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ON p-CHIEF FACTORS AND **EXTENSIONS OF K** *G*-MODULES

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ON p-CHIEF FACTORS AND EXTENSIONS
OF $\mathbb{K}G$-MODULES

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Abstract

The subject of this paper is the relationship between the set of chief factors of a finite group G and extensions of an irreducible $\mathbb{K}G$-module U ($\mathbb{K}$ a field). Let H/L be a p-chief factor of G. We prove that, if H/L is complemented in a vertex of U, then there is a short exact sequence of Ext-functors for the module U and any $\mathbb{K}G$-module V. In some special cases, we prove the converse, which is false in general. We also consider the intersection of the centralizers of all the extensions of U by an irreducible module and provide new bounds for this group. *2000 Math. Subj. Class.:* Primary: 20C20; secondary: 20J05, 20J06

1. Introduction.

The first motivation to investigate the relationship between the set of chief factors of a finite group and extensions of irreducible modules is a well known theorem by Gaschütz for p-solvable groups. This theorem can be extended to an arbitrary group and be stated in the following way. Let p be a prime, H/L a chief factor of a finite group G, $\mathbb{K}$ a field of characteristic p and V an irreducible $\mathbb{K}G$-module. We also denote by $\mathbb{K}$ the module of dimension one on which the action of the group is trivial. If H/L is a p-chief factor of G, then
if H/L is complemented in G there is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(\mathbb{K}, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G/L}(\mathbb{K}, V) \to \mathrm{Hom}_{\mathbb{K}G}((H/L)_{\mathbb{K}}, V) \to 0;$$

in the other case, and also if H/L is a chief factor of order not divisible by p,

$$\mathrm{Ext}^1{}_{\mathbb{K}G/L}(\mathbb{K}, V) = \mathrm{Ext}^1{}_{\mathbb{K}G/H}(\mathbb{K}, V).$$

Recall that for $\mathbb{K}G$-modules U and V, $\mathrm{Ext}^1{}_{\mathbb{K}G}(U, V)$ can be seen as the set of equivalence clases of extensions of U by V. The previous result can be obtained from the inflation-restriction sequence (hereafter inf-res) of Ext-groups associated to the normal subgroup H/L of G/L. Here we also consider the inf-res sequence for the modules U and V. We say that the p-chief factor H/L is influential on ext-U if $H \le \mathrm{C}_G(V)$ and for some irreducible module V with $H \le \mathrm{C}_G(V)$ one has

$$\mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) < \mathrm{Ext}^1{}_{\mathbb{K}G/L}(U, V)$$

We can interprete the result of Gaschütz by saying that H/L is influential on ext-$\mathbb{K}$ if and only if H/L is complemented in G. Fix an irreducible module U and $N = \mathrm{C}_G(U)$. We are interested in the following question: When is a p-chief factor of G which appears under N influential on ext-U? An answer to this question in terms of the group would yield a characterization of the intersection of the centralizers of the non split extensions of U by an irreducible $\mathbb{K}G$-module in terms of the group (see (2.4) below).

As in the case of extensions of $\mathbb{K}$, in the general case one can relate the fact that a chief factor is complemented in a certain subgroup of G (as G itself or N) with its influence on the extensions of U by an irreducible module ([4], [5], [6], [7]). A consequence of this is

$$\Phi_p(N) \leq \mathrm{C}_G(\mathrm{P}_G(U)/\,\mathrm{P}_G(U)J^2) \leq \Phi_p(G) \cap N$$

where $\mathrm{P}_G(U)$ is the projective cover of U, $J = J(\mathbb{K}G)$ and $\Phi_p(G) = \Phi(G \bmod O_{p'}(G))$. This inequality was obtained by Lafuente ([4]) and Stammbach ([8]). It was shown in [5] that both inequalities can be strict, even for p-solvable groups.

In this paper, we consider the vertices of the module U as G/N-module. We prove that if a p-chief factor H/L is complemented in those vertices, then the inf-res sequence associated to H/L, the fixed module U and any module V yields a short exact sequence. Using this fact we give new bounds for the subgroup $\mathrm{C}_G(\mathrm{P}_G(U)/\,\mathrm{P}_G(U)J^2)$. However, these new bounds can also be strict, at least in the not p-solvable case (examples (3.1) and (3.3)). In the last paragraph we consider a chief factor H/L and assume that for any module V the inf-res sequence assoeciated to H/L, U and V yields a short exact sequence. We prove that, under certain conditions, that forces H/L to be complemented in the vertices of U. This is nor true in general. In particular, our conditions hold if either S/N (vertex of U as G/N-module) is cyclic or if G/N is p-solvable.

