\Sigma P_h^Q$$

will be of relevance in the sequel. Associated to these spaces we have a finite element procedure $\mathrm{FEM} : X_h^\Sigma \to P_h^Q$

$$\mathrm{FEM}(g_h^\Sigma) = u_h, \qquad \left| \begin{array}{l} u_h \in P_h^Q, \quad \gamma_\Sigma u_h = g_h^\Sigma, \\ \displaystyle\int_Q \nabla u_h \cdot \nabla v_h = 0, \quad \forall v_h \in P_h^Q \cap H_0^1(Q). \end{array} \right. \tag{3}$$

For the interior boundary we consider a family of subspaces $S_h^\Gamma \subset H^{-1/2}(\Gamma)$, in conditions to be fixed later. In general, this space can be composed of finite elements, trigonometric polynomials (in 2 dimensions) or spherical harmonics (in 3 dimensions), etc. We then define the Galerkin boundary element procedure: $\mathrm{BEM} : H^{1/2}(\Gamma) \to S_h^\Gamma$

$$\mathrm{BEM}(g_\Gamma) = \psi_h, \qquad \left| \begin{array}{l} \psi_h \in S_h^\Gamma, \\ \langle V_\Gamma \psi_h, \xi_h \rangle = \langle g_\Gamma, \xi_h \rangle, \qquad \forall \xi_h \in S_h^\Gamma, \end{array} \right.$$

being $\langle \cdot\,, \cdot \rangle$ the $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ duality product.

Finally, to define the iterative process, we introduce the operator $V_{\Sigma\Gamma} : H^{-1/2}(\Gamma) \to H^{1/2}(\Sigma)$

$$V_{\Sigma\Gamma}\psi := \gamma_\Sigma \mathcal{S}_\Gamma \psi = \int_\Gamma \Phi(\,\cdot\,, y)\psi(y)\mathrm{d}\sigma(y) : \Sigma \to \mathbb{R}$$

and the nodal Lagrange interpolation operator $I_h^\Sigma : \mathcal{C}(\Sigma) \to X_h^\Sigma$, which is well defined for the finite elements considered above.

The iteration goes as follows: first, we choose

$$X_h^\Sigma \ni g_{0,h}^\Sigma \approx g_\Sigma$$

(how this function is chosen is not relevant for convergence of the iteration, but it is for convergence as $h \to 0$); then we do

$$\begin{aligned}
&u_{0,h} := \mathrm{FEM}(g_{0,h}^\Sigma)\\
&\psi_{0,h} := \mathrm{BEM}(g_\Gamma)\\
&\text{for } n \geq 0\\
&\qquad u_{n+1,h} := \mathrm{FEM}(g_{0,h}^\Sigma - I_h^\Sigma V_{\Sigma\Gamma}\psi_{n,h}) = u_{0,h} - \mathrm{FEM}(I_h^\Sigma V_{\Sigma\Gamma}\psi_{n,h})\\
&\qquad \psi_{n+1,h} := \mathrm{BEM}(g_\Gamma - \gamma_\Gamma u_{n,h}) = \psi_{0,h} - \mathrm{BEM}(\gamma_\Gamma u_{n,h})
\end{aligned} \tag{4}$$

The approximation to the exact solution is given by the expressions

$$w_{n,h} := u_{n,h} + \mathcal{S}_\Gamma \psi_{n,h} : \Omega \to \mathbb{R}.$$

Assuming the existence of a limit as $n \to \infty$, say $u_h = \lim u_{n,h}$ and $\psi_h = \lim \psi_{n,h}$, it is simple to show that these functions satisfy the discrete equations

$$\begin{bmatrix} I & \mathrm{FEM} I_h^\Sigma V_{\Sigma\Gamma} \\ \mathrm{BEM}\gamma_\Gamma & I \end{bmatrix} \begin{bmatrix} u_h \\ \psi_h \end{bmatrix} = \begin{bmatrix} u_{0,h} \\ \psi_{0,h} \end{bmatrix}. \tag{5}$$

The iteration (4) is a block Jacobi method for this system. Notice that because of its particular form, the Jacobi method is equivalent to this block partitioned method. A Gauss–Seidel type iteration is also possible, although it losses the parallel structure of the Jacobi method.

Moreover, we can think of equations (5) as a discretization of the model problem (1), by defining finally an approximation $w_h = u_h + \mathcal{S}_\Gamma \psi_h$. We thus can solve these discrete equations by other iterative schemes requiring only multiplication by the matrix (i.e., solution of a finite element and a boundary element problem), such as GMRES (see [8, 13]).

3 An equivalent formulation

The iterative method above can be written in an equivalent way, which emphasizes its proximity to the interior–exterior Schwarz algorithm developed in [3]. This reformulation will be used to give a convergence analysis both as $n \to \infty$ and as $h \to 0$.

Let $R : H^{1/2}(\Sigma) \to H^1(Q)$ be the harmonic lifting operator

$$\left| \begin{array}{ll} R g_\Sigma \in H^1(Q), \quad \gamma_\Sigma R g_\Sigma = g_\Sigma, & \\ \displaystyle\int_Q \nabla R g_\Sigma \cdot \nabla v = 0, & \forall v \in H_0^1(Q). \end{array} \right. \tag{6}$$

Some relevant properties of R are reviewed in the appendix. We then consider the operators $K_{\Gamma\Sigma} := \gamma_\Gamma R : H^{1/2}(\Sigma) \to H^{1/2}(\Gamma)$ and $K_{\Sigma\Gamma} := V_{\Sigma\Gamma} V_\Gamma^{-1} : H^{1/2}(\Gamma) \to H^{1/2}(\Sigma)$ and two matrices of operators $\mathcal{A}, \mathcal{K} : H^{1/2}(\Sigma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$

$$\mathcal{K} := \left[\begin{array}{cc} 0 & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & 0 \end{array} \right], \qquad \mathcal{A} := \mathcal{I} + \mathcal{K} = \left[\begin{array}{cc} I & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & I \end{array} \right].$$

It is simple to prove that $\mathcal{K}$ is compact.

Proposition 2 ([3] Theorem 2.1) *If V_Γ is inversible, $\mathcal{A}$ is inversible.*

Proposition 3 ([3] Theorem 2.2) *The spectral radius of the compact operator $\rho(\mathcal{K}) = (\rho(K_{\Sigma\Gamma} K_{\Gamma\Sigma}))^{1/2}$ is less than one: (a) in three dimensions; (b) in two dimensions if the logarithmic capacity of Σ is less than $1/r$.*

Notice that the logarithmic capacity of the complete domain is relevant in this result (the hypothesis can be however somewhat weakened [3] Proposition 3.3). The lack of good properties for the two dimensional case can be easily mended by taking a smaller value of r or changing units in space to make the domain smaller.