Given $\xi \in \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V)$ and E a module extension of U by V representing the class of ξ, then we write $\mathrm{C}_G(\xi) = \mathrm{C}_G(E)$. All the tensor products we consider are as $\mathbb{K}$-vector spaces, so by the symbol $\otimes$ we will denote $\otimes_{\mathbb{K}}$. For a $\mathbb{K}G$-module V, $V^* = \mathrm{Hom}_{\mathbb{K}}(V,\mathbb{K})$ denotes the dual module, we also put ξ^* for the class of E^*. If $\mathbb{F} : \mathbb{K}$ is a field extension, $V_{\mathbb{F}} = V \otimes \mathbb{F}$ is the module obtained from V by extension of scalars. For all the concepts of representation theory, we refer to [1], for those of cohomolgy to [3].

2. Conditions on chief factors and new bounds.

Throughout the paper G denotes a finite group, U a fixed irreducible $\mathbb{K}G$-module, and $N = \mathrm{C}_G(U)$. The term chief factor always refers to a chief factor of G and will be written p-ch.f.

Definition 2.1. Let $F = H/L$ be a chief factor of G. We say that F is influential on ext-U if $H \leq N$ and for some irreducible module V with $H \leq \mathrm{C}_G(V)$ one has

$$\mathrm{Ext}^1{}_{\mathbb{K}G/H}(U,V) < \mathrm{Ext}^1{}_{\mathbb{K}G/L}(U,V).$$

We also put, for $\mathrm{P}_G(U)$ the projective cover of U and $J = J(\mathbb{K}G)$,

$$\begin{aligned}\mathrm{C}_{ext_G}(U) &= \mathrm{C}_G(\mathrm{P}_G(U)/\,\mathrm{P}_G(U)J^2) = \\ &= \cap\{\mathrm{C}_G(\xi); 0 \neq \xi \in \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V),\ V \text{ irreducible}\}.\end{aligned}$$

Definition 2.2. Let $M \trianglelefteq G$ and H/L p-ch.f. with $H \leq M$.

We say that a p-ch.f. R/K is over H/L in M if $R \leq M$, $L \leq K$ and $R = HK$. We call H/L upper in M if there is no p-ch.f. over H/L in M other than H/L.

For any property P, we call H/L upper in M as P if it verifies P and for any R/K p-ch.f. over H/L in M with $L < K$, R/K does not verify P.

Lemma 2.3. *Let F be a chief factor which is influential on ext-U. Then* *a) $p||F|$. If F is abelian, it is complemented in N.*

b) There exist some p-ch.f. upper in N as influential on ext-U which is over F.

Proof.

a) We may assume $F \trianglelefteq_{\min} G$. By [5; 3.2] $F \not\leq \Phi_p(N)$, therefore $p||F|$. Assume F is abelian. Let $M <_{\max} N$ with $FM = N$, then $F = F \cap M \times T$ for $T \trianglelefteq_{\min} N$ and for certain elements $x_2,\ldots,x_s$ in G, $F \cap M = T^{x_2} \times \ldots \times T^{x_s}$. Using that $MT = N$, $M \cap T = 1$, it is easy to prove that $M \cap M^{x_2} \cap \ldots \cap M^{x_s}$ is a complement of F in N.

b) This is trivial. **q.e.d.**

Our first claim is that the group $\mathrm{C}_{ext_G}(U)$ can be described in terms of influential chief factors. We denote by $\mathrm{R}_p(G)$ the p-solvable radical of G.

Proposition 2.4.

$$\mathrm{R}_p(N) \cap \{L; H/L \text{ } p\text{-ch.f. influential on ext-}U\} \leq \mathrm{C}_{ext_G}(U) =$$
$$\mathrm{R}_p(N) \cap \{L; H/L \text{ } p\text{-ch.f. upper on } N \text{ as influential on ext-}U\}$$

Proof. We put

$$R = \mathrm{R}_p(N) \cap \{L; H/L \text{ } p\text{-ch.f. upper on } N \text{ as influential on ext-}U\}.$$

Obviously, $\mathrm{R}_p(N) \cap \{L; H/L$ p-ch.f. influential on ext-$U\} \leq R$, so we only have to prove that this last group is equal to $\mathrm{C}_{ext_G}(U)$. Let H/L be a p-ch.f. upper on N as influential on ext-U. Since it is influential on ext-U, there exists a $0 \neq \xi \in \mathrm{Ext}^1{}_{\mathbb{K}G/L}(U,V)$ for V irreducible such that $H \not\leq \mathrm{C}_G(\xi)$. This implies that $H\,\mathrm{C}_G(\xi)/\,\mathrm{C}_G(\xi)$ is influential on ext-U. But this factor is over H/L, thus $L = \mathrm{C}_G(\xi)$. So we have proved

$$\mathrm{C}_{ext_G}(U) \leq R.$$

Now, let V be irreducible and $0 \neq \xi \in \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V)$. If $\mathrm{C}_G(\xi) = N$ then $R \leq \mathrm{C}_G(\xi)$, so we may assume $\mathrm{C}_G(\xi) < N$. For any $\bar{H} = H/\,\mathrm{C}_G(\xi) \trianglelefteq_{\min} G/\,\mathrm{C}_G(\xi)$ with $H \leq N$ we define a group $K(H)$ in the following way

if $\bar{H}$ is p-ch.f., let $K(H)$ be a subgroup such that $K(H)H/K(H)$ is a p-ch.f. upper in N as influential on ext-U which is over $\bar{H}$ (see (2.3) b));

if $\bar{H}$ is non-abelian of order a multiple of p, $K(H) = N \cap \mathrm{C}_G(\bar{H})$.

Note that by (2.3) a) these are the only possibilities for $\bar{H}$. Next, we claim that

$$\mathrm{C}_G(\xi) = \cap\{K(H); H/\,\mathrm{C}_G(\xi) \trianglelefteq_{\min} G/\,\mathrm{C}_G(\xi), H \leq N\}.$$

Obviously, $\mathrm{C}_G(\xi) \leq K(H)$ for any H. If there is some H_1 with $\mathrm{C}_G(\xi) < H_1 \leq \cap\{K(H); H/\,\mathrm{C}_G(\xi) \trianglelefteq_{\min} G/\,\mathrm{C}_G(\xi), H \leq N\}$, then $H_1 \leq K(H_1)$. This contradiction proves the claim. Now, since $R \leq K(H)$ for any such H, one has $R \leq \mathrm{C}_G(\xi)$, and thus the result follows. **q.e.d.**

Note that, in the previous result, we could have written "upper in $N \cap \mathrm{C}_G(H/L)$" instead of "upper in N", but this makes no difference, since if H/L is a p-ch.f. under N, then H/L and any p-ch.f. over it in N are obviously under $N \cap \mathrm{C}_G(H/L)$.

The problem to use the description given for $\mathrm{C}_{ext_G}(U)$ is that it is difficult in general to determine whether a p-ch.f. is influential on ext-U or not. Next, we prove that if the chief factor is complemented in certain subgroups of G, it must be influential.

Lemma 2.5. *Let U, U_1, V, V_1 be $\mathbb{K}G$-modules such that $U|U_1$ and $V|V_1$. Assume that for $H \trianglelefteq G$, $H \leq \mathrm{C}_G(U_1) \cap \mathrm{C}_G(V_1)$ there is a short exact sequence*

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U_1, V_1) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U_1, V_1) \to \mathrm{Ext}^1{}_{\mathbb{K}H}(U_1, V_1)^G \to 0$$

then there also is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V) \to \mathrm{Ext}^1{}_{\mathbb{K}H}(U, V)^G \to 0.$$

Proof. The functor Ext is additive. **q.e.d.**

For any $S \leq G$, we say that U is S-projective if it is projective relative to S ([1; I, 4]).

Proposition 2.6. *Let $H \trianglelefteq G$ be a p-elementary abelian subgroup of G and $H \leq S \leq G$ such that U is S-projective. If H has a complement in S, then for any module V with $H \leq \mathrm{C}_G(V)$ there is a short exact sequence*

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V) \to 0.$$

Proof. Note that $H \leq N$. Since U is S-projective, $U|\ W{\uparrow}^G$ for some $\mathbb{K}S$-module W. By [5; 3.1], the fact that H is complemented in S implies that for any $\mathbb{K}G$-module V centralized by H, there is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}S/H}(W, V) \to \mathrm{Ext}^1{}_{\mathbb{K}S}(W, V) \to \mathrm{Hom}_{\mathbb{K}S}(H_{\mathbb{K}} \otimes W, V) \to 0.$$

Aplying now the Eckmann-Shapiro lemma we obtain

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(W{\uparrow}^G, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(W{\uparrow}^G, V) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes W{\uparrow}^G, V) \to 0.$$

It thus suffices to use the previous lemma. **q.e.d.**

Corollary 2.7. *Let $F = H/L$ be a p-chief factor of G with $H \leq N$ and $H \leq S \leq G$ such that U is S-projective. If F has a complement in S, then F is influential on ext-U.*

Proof. Choose V irreducible with $\mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V) \neq 0$; note that this implies $H \leq \mathrm{C}_G(V)$. It suffices to apply (2.6) to G/L. **q.e.d.**

Corolary 2.8. *Let $F = H/L$ be a p-chief factor of G with $H \leq N$ and $H \leq S \leq G$ such that U is S-projective. Let V be an irreducible module with $0 \neq \mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V)$. If F has a complement in S, then F has also a complement in $\mathrm{C}_G(V)$.*

Proof. First note that these conditions imply that $H \leq \mathrm{C}_G(V)$. By (2.6), there exists $0 \neq \xi \in \mathrm{Ext}^1{}_{\mathbb{K}G/L}(U, V)$ with $H \not\leq \mathrm{C}_G(\xi)$. Dualizing, we have an element $0 \neq \xi^* \in \mathrm{Ext}^1{}_{\mathbb{K}G/L}(V^*, U^*)$ such that $H \not\leq \mathrm{C}_G(\xi^*) = \mathrm{C}_G(\xi)$. Thus F is influential on ext-V^*, so by (2.3) it is complemented in $\mathrm{C}_G(V) = \mathrm{C}_G(V^*)$.**q.e.d.**

We also have the next corollary, which is an old and well known result by Gaschütz ([2; 2]).