In addition, if we consider the uniquely solvable system

$$\left[\begin{array}{cc} I & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & I \end{array} \right] \left[\begin{array}{c} f_\Sigma \\ f_\Gamma \end{array} \right] = \left[\begin{array}{c} g_\Sigma \\ g_\Gamma \end{array} \right] \qquad (7)$$

then the solution to (1) can be decomposed as

$$w = R f_\Sigma + \mathcal{S}_\Gamma V_\Gamma^{-1} f_\Gamma,$$

i.e., the equations provide the unique decomposition of w as a sum of a single–layer potential plus a solution in the whole set Q. This idea is quite common in the field of time–harmonic waves, where $R f_\Sigma$ plays the role of an incident wave and the other term is the scattering produced by Γ. We will see that the iterative method can be understood as a discretization of the Jacobi method for (7) (which coincides with the Neumann series iteration), which was studied in [3].

New discrete spaces and operators. In order to write the iteration and its possible limit more appropriately, we introduce some new discrete elements:

(a) Let $X_h^\Gamma := V_\Gamma S_h^\Gamma$. This space, isomorphic to S_h^Γ, appears just for the sake of the analysis, but does not have to be constructed or used in practice.

(b) Let $R_h : X_h^\Sigma \to P_h^Q$, be the discrete harmonic lifting (up to now denoted by FEM). Some properties of R_h are given in the appendix.

(c) Let $L_h : H^{1/2}(\Gamma) \to X_h^\Gamma$ be given by $L_h := V_\Gamma \mathrm{BEM}$, i.e.

$$\left| \begin{array}{l} L_h g_\Gamma \in X_h^\Gamma, \\ \langle L_h g_\Gamma, \xi_h \rangle = \langle g_\Gamma, \xi_h \rangle, \qquad \forall \xi_h \in S_h^\Gamma. \end{array} \right.$$

(d) Finally consider any projection $N_h : H^{1/2}(\Sigma) \to X_h^\Sigma$, that is uniformly bounded as an operator in $H^{1/2}(\Sigma)$. At this stage, the $H^{1/2}(\Sigma)-$orthogonal projection onto X_h^Σ is a suitable choice for N_h. The operator itself is immaterial for what follows.

These operators lead to consider approximations of $K_{\Sigma\Gamma}$ and $K_{\Gamma\Sigma}$

$$\begin{aligned} K^h_{\Sigma\Gamma} &:= I_h^\Sigma K_{\Sigma\Gamma} &:& \quad H^{1/2}(\Gamma) \to X_h^\Sigma \subset H^{1/2}(\Sigma), \\ K^h_{\Gamma\Sigma} &:= L_h \gamma_\Gamma R_h N_h &:& \quad H^{1/2}(\Sigma) \to X_h^\Gamma \subset H^{1/2}(\Gamma), \end{aligned}$$

the corresponding matrix of operators

$$\mathcal{K}_h := \begin{bmatrix} 0 & K^h_{\Sigma\Gamma} \\ K^h_{\Gamma\Sigma} & 0 \end{bmatrix},$$

and $\mathcal{A}_h := \mathcal{I} + \mathcal{K}_h : H^{1/2}(\Sigma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Sigma) \times H^{1/2}(\Gamma)$.

Proposition 4 *Consider the iteration* (4) *and let*

$$g^\Sigma_{n,h} := \gamma_\Sigma u_{n,h} \in X_h^\Sigma \qquad g^\Gamma_{n,h} := V_\Gamma \psi^\Gamma_{n,h} \in X_h^\Gamma.$$

Then

$$\begin{bmatrix} g^\Sigma_{n+1,h} \\ g^\Gamma_{n+1,h} \end{bmatrix} = \begin{bmatrix} g^\Sigma_{0,h} \\ g^\Gamma_{0,h} \end{bmatrix} - \begin{bmatrix} 0 & K^h_{\Sigma\Gamma} \\ K^h_{\Gamma\Sigma} & 0 \end{bmatrix} \begin{bmatrix} g^\Sigma_{n,h} \\ g^\Gamma_{n,h} \end{bmatrix}.$$

Proof. A very simple manipulation of the expressions in the iteration (4), noticing that $u_{n,h} = \mathrm{FEM}(g^\Sigma_{n,h})$, shows that this one is equivalent to:

$$\begin{aligned} & g^\Sigma_{0,h} \approx g_\Sigma \\ & g^\Gamma_{0,h} := V_\Gamma \mathrm{BEM}(g_\Gamma) \\ & \text{for } n \geq 0 \\ & \qquad g^\Sigma_{n+1,h} := g^\Sigma_{0,h} - I_h^\Sigma K_{\Sigma\Gamma} g^\Gamma_{n,h} \\ & \qquad g^\Gamma_{n+1,h} := g^\Gamma_{0,h} - V_\Gamma \mathrm{BEM}(\gamma_\Gamma \mathrm{FEM}(g^\Sigma_{n,h})). \end{aligned} \tag{8}$$

The result is then a straightforward consequence of the notations and of the fact that both L_h and N_h are projections onto the respective discrete spaces X_h^Γ and X_h^Σ. □

This result shows that this is the Jacobi method for the equations (compare to (7))

$$\begin{bmatrix} I & K^h_{\Sigma\Gamma} \\ K^h_{\Gamma\Sigma} & I \end{bmatrix} \begin{bmatrix} f_h^\Sigma \\ f_h^\Gamma \end{bmatrix} = \begin{bmatrix} g^\Sigma_{0,h} \\ g^\Gamma_{0,h} \end{bmatrix}. \tag{9}$$

Notice that the operator $\mathcal{A}_h$ is defined in the whole continuous space. However, when the right–hand side belongs to the discrete space (as in (9)), both the solution and all the iterations are in the discrete space. Therefore, the whole of the iterative process happens in a finite–dimensional setting.

4 Convergence of the iterative process

Hypothesis 1 *As $h \to 0$, we assume that*

$$\inf_{\psi_h \in S_h^\Gamma} \|\psi - \psi_h\|_{-1/2,\Gamma} \to 0, \qquad \forall \psi \in H^{-1/2}(\Gamma). \tag{10}$$

Since V_Γ is strongly elliptic, this hypothesis is equivalent to convergence of the boundary element discretization. When $h \to 0$ we are implicitly assuming that the triangulation becomes finer. Therefore

$$\inf_{u_h \in P_h^Q} \|u - u_h\|_{1,Q} \to 0, \qquad \forall u \in H^1(Q). \tag{11}$$

As a general notation we will write for a family of operators Ξ_h converging pointwise to Ξ (i.e., $\Xi_h g \to \Xi g$ for all g),

$$\Xi_h \stackrel{\text{pt}}{\to} \Xi.$$

The arrow notation $\Xi_h \to \Xi$ denotes norm convergence.