Corollary 2.9. *Let $H \trianglelefteq G$ be p-elementary abelian and S be a p-Sylow subgroup of G. Then, H is complemented in G if and only if it is complemented in S.*

Proof. Assume H is complemented in S. S is a vertex of the trivial module $\mathbb{K}$. We may apply (2.6) to $\mathbb{K}$ and $H_{\mathbb{K}}$ and obtain a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(\mathbb{K}, H_{\mathbb{K}}) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(\mathbb{K}, H_{\mathbb{K}}) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}}, H_{\mathbb{K}}) \to 0.$$

By a well known result which we have stated as a lemma in (4.1), this implies that H is complemented in G. The converse is obvious. **q.e.d.**

Using (2.6) we can give a new upper bound for the group $\mathrm{C}_{ext_G}(U)$. In fact, (see (2.11) and (2.12) below) we can give a bound $\Phi(U, S)$ for each subgroup $S \leq G$ such that U is S-projective and $N \leq S$ (so S contains all the chief factors under N). The group $\Phi(U, S)$ will be smallest if we choose S minimal with these conditions, that is, if S/N is a vertex of U as $\mathbb{K}G/N$-module. Notice also that by b) in the next lemma, there is no loss by restricting ourselves to these groups.

Lemma 2.10. *Let Q be a vertex of U, then*

a) QN/N is a vertex of U as $\mathbb{K}G/N$-module.

b) For any $H \trianglelefteq G$ p-elementary abelian with $H \leq N$, H is complemented in QN if and only if it is complemented in Q.

Proof.

a) It is clear that for some vertex M/N of U as $\mathbb{K}G/N$-module, $M \leq QN$. By [1; III, 4.12], $Q \cap N$ is a p-Sylow subgroup of N. Then, since $N(Q \cap M) = M$, $p \not| |M : S \cap M|$. Hence $U\downarrow_M$ is $Q \cap M$-projective. As U is M-projective we deduce that it is also Q-projective so $Q \cap M = Q$, that is, $M = QN$.

b) Notice that $H \leq Q$. Since Q is a p-Sylow subgroup of QN, the claim follows from (2.9). **q.e.d.**

Definition 2.11. Let $N \leq S \leq G$, we say that a p-ch.f. H/L verifies the property (1) with respect to S and U if

(1) $H \leq N$, H/L is complemented in S and is either upper in N as complemented in N or, for some irreducible $\mathbb{K}G$-module V with $0 \neq \mathrm{Hom}_{\mathbb{K}G}((H/L)_{\mathbb{K}} \otimes U, V)$, H/L is upper in $\mathrm{C}_G(V)$ as complemented in $\mathrm{C}_G(V)$ (see (2.8)). We put

$$\phi(U, S) = \mathrm{R}_p(N) \cap \{L; H/L \text{ } p\text{-ch.f. verifying (1) respect to } S \text{ and } U\}$$
and, for $\hat{S}/N$ a vertex of U as G/N-module, $\phi(U) = \phi(U, \hat{S})$.

Recall that all the vertices are conjugated, and therefore the subgroup $\phi(U)$ is well defined.

Theorem 2.12. *Let $S \leq G$ be a subgroup such that U is S-projective. Then*

$$\mathrm{C}_{ext_G}(U) \leq \phi(U) \leq \phi(U, S).$$

Proof. Let $\hat{S}/N$ be a vertex of U as G/N-module. For the second inequality, it suffices to note that, up to conjugacy, we may assume $\hat{S} \leq S$. Let $F = H/L$ be a p-ch.f. which verifies (1) with respect to $\hat{S}$ and U. Then, by (2.7) F is influential on ext-U. If, moreover, F is upper in N as complemented in N, by (2.3) a) any p-ch.f. over F in N and other than F can not be influential on ext-U. That means that F is upper in N as influential on ext-U. Thus by (2.4) $\mathrm{C}_{ext_G}(U) \leq L$.

Now, assume that for some V irreducible with $0 \neq \mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V)$, F is upper in $\mathrm{C}_G(V)$ as complemented in $\mathrm{C}_G(V)$. By (2.6) there exists a $0 \neq \xi \in \mathrm{Ext}^1{}_{\mathbb{K}G/L}(U, V)$ with $H \not\leq \mathrm{C}_G(\xi)$. Dualizing we have an element $0 \neq \xi^* \in \mathrm{Ext}^1{}_{\mathbb{K}G/L}(V^*, U^*)$ such that $H \not\leq \mathrm{C}_G(\xi^*) = \mathrm{C}_G(\xi)$. Consider the p-ch.f.

$H\,\mathrm{C}_G(\xi)/\,\mathrm{C}_G(\xi)$. It is over F and is influential on ext-V^*, thus is complemented in $\mathrm{C}_G(V^*) = \mathrm{C}_G(V)$. By our assumption on F, $\mathrm{C}_{ext_G}(U) \leq \mathrm{C}_G(\xi) = L$. **q.e.d.**

Using duality and (2.6), we can refine a little bit more the inequality $\mathrm{C}_{ext_G}(U) \leq \phi(U)$. Let V be an irreducible module such that for $F = H/L$ a p-chief factor of G upper in N, one has $\mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V) \neq 0$. Moreover let $Q/\,\mathrm{C}_G(V)$ be a vertex of V as $G/\,\mathrm{C}_G(V)$-module. Assume that F is complemented in Q. Obviously $Q/\,\mathrm{C}_G(V)$ is also a vertex of V^* as $G/\,\mathrm{C}_G(V)$-module, and therefore for any module, in particular for U^*, there exists a short exact sequence

$$\begin{array}{ccccccc} 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}G/H}(V^*, U^*) & \to & \mathrm{Ext}^1{}_{\mathbb{K}G/L}(V^*, U^*) & \to & \mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes V^*, U^*) & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) & \to & \mathrm{Ext}^1{}_{\mathbb{K}G/L}(U, V) & \to & \mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V) & \to 0 \end{array}$$

Thus F is influential on ext-U.

Definition 2.13. We say that a p-ch.f. H/L verifies the property (2) with respect to U if

(2) $H \leq N$ and for some irreducible module U such that $\mathrm{Hom}_{\mathbb{K}G}((H/L)_{\mathbb{K}} \otimes U, V) \neq 0$, if $Q/\,\mathrm{C}_G(V)$ is a vertex of V as $G/\,\mathrm{C}_G(V)$-module, then H/L is complemented in Q and is either upper in N as complemented in N (see (2.8)) or upper in $\mathrm{C}_G(V)$ as complemented in $\mathrm{C}_G(V)$.

Put

$$\delta(U) = \cap\{L; H/L \ p\text{-ch.f. verifying (2) with respect to } U\} \cap \phi(U).$$

We have

Theorem 2.14.

$$\mathrm{C}_{ext_G}(U) \leq \delta(U) \leq \phi(U) \leq \Phi_p(G) \cap N.$$

Proof. The first inequality follows from an argument similar to the one used in the proof of (2.12), using duality. The inequality $\delta(U) \leq \phi(U)$ is trivial. For the last part, note that

$$\Phi_p(G) = \cap\{\mathrm{D}_G(F); F \ p\text{-ch.f. complemented in } G\} \text{ where}$$
$$\mathrm{D}_G(F) = G \cap \{L; \mathrm{C}_G(F)/L \cong F \text{ and complemented in } G\}.$$

Let $\hat{S}/N$ be a vertex of U as G/N-module. For any F,

$$\mathrm{D}_G(F) \cap N = \cap\{L \cap N; \mathrm{C}_G(F)/L \cong F \text{ and complemented in } G\}.$$

Consider $\mathrm{C}_G(F)/L \cong F$ and complemented in G. If $\mathrm{C}_G(F) \cap N/L \cap N \neq 1$, it is a p-ch.f. upper in N and complemented in G, so it is also complemented in $\hat{S}$ and verifies the property (1) with respect to U. Thus $\phi(U) \leq L \cap N$. Now assume $\mathrm{C}_G(F) \cap N = L \cap N$. Since $\mathrm{C}_G(F)/L \cong F$ this implies that N centralizes F, that is, $\phi(U) \leq N = \mathrm{C}_G(F) \cap N = L \cap N \leq L$. **q.e.d.**

3. Some examples.

In this paragraph we prove that the inequalities of (2.14) can be strict, the second one even in the p-solvable case. First we consider an example for which $\delta(U) < \phi(U)$

Example 3.1. Let $A = SL(2,3)$, $\mathbb{K} = \mathbb{F}_3$. There are three irreducible modules X_1, X_2 and X_3 of dimensions 1, 2 and 3 respectively. These modules satisfy: X_3 is projective (it is the Steinberg module), X_2 is faithful, $X_2 \otimes X_2 \cong X_1 \oplus X_3$ and there exists a module N which is a non-split extension of X_2 by X_2. Let $G = A[N]$ and $F \trianglelefteq G$, $X_2 \cong_A F \leq N$ and put $X_2 = U$. Let S be a 3-Sylow subgroup of G, S is also a vertex of U. There are two chief factors under $N = \mathrm{C}_G(U)$, namely F and N/F and both are upper in N (notice that N is a non-split extension). N/F is complemented in G therefore also in S, but F can not be complemented in S by (2.9). Hence

$$\phi(U) = F.$$

Let $V = X_3$. N is a vertex of V, $0 \neq \mathrm{Hom}_{\mathbb{K}G}(F \otimes U, V)$ and F is complemented in N. Thus

$$\delta(U) = 1 < \phi(U).$$

Next, we consider the inequality $\mathrm{C}_{ext_G}(U) \leq \delta(U)$. To construct an example for which it is strict, we are going to use the following result.