Proposition 5 *If Hypothesis 1 holds, for the families $L_h : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $N_h : H^{1/2}(\Sigma) \to H^{1/2}(\Sigma)$ we have the convergences:*

$$L_h \stackrel{\text{pt}}{\to} I, \qquad N_h \stackrel{\text{pt}}{\to} I.$$

Proof. The first one is an immediate consequence of the approximation property (10). Notice also that (11) implies

$$\inf_{g_h \in X_h^\Sigma} \|g - g_h\|_{1/2,\Sigma} \to 0, \qquad \forall g \in H^{1/2}(\Sigma), \tag{12}$$

which is equivalent to the second property. □

Proposition 6 *As a class of operators from $H^{1/2}(\Sigma) \to H^1(Q)$ we have*

$$(R_h - R)N_h \stackrel{\text{pt}}{\to} 0.$$

Proof. Denoting $M_h : H^1(\Sigma) \to X_h^\Sigma$ to the orthogonal projection onto X_h^Σ

$$\begin{aligned} \|(R - R_h)N_h g\|_{1,Q} &\leq \|(R - R_h)N_h(I - M_h)g\|_{1,Q} + \|(R - R_h)N_h M_h g\|_{1,Q} \\ &\leq C\|g - M_h g\|_{1/2,\Sigma} + \|(R - R_h)M_h g\|_{1,Q} \\ &\leq C'\Big(\|g - M_h g\|_{1,\Sigma} + h^{1/2}\|g\|_{1,\Sigma}\Big), \end{aligned}$$

for all $g \in H^1(\Sigma)$ (see Proposition 20). The first addendum converges to zero, since we are dealing with finite elements on the boundary with a refining set of grids. The density of $H^1(\Sigma)$ in $H^{1/2}(\Sigma)$ proves the result. □

Theorem 7 *If Hypothesis 1 holds, as $h \to 0$ we have the following convergences:*

(a) $K^h_{\Sigma\Gamma} \to K_{\Sigma\Gamma}$,

(b) $K^h_{\Gamma\Sigma} \stackrel{\text{pt}}{\to} K_{\Gamma\Sigma}$,

(c) $\mathcal{K}_h \stackrel{\text{pt}}{\to} \mathcal{K}$,

(d) $K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma} \to K_{\Gamma\Sigma} K_{\Sigma\Gamma}$.

Proof. We first remark that $K_{\Sigma\Gamma} - K^h_{\Sigma\Gamma} = (I - I_h^\Sigma) K_{\Sigma\Gamma} = (I - I_h^\Sigma)\gamma_\Sigma \mathcal{S}_\Gamma V_\Gamma^{-1}$. Then (a) is a simple consequence of Proposition 24.

To prove (b), we first decompose

$$K_{\Gamma\Sigma} - K^h_{\Gamma\Sigma} = K_{\Gamma\Sigma}(I - N_h) + (I - L_h) K_{\Gamma\Sigma} N_h + L_h \gamma_\Gamma (R - R_h) N_h. \tag{13}$$

By Proposition 5, we have that $K_{\Gamma\Sigma}(I - N_h) \stackrel{\text{pt}}{\to} 0$. The compactness of $K_{\Gamma\Sigma}$ and Proposition 5 prove that $(I - L_h) K_{\Gamma\Sigma} N_h \to 0$ (see [11] Theorem 10.7). For the pointwise convergence of the third term in (13) we simply apply Proposition 6 and the uniform boundedness of L_h.

Convergence (c) is an obvious consequence of (a) and (b). Finally, to prove (d), we write

$$K_{\Gamma\Sigma} K_{\Sigma\Gamma} - K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma} = K_{\Gamma\Sigma}(K_{\Sigma\Gamma} - K^h_{\Sigma\Gamma}) + (K_{\Gamma\Sigma} - K^h_{\Gamma\Sigma}) K^h_{\Sigma\Gamma}. \tag{14}$$

It is clear that the first addendum converges to zero by (a). It is easy to prove that $K^h_{\Sigma\Gamma} : H^{1/2}(\Gamma) \to H^1(\Sigma)$ is uniformly bounded (see Proposition 24). The family $\{K^h_{\Sigma\Gamma}\}$ is therefore collectively compact (see [11] Chapter 10). Then (b) and the collective compactness of this family ensure (see [11] Theorem 10.7) uniform convergence of the second term in (14).

□

Corollary 8 *If V_Γ is elliptic, $\rho(\mathcal{K}) < 1$ and Hypothesis 1 holds, then for h small enough, $\rho(K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma}) \le \rho_0 < 1$.*

Proof. In $H^{1/2}(\Gamma)$ we consider the equivalent norm $\langle V_\Gamma^{-1} \cdot, \cdot \rangle^{1/2}$ and the corresponding operator norm $|||\cdot|||$

$$|||B||| := \sup_{0 \neq g \in H^{-1/2}(\Gamma)} \frac{\langle V_\Gamma^{-1} B g, B g\rangle^{1/2}}{\langle V_\Gamma^{-1} g, g\rangle^{1/2}} \tag{15}$$

in the set of bounded linear maps from $H^{1/2}(\Gamma)$ into itself, equivalent to the common one. Taking the logical definitions for V_Σ and $V_{\Gamma\Sigma}$, which is the transposed of $V_{\Sigma\Gamma}$, we have that $V_\Gamma^{-1} K_{\Gamma\Sigma} K_{\Sigma\Gamma} = V_\Gamma^{-1} V_{\Gamma\Sigma} V_\Sigma^{-1} V_{\Sigma\Gamma} V_\Gamma^{-1}$ is compact and selfadjoint in this new inner product (see [3] Theorem 3.2, for the necessary adjustments if V_Σ is not inversible). Therefore it follows that

$$\rho(K_{\Gamma\Sigma} K_{\Sigma\Gamma}) = |||K_{\Gamma\Sigma} K_{\Sigma\Gamma}||| < 1.$$

Hence

$$\rho(K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma}) \le |||K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma}||| \le |||K_{\Gamma\Sigma} K_{\Sigma\Gamma}||| + |||K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma} - K_{\Gamma\Sigma} K_{\Sigma\Gamma}||| \to |||K_{\Gamma\Sigma} K_{\Sigma\Gamma}|||.$$

□

Theorem 9 *If V_Γ is elliptic, $\rho(\mathcal{K}) < 1$ and Hypothesis 1 holds, then $\mathcal{A}_h$ is inversible for h small enough and the iterative process converges with $h-$independent velocity of convergence.*

Proof. As a consequence of the preceding result, for h small enough

$$\rho(\mathcal{K}_h)^2 = \rho(K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma}) \le \rho_0 < 1.$$

This inequality proves the result. □

Theorem 10 *If V_Γ is elliptic, $\rho(\mathcal{K}) < 1$ and Hypothesis 1 holds, then the inverse of $\mathcal{A}_h$ is uniformly bounded for h small enough.*

Proof. We consider in $H^{1/2}(\Gamma)$ the non–standard norm $\langle V_\Gamma^{-1}\,\cdot\,,\,\cdot\,\rangle^{1/2}$ defined in the proof of Corollary 8, and the associated operator norm (15). Since $|||K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma}||| \le \tau < 1$, with τ independent of h, it is clear that

$$\|(K^h_{\Sigma\Gamma} K^h_{\Gamma\Sigma})^N\| \le \|K^h_{\Sigma\Gamma}\|\,\|(K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma})^{N-1}\|\,\|K^h_{\Gamma\Sigma}\| \le C|||K^h_{\Gamma\Sigma} K^h_{\Sigma\Gamma}|||^{N-1} \le C\tau^{N-1}.$$

Taking N large enough we obtain $\|\mathcal{K}_h^{2N}\| \le \tau' < 1$, which implies the result. □

5 Overlapped finite and boundary elements

Our next step is to show that Theorem 10 can be proven without assuming that $\rho(\mathcal{K}) < 1$, but only inversibility of V_Γ, i.e. of $\mathcal{A}$. This can be accomplished by assuming an additional approximation property for S^Γ_h. The effect on the analysis will be the fact that we will be able to consider the discrete method (9) as an approximation scheme in itself, not only as the limit of the Schwarz method. This method is then the overlapping of a finite element and a boundary element discretization. Since the restriction of $\mathcal{A}_h$ to $X^\Sigma_h \times X^\Gamma_h$ has images on this same space, in practice the equations (9) can be understood in this discrete space. Therefore N_h is never used in implementation. Besides, we can always revert to the equivalent formulation (5).

The following hypothesis implies Hypothesis 1.

Hypothesis 2 *There exists α such that $\alpha(h) \to 0$ as $h \to 0$ and*

$$\inf_{\psi_h \in S^\Gamma_h} \|\psi - \psi_h\|_{-1/2,\Gamma} \le \alpha(h)\|\psi\|_{0,\Gamma}, \qquad \forall \psi \in H^0(\Gamma).$$

Proposition 11 *If Hypothesis 2 holds, then there exists β such that $\beta(h) \to 0$ as $h \to 0$ and*

$$\|L_h f - f\|_{0,\Gamma} \le \beta(h)\|f\|_{1/2,\Gamma}, \qquad \forall f \in H^{1/2}(\Gamma).$$

Proof. Let $\psi := V_\Gamma^{-1} f$ and $\psi_h = V_\Gamma^{-1} L_h f$. Then $\|L_h f - f\|_{0,\Gamma} \le C\|\psi - \psi_h\|_{-1,\Gamma}$, since $V_\Gamma : H^{-1}(\Gamma) \to H^0(\Gamma)$ is bounded (see [7]). The result is then a simple consequence of the use of Aubin–Nitsche type estimates (see [15] Chapter 1). □

Let $N_h : H^{1/2}(\Sigma) \to X_h^\Sigma \subset H^{1/2}(\Sigma)$ be now any projection onto X_h^Σ such that it is $h-$uniformly bounded and

$$\|N_h g - g\|_{0,\Sigma} \le \delta(h)\|g\|_{1/2,\Sigma}, \qquad \forall g \in H^{1/2}(\Sigma), \qquad \lim_{h\to 0} \delta(h) = 0. \tag{16}$$

The existence of such a projection is guaranteed by Proposition 22. Moreover, if we renew the definition of $K_{\Gamma\Sigma}^h$, then Propositions 4, 5, 6 and Theorem 7 still hold.

Theorem 12 *If Hypothesis 2 holds then*

$$K_{\Sigma\Gamma}^h K_{\Gamma\Sigma}^h \to K_{\Sigma\Gamma} K_{\Gamma\Sigma}.$$

Hence $\mathcal{K}_h^2 \to \mathcal{K}^2$.

Proof. Since by Theorem 7(a) we have that $K_{\Sigma\Gamma}^h \to K_{\Sigma\Gamma}$, we just have to prove that $K_{\Sigma\Gamma} K_{\Gamma\Sigma}^h \to K_{\Sigma\Gamma} K_{\Gamma\Sigma}$. Since $K_{\Sigma\Gamma} : H^0(\Gamma) \to H^{1/2}(\Sigma)$ is bounded we can simply prove that

$$\|(K_{\Gamma\Sigma}^h - K_{\Gamma\Sigma})g\|_{0,\Gamma} \le \varepsilon(h)\|g\|_{1/2,\Sigma} \tag{17}$$

with $\varepsilon(h) \to 0$. By the definitions of the operators involved

$$\|(K_{\Gamma\Sigma}^h - K_{\Gamma\Sigma})g\|_{0,\Gamma} \le \|(L_h - I)\gamma_\Gamma R_h N_h g\|_{0,\Gamma} + \|\gamma_\Gamma(R_h N_h - R)g\|_{0,\Gamma}. \tag{18}$$

For the first term we use Proposition 11 and the uniform boundedness of R_h (see Proposition 20)

$$\|(L_h - I)\gamma_\Gamma R_h N_h g\|_{0,\Gamma} \le \beta(h)\|\gamma_\Gamma R_h N_h g\|_{1/2,\Gamma} \le \beta(h) C\|g\|_{1/2,\Sigma}. \tag{19}$$

On the other hand, for arbitrary $\eta \in (0, 1/2)$

$$\begin{aligned} \|\gamma_\Gamma(R_h N_h - R)g\|_{0,\Gamma} &\le C\|(R_h N_h - R)g\|_{1/2+\eta,Q} \\ &\le C\|(R_h - R)N_h g\|_{1/2+\eta,Q} + C\|N_h g - g\|_{\eta,\Sigma} \end{aligned} \tag{20}$$

where we have applied Propositions 17 and 18. By interpolation properties of Sobolev spaces, Proposition 21 and Proposition 20 (with $s = 0$) we have

$$\begin{aligned} \|(R_h - R)N_h g\|_{1/2+\eta,Q} &\le \|(R_h - R)N_h g\|_{0,Q}^{1/2-\eta}\|(R_h - R)N_h g\|_{1,Q}^{1/2+\eta} \\ &\le \gamma(h)^{1/2-\eta} C\|N_h g\|_{1/2,\Sigma}. \end{aligned} \tag{21}$$

The remaining bound follows again by interpolation of Sobolev spaces and (16)

$$\|N_h g - g\|_{\eta,\Sigma} \le \|N_h g - g\|_{0,\Sigma}^{1-2\eta}\|N_h g - g\|_{1/2,\Sigma}^{2\eta} \le \delta(h)^{1-2\eta} C\|g\|_{1/2,\Sigma}. \tag{22}$$

Gathering (18–22) we prove (17) and thus the first result. The second one is a simple consequence of this and Theorem 7(d). □

Theorem 13 *If V_Γ^{-1} exists and Hypothesis 2 holds, then $\mathcal{A}_h$ is inversible for h small enough and its inverse is $h-$uniformly bounded.*