Lemma 3.2. *Let A be a group and U a faithful $\mathbb{K}A$-module. Assume $G = A[N]$ with N a p-elementary abelian group such that $N_{\mathbb{K}} \leq U^* \otimes V$ for some irreducible $\mathbb{K}A$-module V. Then*

$$\mathrm{C}_{ext_G}(U) = 1.$$

Proof. By [5; 3.1] (or by (2.6)) the inf-res sequence yields a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/N}(U,V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V) \overset{\mathrm{res}}{\to} \mathrm{Hom}_{\mathbb{K}G}(N_{\mathbb{K}} \otimes U, V) \to 0.$$

We have $\mathrm{Hom}_{\mathbb{K}G}(N_{\mathbb{K}} \otimes U, V) \cong \mathrm{Hom}_{\mathbb{K}G}(N_{\mathbb{K}}, U^* \otimes V) \cong \mathrm{Hom}_G(N, U^* \otimes V)$. We may choose $f \in \mathrm{Hom}_G(N, U^* \otimes V)$ with $\mathrm{Ker} f = 1$ and $\xi \in \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V)$ with $f = \mathrm{res}(\xi)$. By [8; 4.1], $N \cap \mathrm{C}_G(\xi) = \mathrm{Ker}(\mathrm{res}(\xi)) = 1$. Since $\mathrm{C}_G(\xi) \leq \mathrm{C}_G(U) = N$, we deduce $\mathrm{C}_G(\xi) = 1$. **q.e.d.**

Example 3.3. Let $A = A_5$, $\mathbb{K} = \mathrm{GF}(2)$ and $\mathbb{F} = \mathrm{GF}(4)$. $\mathbb{F}$ is a spliting field for A and the irreducible $\mathbb{F}A$-modules are X_1, X_2, X_3 and X_4 of dimensions 1, 2, 2 and 4 respectively. There is an irreducible $\mathbb{K}A$-module, which we will denote by U, such that $U_{\mathbb{F}} = X_2 \oplus X_3$. It is easy to see the following facts: all these modules are self dual, $X_2 \otimes X_3 \cong X_4$ and $X_i \otimes X_i$ is a module with upper Loewy series X_1, X_j, X_1 for $i, j = 2, 3$, $i \neq j$.

Let $P \cong C_2 \times C_2$ be a 2-Sylow subgroup of A, then for any $1 \neq Q < P$, $C_2 \cong Q$, thus a faithful $\mathbb{F}Q$-module of dimension 2 must be projective. This implies that $X_2{\downarrow_Q}$, $X_3{\downarrow_Q}$ and $U{\downarrow_Q}$ are projective and therefore P is a vertex of U.

Let $N = (U \otimes U)J(\mathbb{K}A)$. Taking into account that the extension of scalars preserves the Loewy structure of a module, one deduces that $N \leq U \otimes U$ is a non-split extension

$$0 \to H \to N \to U \to 0$$

where $H \cong \mathbb{K} \oplus \mathbb{K}$ and that $N_{\mathbb{F}}$ is the sum of two non split extensions of X_i by X_1 for $i = 2, 3$.

Let $G = A[N]$. By (3.2), $\mathrm{C}_{ext_G}(U) = 1$. N/H is upper in $N = \mathrm{C}_G(U)$ and complemented in G, so $\delta(U) \leq H$. Let $H/L \cong_G \mathbb{K}$ be a chief factor. Let V be an irreducible $\mathbb{F}G$-module with $\mathrm{Hom}_{\mathbb{K}G}((H/L)_{\mathbb{K}} \otimes U, V) \neq 0$, then $V \cong U$. So PN/N is a vertex of V as G/N-module. Assume that H/L is complemented in PN, then by (2.9) is complemented in G so the extension

$$0 \to H/L \to N/L \to U \to 0$$

splits. So there is a quotient of $N_{\mathbb{F}}$ isomorphic to $\mathbb{F}$. This is a contradiction. Hence H/L is not complemented in PN and we deduce

$$1 = \mathrm{C}_{ext_G}(U) < H = \delta(U).$$

4. Particular cases.

As we have remarked in the introduction, the set of chief factors which are influential on ext-$\mathbb{K}$ is precisely the set of chief factors which are complemented in G. Thus in this case we know how to determine whether a chief factor is influential or not. This can be proved using the following well known result, which for convenience we state as a lemma.