Proof. Let $\mathcal{B} := \mathcal{I} - \mathcal{K}$ and $\mathcal{B}_h := \mathcal{I} - \mathcal{K}_h$. By the particular matrix form of $\mathcal{K}$, it is clear that its spectrum is symmetric with respect to the origin. Hence, $\mathcal{B}$ is one–to–one and by the Fredholm alternative, $\mathcal{B}$ is an isomorphism. Moreover

$$\mathcal{A}_h\mathcal{B}_h = (\mathcal{I} + \mathcal{K}_h)(\mathcal{I} - \mathcal{K}_h) = \mathcal{I} - \mathcal{K}_h^2 \to \mathcal{I} - \mathcal{K}^2 = \mathcal{A}\mathcal{B}$$

and therefore $\mathcal{A}_h\mathcal{B}_h$ is inversible for h small enough, with uniformly bounded inverse. The family of operators $\mathcal{C}_h := \mathcal{B}_h(\mathcal{A}_h\mathcal{B}_h)^{-1}$ is uniformly bounded and $\mathcal{A}_h\mathcal{C}_h = \mathcal{I}$. This implies, by the Fredholm alternative, that $\mathcal{A}_h$ is inversible and hence $\mathcal{C}_h = \mathcal{A}_h^{-1}$. □

6 Approximation properties

In this section we develop an asymptotic analysis of the approximation of the solution of (7) by that of (9), valid in the hypotheses of the preceding sections. We will in fact use that $\mathcal{A}_h$ has uniformly bounded inverse (Theorems 10 and 13).

We simply recall that $g_{0,h}^\Sigma \approx g_\Sigma$ is an initial choice, but $g_{0,h}^\Gamma = V_\Gamma \mathrm{BEM}(g_\Gamma)$ is the image by V_Γ of the density calculated by the boundary element method for the data on Γ.

If (f_Σ, f_Γ) solves (7), then the solution $w \in H^1(\Omega)$ to (1) is decomposed as

$$w = u + \mathcal{S}_\Gamma\psi, \qquad u := Rf_\Sigma, \qquad \psi := V_\Gamma^{-1} f_\Gamma. \tag{23}$$

The discrete version of this is

$$w_h := u_h + \mathcal{S}_\Gamma\psi_h, \qquad u_h := R_h f_h^\Sigma, \qquad \psi_h := V_\Gamma^{-1} f_h^\Gamma,$$

(f_h^Σ, f_h^Γ) being the solution of (9). Therefore, (u_h, ψ_h) is the solution to (5).

Proposition 14 *If $g_{0,h}^\Sigma \to g_\Sigma$ in $H^{1/2}(\Sigma)$ as $h \to 0$, then*

$$\lim_{h\to 0}\Big(\|f_h^\Sigma - f_\Sigma\|_{1/2,\Sigma} + \|f_h^\Gamma - f_\Gamma\|_{1/2,\Gamma}\Big) = 0 \tag{24}$$

and therefore

$$\lim_{h\to 0}\|w - (u_h + \mathcal{S}_\Gamma\psi_h)\|_{1,\Omega} = 0. \tag{25}$$

Proof. Notice that $g_{0,h}^\Gamma \to g_\Gamma$ in $H^{1/2}(\Gamma)$ (see Hypothesis 1). Since $\mathcal{A}_h \stackrel{\mathrm{pt}}{\to} \mathcal{A}$ and $\mathcal{A}_h^{-1}$ is uniformly bounded, it follows that $\mathcal{A}_h^{-1} \stackrel{\mathrm{pt}}{\to} \mathcal{A}^{-1}$, which proves (24). Then (25) is a simple consequence of the boundedness of $\mathcal{S}_\Gamma V_\Gamma^{-1} : H^{1/2}(\Gamma) \to H^1(\Omega)$ and of Propositions 6 and 20. □

Theorem 15 *In the notations above,*

$$\begin{aligned}\|w-w_h\|_{1,\Omega} &\le C\Big[\|g_\Sigma - g^\Sigma_{0,h}\|_{1/2,\Sigma} + \inf_{\psi_h\in S^\Gamma_h}\|V_\Gamma^{-1}g_\Gamma-\psi_h\|_{-1/2,\Gamma}\\ &\qquad + \inf_{p_h\in P^Q_h}\|u-p_h\|_{1,Q} + \inf_{\psi_h\in S^\Gamma_h}\|\psi-\psi_h\|_{-1/2,\Gamma} + h^{k+1/2}\|\psi\|_{-1/2,\Gamma}\Big].\end{aligned}$$

Proof. Since $R_hN_h\gamma_\Sigma u_h = u_h$, it follows readily that

$$\begin{aligned}\|u-u_h\|_{1,Q} &\le \|u-R_hN_h\gamma_\Sigma u\|_{1,Q} + \|R_hN_h(\gamma_\Sigma u - \gamma_\Sigma u_h)\|_{1,Q}\\ &\le C\Big[\inf_{p_h\in P^Q_h}\|u-p_h\|_{1,Q} + \|f_\Sigma - f^\Sigma_h\|_{1/2,\Sigma}\Big] \qquad (26)\end{aligned}$$

where we have applied Proposition 23 and the uniform boundedness of R_h and N_h. It is also clear that

$$\|\mathcal{S}_\Gamma\psi - \mathcal{S}_\Gamma\psi_h\|_{1,\Omega} \le C\|f_\Gamma - f^\Gamma_h\|_{1/2,\Gamma}. \qquad (27)$$

If we use the uniform boundedness of $\mathcal{A}_h^{-1}$ and the identities

$$\begin{aligned}\mathcal{A}_h\begin{bmatrix} f_\Sigma - f^\Sigma_h \\ f_\Gamma - f^\Gamma_h\end{bmatrix} &= \begin{bmatrix} g_\Sigma \\ g_\Gamma\end{bmatrix} - \begin{bmatrix} g^\Sigma_{0,h} \\ g^\Gamma_{0,h}\end{bmatrix} + (\mathcal{K}_h - \mathcal{K})\begin{bmatrix} f_\Sigma \\ f_\Gamma\end{bmatrix}\\ &= \begin{bmatrix} g_\Sigma - g^\Sigma_{0,h} \\ g_\Gamma - L_h g_\Gamma\end{bmatrix} + \begin{bmatrix} (K^h_{\Sigma\Gamma} - K_{\Sigma\Gamma})f_\Gamma \\ (K^h_{\Gamma\Sigma} - K_{\Gamma\Sigma})f_\Sigma\end{bmatrix}\end{aligned}$$

and

$$(K^h_{\Gamma\Sigma} - K_{\Gamma\Sigma})f_\Sigma = (L_h - I)(g_\Gamma - f_\Gamma) + L_h\gamma_\Gamma(R_hN_h - R)f_\Sigma$$

it follows that

$$\begin{aligned}\|f_\Sigma - f^\Sigma_h\|_{1/2,\Sigma} + \|f_\Gamma - f^\Gamma_h\|_{1/2,\Gamma} &\le C\Big[\|g_\Sigma - g^\Sigma_{0,h}\|_{1/2,\Sigma} + \|g_\Gamma - L_h g_\Gamma\|_{1/2,\Gamma}\\ &\quad + \|(I^\Sigma_h - I)\gamma_\Sigma\mathcal{S}_\Gamma\psi\|_{1/2,\Sigma} + \|f_\Gamma - L_h f_\Gamma\|_{1/2,\Gamma} \qquad (28)\\ &\quad + \|L_h\gamma_\Gamma(R_hN_h - R)f_\Sigma\|_{1/2,\Gamma}\Big].\end{aligned}$$

Since L_h is a uniformly bounded projection onto $X^\Gamma_h = V_\Gamma S^\Gamma_h$, we obtain

$$\|g_\Gamma - L_h g_\Gamma\|_{1/2,\Gamma} \le C\inf_{r_h\in X^\Gamma_h}\|g_\Gamma - r_h\|_{1/2,\Gamma} \le C'\inf_{\psi_h\in S^\Gamma_h}\|V_\Gamma^{-1}g_\Gamma - \psi_h\|_{-1/2,\Gamma}.$$

The fourth term in (28) is bounded similarly. For the third term we apply Proposition 24 and for the fifth one, the uniform boundedness of L_h and Proposition 23. Gathering these bounds, (26), (27) and (28) the result follows readily. □

Proposition 16 *If the solution of* (1)*, w, is decomposed as in* (23)*, then for $s > 1$, $w \in H^s(\Omega)$ if and only if $u \in H^s(Q)$ and $\mathcal{S}_\Gamma\psi \in H^s(\Omega)$.*

Proof. It is a simple consequence of the fact that $u \in \mathcal{C}^\infty(Q)$ and $\mathcal{S}_\Gamma\psi \in \mathcal{C}^\infty(\mathbb{R}^d\backslash\Gamma)$. □

Finite element error. Because of Theorem 15, we can now study the discretization error as a sum of different effects, part of them depending on the finite element discretization and the remaining, on the boundary elements. We now denote h_Q to the maximum diameter in the triangulation. Assuming that $w \in H^{k+1}(\Omega)$ and taking $g^{\Sigma}_{0,h} := I^{\Sigma}_h g_\Sigma$, we obtain that

$$\inf_{p_h \in P^Q_h} \|u - p_h\|_{1,Q} \leq C h^k_Q |u|_{k+1,Q}$$

and

$$\|g_\Sigma - g^{\Sigma}_{h,0}\|_{1/2,\Sigma} \leq C h^k_Q \|g\|_{k+1/2,\Sigma^*}$$

where the norm of g in the right hand side of the last expression is taking as the sum of the norms of each side/face of Σ.

Notice that this last bound can be put in reference to the diameter of the inherited triangulation of Σ instead of the interior diameter h_Q. The fifth addendum of the error in Theorem 15 uses this same quantity, because it proceeds from the boundary interpolation error.

Boundary element error. If Γ is a smooth boundary (let us assume it to be $\mathcal{C}^\infty$ for simplicity), then $w \in H^{k+1}(\Omega)$ implies that ψ and $V_\Gamma^{-1} g_\Gamma$ belong to $H^{k-1/2}(\Gamma)$. We can take S^Γ_h to be a space of piecewise polynomials of degree $k-1$ (not necessarily continuous). Hypotheses 1 and 2 are then trivially satisfied.

Then, the terms related to the boundary discretization provide an error behaving like $\mathcal{O}(h^k_\Gamma)$ (see [5, 20]). In fact, by assuming some further regularity, namely $w \in H^{k+3/2}(\Omega)$, this bound can be improved by one half.

Although difficult to implement, trigonometric polynomials/spherical harmonics provide very good error bounds. If Γ is composed of several disconnected smooth closed curves/surfaces, diffeomorphic to circumferences/spheres, and we have parameterizations of them, then we can use this kind of spectral approximation. If the data on Γ satisfies $g_\Gamma \in \mathcal{C}^\infty(\Gamma)$, then $f_\Gamma = g_\Gamma - K_{\Gamma\Sigma} f_\Sigma \in \mathcal{C}^\infty(\Gamma)$ and we can bound both error terms by $C_m N^{-m}$, being N the number of degrees of freedom on Γ and m any positive integer. The bounds can be improved to give exponential order with additional analyticity assumptions. This means that very few degrees of freedom are needed in Γ. For more on spectral boundary element discretizations in two dimensions see [17] (see also [14] for spherical harmonics in relationship with Sobolev spaces).

7 Some numerical experiments

We illustrate some of the foregoing results by means of a simple two dimensional example. The complete domain Q is the rectangle $[0, 4] \times [0, 1]$. The obstacles are three circles with centers on $(2, 0.75)$, $(1.25, 0.4)$ and $(2.75, 0.4)$ and respective radii 0.12, 0.09, 0.09. Their boundaries are denoted by Γ_k $(k = 1, 2, 3)$. We take the Dirichlet data so that the exact solution of (1) is

$$\begin{aligned} w(x,y) &= (x-2)^4 - 6y^2(x-2)^2 + y^4 \\ &\quad + \log\left((x-2.05)^2 + (y-0.7)^2\right) + \log\left((x-1.3)^2) + (y-0.42)^2\right). \end{aligned}$$

The decomposition (23) of w that the algorithm finds, takes $u(x,y) = (x-2)^4 - 6y^2(x-2)^2 + y^4$.

In Q we define a uniform triangular mesh based on taking N equal partitions on the vertical side and $4\,N$ on the horizontal direction and continuous $\mathbb{P}_1$ elements. Notice that the stiffness matrix of the FEM procedure is (up to a multiplicative factor) the same as that of the central finite difference method on the uniform grid formed by the vertices of all triangles (see [10]). The trace on the exterior boundary $g_{0,h}^{\Sigma}$ is that given by linear interpolation. We profit from the simple structure of the FE problem and from the values of N that will be chosen to use a FFT technique to solve the corresponding linear systems (see [13] Chapter 4).

The single layer potential is taken in a parameterized form. Let $\gamma_k(t) := (c_x^k, c_y^k) + r_k\,(\cos 2\pi t, \sin 2\pi t)$, (c_x^k, c_y^k) denoting the center of Γ_k and r_k its radius. We take

$$\sum_{k=1}^{3} \int_0^1 \log|x - \gamma_k(t)|\psi_k(t)\mathrm{d}t$$

as solution of the exterior problem, with densities ψ_k to be determined.