Lemma 4.1. *Let $H \trianglelefteq G$ p-elementary abelian. Then H is complemented in G if and only if there is a short exact sequence*

$$0 \to \mathrm{H}^1(G/H, H_{\mathbb{K}}) \to \mathrm{H}^1(G, H_{\mathbb{K}}) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}}, H_{\mathbb{K}}) \to 0$$

Proof. It is a direct consequence of [3; VI, 10]. **q.e.d.**

This can be generalized in certain cases to any irreducible module U in the following way

Theorem 4.2. *Let S/N be a vertex of U as G/N-module and $T \leq G$ with $S \leq T \leq G$, then U is T-projective. Assume that there exists a $\mathbb{K}T$-module W such that $U | W{\uparrow}^G$ and $p \nmid \dim W$. Then for $H \trianglelefteq G$ p-elementary abelian the following conditions are equivalent:*

i) For any $\mathbb{K}G$-module V with $H \leq \mathrm{C}_G(V)$ there is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V) \to 0;$$

ii) There is a short exact sequence for $V_1 = H_{\mathbb{K}} \otimes U$

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_1) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_1) \to \mathrm{Hom}_{\mathbb{K}G}(V_1, V_1) \to 0;$$

iii) There is a short exact sequence for $V_2 = H_{\mathbb{K}} \otimes W{\uparrow}^G$

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_2) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_2) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V_2) \to 0;$$

iv) H is complemented in S.

Proof. The implication i)$\Rightarrow$ ii) is trivial, ii)$\Rightarrow$ iii) follows from (2.5) and iv)$\Rightarrow$ i) is (2.6). So it remains to prove iii)$\Rightarrow$ iv). Let $W_1 = H_{\mathbb{K}} \otimes W$. From the Eckmann-Shapiro lemma and the given short exact sequence we obtain

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_2) & \to & \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_2) & \to & \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V_2) & \to 0 \\ & \| & & \| & & \| & \\ 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}T/H}(U, W_1) & \to & \mathrm{Ext}^1{}_{\mathbb{K}T}(U, W_1) & \to & \mathrm{Hom}_{\mathbb{K}T}(H_{\mathbb{K}} \otimes U, W_1) & \to 0. \end{array}$$

Since $W|\, U{\downarrow_T}$, it follows by (2.5) that

$$\begin{array}{ccccccc} 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}T/H}(W, W_1) \to & \mathrm{Ext}^1{}_{\mathbb{K}T}(W, W_1) \to & \mathrm{Hom}_{\mathbb{K}T}(W_1, W_1) \to & 0 \\ & \| & \| & \| & \\ 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}T/H}(\mathbb{K}, W_1 \otimes W^*) \to & \mathrm{Ext}^1{}_{\mathbb{K}T}(\mathbb{K}, W_1 \otimes W^*) \to & \mathrm{Hom}_{\mathbb{K}T}(H_{\mathbb{K}}, W_1 \otimes W^*) \to & 0. \end{array}$$

Now, since $p \not| \dim W$, we have $\mathbb{K}|W \otimes W^*$. Thus $H_{\mathbb{K}}|H_{\mathbb{K}} \otimes W \otimes W^* = W_1 \otimes W^*$. Applying again (2.5) we obtain the short exact sequence

$$\begin{array}{ccccc} 0 \to & \mathrm{Ext}^1{}_{\mathbb{K}T/H}(\mathbb{K}, H_{\mathbb{K}}) \to & \mathrm{Ext}^1{}_{\mathbb{K}T}(\mathbb{K}, H_{\mathbb{K}}) \to & \mathrm{Hom}_{\mathbb{K}T}(H_{\mathbb{K}}, H_{\mathbb{K}}) \to & 0 \\ & \| & \| & \| & \\ 0 \to & \mathrm{H}^1(T/H, H_{\mathbb{K}}) \to & \mathrm{H}^1(T, H_{\mathbb{K}}) \to & \mathrm{Hom}_{\mathbb{K}T}(H_{\mathbb{K}}, H_{\mathbb{K}}) \to & 0 \end{array}$$

By (4.1), H is complemented in T, therefore it is also complemented in S. **q.e.d.**

Now, we consider two particular cases in wich the hypotheses of this proposition are satisfied.

Corollary 4.3. *Assume that for the vertex S/N of U as G/N-module, S/N is cyclic. Then for $H \trianglelefteq G$ p-elementary abelian the following conditions are equivalent:*

i) For any $\mathbb{K}G$-module V with $H \leq \mathrm{C}_G(V)$ there is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V) \to 0;$$

ii) There is a short exact sequence for $V_1 = H_{\mathbb{K}} \otimes U$

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_1) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_1) \to \mathrm{Hom}_{\mathbb{K}G}(V_1, V_1) \to 0;$$

iii) There is a short exact sequence for $V_2 = H_{\mathbb{K}} \otimes W{\uparrow^G}$

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_2) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_2) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V_2) \to 0;$$

iv) H is complemented in S.