The BEM procedure we consider here is a non–conforming Petrov–Galerkin delta–delta method, which does not fit in the hypotheses but has the advantage of its great simplicity. The method appears as a quadrature method in [16] and is also derived in a very different setting in [4]. We take M equally spaced points on the interval used for parameterization of each curve: $t_i := i\,h$ $(i = 1, \ldots, M,\ h := 1/M)$ and also the displaced grid $s_i := (i + 1/6)\,h$. With right hand side function $g : \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \to \mathbb{R}$ we have to solve a $3M \times 3M$ system given by the equations:

$$\sum_{k=1}^{3} \sum_{j=1}^{M} \log|\gamma_\ell(s_i) - \gamma_k(t_j)|\,\psi_{k,j} = g(\gamma_\ell(s_i)), \qquad \ell = 1\ldots,3\,;\ i = 1, \ldots, M.$$

The corresponding potential is then

$$\sum_{k=1}^{3} \sum_{j=1}^{M} \log|x - \gamma_k(t_j)|\,\psi_{k,j},$$

which has singularities on the points $\gamma_k(t_j) \in \Gamma_k$. These singularities provoke a lose in the order of convergence when we approach the curves, but do not affect convergence away from the obstacles.

In Figure 2 we show the contour lines of the interior solution u, the exterior solution $\mathcal{S}_\Gamma\psi$ and their sum, computed with $N = 32$ and $M = 40$.

For different values of N and M, we apply the Jacobi–Scwharz iteration (4) and also GMRES to the equivalent system (5). Convergence of the exact Schwarz iteration is ensured by [3] Proposition 3.3. The stopping criterion for both iterations is based on the relative error of the residual. Table 1 shows the number of Schwarz and GMRES iterations. Notice that the number of iterations is independent of N and M. The spectral radius of $\mathcal{K}_h$ is approximately 0.66. Table 1 also shows the maximum error of the full solution on nodes placed on Ω and also the maximum nodal error in nodes such that their distance with the interior boundaries is bigger than 0.25, where clearly the attained convergence order is 2.

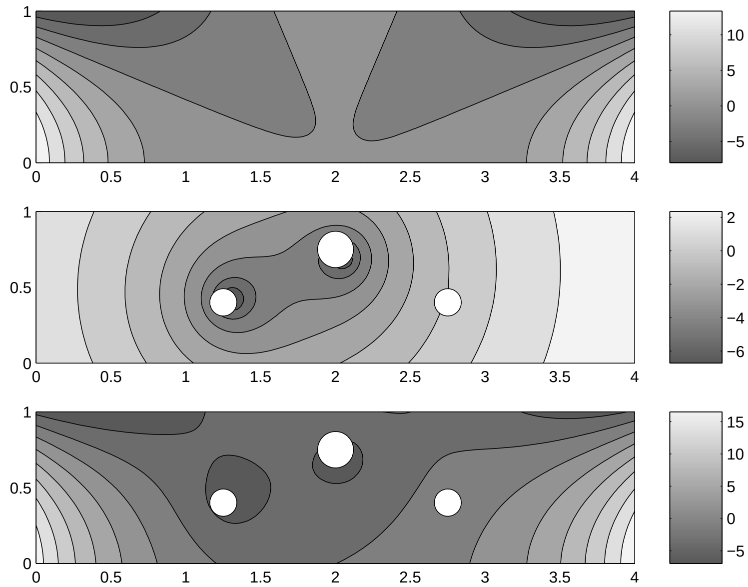

Figure 2: Interior, exterior and full solution of the problem

N	M	# Schwarz it.	# GMRES it.	Nodal error	Nodal error away from Γ
8	10	25	10	2.80E–1	1.44E–2
16	20	25	10	8.39E–2	3.96E–3
32	40	25	10	2.83E–2	1.03E–3
64	80	25	10	3.76E–3	2.63E–4

Table 1: Number of iterations and errors

8 Appendix

In this section we gather some results which are of use in the preceding work and state them in the precise form that is needed. The domain Q considered here has a polygonal/polyhedral boundary Σ (Q may not be simply connected) and the corresponding harmonic lifting is denoted by R (see (6)). At the discrete level, we consider the finite element space P_h^Q of Lagrangian simplices of degree k, its trace $X_h^\Sigma = \gamma_\Sigma P_h^Q$ and the discrete harmonic lifting R_h (see (3)).

Proposition 17 ([7, 12]) *The trace operator $\gamma_\Sigma : H^{s+1/2}(Q) \to H^s(\Sigma)$ is well defined and bounded for all $s \in (0,1)$.*

Obviously, this result also holds for the interior trace on Γ.

Proposition 18 ([7, 12]) *The harmonic lifting $R : H^s(\Sigma) \to H^{s+1/2}(Q)$ is well defined and bounded for all $s \in [0,1]$.*

The following result refers to the Scott–Zhang projection onto finite element spaces. The exact definition of this operator is immaterial for our needs; only some of its properties are used.

Proposition 19 ([18]) *There exists a linear projection* $\Pi_h : H^1(Q) \to P_h^Q$ *such that:*

(a) $\Pi_h u \in H_0^1(Q)$ *for all* $u \in H_0^1(Q)$.

(b) *For all* $0 \le s \le 1 \le t \le d+1$, *there exists a constant* C *independent of* h *such that*

$$\|\Pi_h u - u\|_{s,Q} \le C h^{t-s} |u|_{t,Q}, \qquad \forall u \in H^t(Q).$$

Proposition 20 *For all* $s \in [0, 1/2]$, *there exists* C *independent of* h *such that*

$$\|Rg_h - R_h g_h\|_{1,Q} \le C h^s \|g_h\|_{1/2+s,\Sigma}, \qquad \forall g_h \in X_h^\Sigma.$$

Proof. Since $Rg_h - R_h g_h \in H_0^1(Q)$ and Π_h is a projection, $\Pi_h R g_h - R_h g_h \in H_0^1(Q) \cap P_h^Q$ and thus

$$\begin{aligned} \int_Q \nabla(R - R_h)g_h \cdot \nabla(R - R_h)g_h &= \int_Q \nabla(R - R_h)g_h \cdot \nabla(R - \Pi_h R)g_h \\ &\le |(R - R_h)g_h|_{1,Q} |(R - \Pi_h R)g_h|_{1,Q}. \end{aligned} \tag{29}$$

On the other hand, by the preceding results, for all $s \in [0, 1/2]$ it holds

$$\|Rg_h - \Pi_h R g_h\|_{1,Q} \le C h^s \|Rg_h\|_{1+s,Q} \le C' h^s \|g_h\|_{1/2+s,\Sigma} \tag{30}$$

for all $g_h \in X_h^\Sigma$. The result then follows from (29, 30) and the Poincaré inequality. □

Proposition 21 *There exists a function* γ *such that* $\gamma(h) \to 0$ *as* $h \to 0$ *and*

$$\|Rg_h - R_h g_h\|_{0,Q} \le \gamma(h) \|g_h\|_{1/2,\Sigma}, \qquad \forall g_h \in X_h^\Sigma.$$

Proof. The quantity we want to estimate can be written

$$\|Rg_h - R_h g_h\|_{0,Q} = \sup_{0 \ne \phi \in L^2(Q)} \frac{1}{\|\phi\|_{0,Q}} \int_Q (Rg_h - R_h g_h)\, \phi. \tag{31}$$

For a given $\phi \in L^2(Q)$, we take the unique $w \in H_0^1(Q)$ such that $\Delta w = \phi$. With the regularity assumed for the boundary we can estimate (see [9])

$$\|w\|_{3/2,Q} \le C \|\phi\|_{0,Q} \tag{32}$$

with C independent of ϕ. Since $\Pi_h w \in H_0^1(Q) \cap P_h^Q$ it follows that

$$\begin{aligned} \int_Q (Rg_h - R_h g_h)\, \Delta w &= \int_Q \nabla(Rg_h - R_h g_h) \cdot \nabla w \\ &= \int_Q \nabla(Rg_h - R_h g_h) \cdot \nabla(w - \Pi_h w) \\ &\le \|Rg_h - R_h g_h\|_{1,Q} \|w - \Pi_h w\|_{1,Q} \\ &\le C h^{1/2} \|g_h\|_{1/2,Q} \|w\|_{3/2,Q} \end{aligned}$$

by Propositions 19 and 20. This inequality, (31) and (32) prove the result. □

The bound (32) is not optimal in the sense that some higher norm of w can be estimated. This would allow for some better value for the function $\gamma(h)$, which is however irrelevant for our purposes.

Proposition 22 *The operator* $N_h := \gamma_\Sigma \Pi_h R$ *satisfies:*

(a) $N_h^2 = N_h$.

(b) *There exists* C *independent of* h *such that*

$$\|N_h g\|_{1/2,\Sigma} \le C\|g\|_{1/2,\Sigma}, \qquad \forall g \in H^{1/2}(\Sigma).$$

(c) *There exists a function* δ *such that* $\delta(h) \to 0$ *as* $h \to 0$ *and*

$$\|N_h g - g\|_{0,\Sigma} \le \delta(h)\|g\|_{1/2,\Sigma}, \qquad \forall g \in H^{1/2}(\Sigma).$$

Proof. Both (a) and (b) are straightforward, using Proposition 19 with $s = t = 1$ for this last. By Propositions 17 and 19, it follows that for all $\varepsilon \in (0, 1/2]$

$$\|N_h g - g\|_{0,\Sigma} = \|\gamma_\Sigma(\Pi_h - I)Rg\|_{0,\Sigma} \le Ch^{1/2-\varepsilon}\|Rg\|_{1,Q} \le C' h^{1/2-\varepsilon}\|g\|_{1/2,\Sigma}$$

for all $g \in H^{1/2}(\Sigma)$. □

Proposition 23 *Let* $N_h : H^{1/2}(\Sigma) \to X_h^\Sigma$ *satisfy properties* (a) *and* (b) *of Proposition 22. Then, there exists* C *independent of* h *such that for all* $u \in H^1(Q)$ *with* $\Delta u = 0$ *in* Q,

$$\|u - R_h N_h \gamma_\Sigma u\|_{1,Q} \le C \inf_{p_h \in P_h^Q} \|u - p_h\|_{1,Q}.$$

Proof. We first prove the result for the particular choice $N_h = \gamma_\Sigma \Pi_h R$. Let then $v_h := R_h(\gamma_\Sigma \Pi_h R)\gamma_\Sigma u = R_h \gamma_\Sigma \Pi_h u$ and notice that $\gamma_\Sigma v_h = \gamma_\Sigma \Pi_h u$ and that hence

$$\int_Q \nabla v_h \cdot \nabla(\Pi_h u - v_h) = 0 = \int_Q \nabla u \cdot \nabla(\Pi_h u - v_h). \tag{33}$$

By (33) and the Poincaré inequality, it follows that

$$\|u - v_h\|_{1,Q} \le \|u - \Pi_h u\|_{1,Q} + \|\Pi_h u - v_h\|_{1,Q} \le C\|\Pi_h u - u\|_{1,Q}.$$

Proceeding in a standard way, the uniform boundedness of Π_h and the fact that it is a projection onto P_h^Q imply

$$\|u - \Pi_h u\|_{1,Q} \le C' \inf_{p_h \in P_h^Q} \|u - p_h\|_{1,Q}.$$

For the general case, we decompose (taking v_h as above)

$$\|u - R_h N_h \gamma_\Sigma u\|_{1,Q} \le \|u - v_h\|_{1,Q} + \|R_h(\gamma_\Sigma \Pi_h - N_h \gamma_\Sigma)u\|_{1,Q}.$$

The uniform boundedness of R_h (Proposition 20) and the fact that N_h is a uniformly bounded projection onto X_h^Σ prove then

$$\begin{aligned} \|R_h(\gamma_\Sigma \Pi_h - N_h \gamma_\Sigma)u\|_{1,Q} &\le C_1\|(\gamma_\Sigma \Pi_h - N_h \gamma_\Sigma)u\|_{1/2,\Sigma} \\ &\le C_2\|\gamma_\Sigma(\Pi_h u - u)\|_{1/2,\Sigma} \le C_3\|\Pi_h u - u\|_{1,Q}. \end{aligned}$$

This finishes the proof. □

Proposition 24 *Let $I_h^\Sigma : \mathcal{C}(\Sigma) \to X_h^\Sigma$ be the nodal Lagrange interpolation operator, let Γ be a Lipschitz curve/surface not intersecting Σ and $\mathcal{S}_\Gamma$ the associated single layer potential* (2)*. Then for all $s \in [0,1]$*

$$\|\gamma_\Sigma \mathcal{S}_\Gamma \psi - I_h^\Sigma \gamma_\Sigma \mathcal{S}_\Gamma \psi\|_{s,\Sigma} \le C h^{k+1-s} \|\psi\|_{-1/2,\Gamma}, \qquad \forall \psi \in H^{-1/2}(\Gamma). \tag{34}$$

Proof. We consider the decomposition of the boundary $\Sigma = \cup_\ell \Sigma_\ell$ in the set of its sides (faces in three dimensions) and the product space

$$\mathcal{H}^t := \prod_\ell H^t(\Sigma_\ell)$$

endowed with its natural norm. It follows from standard Lagrange interpolation theory that

$$\|g - I_h^\Sigma g\|_{s,\Sigma} \le C h^{t-s} \|g\|_{\mathcal{H}^t}, \qquad \forall g \in \mathcal{H}^t \cap \mathcal{C}(\Sigma) \tag{35}$$

for $0 \le s \le 1$, $(d-1)/2 < t \le k+1$. From classical theory of surface potentials, we know that that $\gamma_\Sigma \mathcal{S}_\Gamma : H^{-1/2}(\Gamma) \to \mathcal{H}^t$ is bounded, the image elements satisfying in fact all available compatibility conditions in corners, vertices, etc. The result is then a straightforward consequence of (35). □

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