Proof. The module U is S-projective. We choose W to be a $\mathbb{K}S$-module which is a source of U. Since $W|\, U{\downarrow_S}$ one has that W is a $\mathbb{K}S/N$-module. Assume $p|\dim W$. The group S/N is cyclic so this would imply that for some $S_1 \leq S$ of index p and some $\mathbb{K}S_1$-module W_1, $W \cong W_1{\uparrow^S}$. This is a contradiction since S/N is a vertex of U as G/N-module. **q.e.d.**

Corollary 4.4. *Assume that G/N is p-solvable. Then for $H \trianglelefteq G$ p-elementary abelian the following conditions are equivalent:*

i) For any $\mathbb{K}G$-module V with $H \leq \mathrm{C}_G(V)$ there is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V) \to 0$$

ii) There is a short exact sequence for $V_1 = H_{\mathbb{K}} \otimes U$

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_1) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_1) \to \mathrm{Hom}_{\mathbb{K}G}(V_1, V_1) \to 0;$$

iii) There is a short exact sequence for $V_2 = H_{\mathbb{K}} \otimes W{\uparrow^G}$

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/H}(U, V_2) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U, V_2) \to \mathrm{Hom}_{\mathbb{K}G}(H_{\mathbb{K}} \otimes U, V_2) \to 0;$$

iv) H is complemented in S.

Proof. By a theorem of Huppert ([7; 10.11], the hypothesis of (4.2) are satisfied. **q.e.d.**

Our last result allows us to prove that (4.3) and (4.4) do not hold in general. Recall that given any module U, a short exact sequence of modules $0 \to W \to E \to V \to 0$ is said to be U-split if the sequence

$$0 \to W \otimes U \to W \otimes E \to W \otimes V \to 0$$

splits.

Proposition 4.5. *Let A be a group and U a faithful $\mathbb{K}A$-module. Assume that $G = A[N]$ with N p-elementary abelian such that $N_{\mathbb{K}} \cong E$ for E an extension $0 \to V_1 \to E \to V_2 \to 0$ which is U-split. Let $F \trianglelefteq G$, $F \leq N$ with $F_{\mathbb{K}} \cong V_1$. Then for any $\mathbb{K}G$-module V there is a short exact sequence*

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/F}(U,V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V) \to \mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V) \to 0.$$

Proof. Since N is complemented in G, there is for any module V a short exact sequence

$$0 \to \mathrm{H}^1(G/N, U^* \otimes V^N) \to \mathrm{H}^1(G, U^* \otimes V) \to \mathrm{H}^1(N, U^* \otimes V)^G \to 0. \tag{1}$$

The sequence $0 \to V_1 \otimes U \to E \otimes U \to V_2 \otimes U \to 0$ splits. Hence both the dual sequence and the tensor product of this last one by V split. Since N is p-elementary abelian we have

$$\begin{array}{ccccccc}
0 \to & V_2^* \otimes U^* \otimes V \to & & E^* \otimes U^* \otimes V \to & & V_1^* \otimes U^* \otimes V \to 0 & \\
& \| & & \| & & \| & \\
0 \to & \mathrm{Hom}_{\mathbb{K}}(V_2, U^* \otimes V) \to & & \mathrm{Hom}_{\mathbb{K}}(E, U^* \otimes V) \to & & \mathrm{Hom}_{\mathbb{K}}(V_1, U^* \otimes V) \to 0 & \\
& \| & & \| & & \| & \\
0 \to & \mathrm{H}^1(N/F, U^* \otimes V) \to & & \mathrm{H}^1(N, U^* \otimes V) \to & & \mathrm{Hom}_{\mathbb{K}N}(F_{\mathbb{K}}, U^* \otimes V) \to 0 & (2)
\end{array}$$

and therefore the sequence (2) splits. This, together with the short exact sequence (1) yields the result, with the same proof as in [6; 1.6]. **q.e.d.**

Example 4.6. Assume we have a group A and for $\mathbb{K} = \mathrm{GF}(p)$ an irreducible $\mathbb{K}A$-module U which is faithful, with $p | \dim U$ and such that the p-Sylow subgroups of G are vertices for U.

Then there is a short exact sequence

$$0 \to \mathbb{K} \to U^* \otimes U \to M \to 0$$

which is not split but is U-split. Let $N = U^* \otimes U$ and $F \leq_A N$ image of $\mathbb{K}$ in the previous sequence. We may apply (4.5) to $G = A[N]$ and obtain that for any $\mathbb{K}G$-module V there is a short exact sequence

$$0 \to \mathrm{Ext}^1{}_{\mathbb{K}G/F}(U,V) \to \mathrm{Ext}^1{}_{\mathbb{K}G}(U,V) \to \mathrm{Hom}_{\mathbb{K}G}(F_{\mathbb{K}} \otimes U, V) \to 0.$$

However, F can not be complemented in the vertices of U, by (2.9).

For example, $A = A_5$ and U the module of (3.3) verify this conditions.

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