PRE-PUBLICACIONES del seminario matematico 2002
"garcia de galdeano"

Dual mixed volumes, isotropic measures and reverse dual isoperimetric inequalities for convex bodies

J. Bastero
M. Romance

n. 8

seminario matemático
garcía de galdeano
Universidad de Zaragoza

Dual mixed volumes, isotropic measures and reverse dual isoperimetric inequalities for convex bodies

Jesús Bastero[1]
Departamento de Matemáticas
Universidad de Zaragoza
Ciudad Universitaria s/n, 50009
Zaragoza (Spain)
E-mail: bastero@posta.unizar.es

and

Miguel Romance[2]
Departamento de Matemáticas
Universidad de Zaragoza
Ciudad Universitaria s/n, 50009
Zaragoza (Spain)
E-mail: mromance@posta.unizar.es

Version: March 18, 2002

We show that there are close relations between extremal problems in dual Brunn-Minkowski theory and isotropic type properties for some Borel measures on the sphere. We also study reverse inequalities for dual mixed volumes which are related with classical positions, like maximal volume position, ℓ-position or isotropic position.

1. INTRODUCTION AND NOTATION

An *isotropic measure* μ in $\mathbb{R}^n$ is a positive, finite Borel measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} x_i x_j d\mu(x) = L^2 \delta_{ij}$$

for all $1 \leq i, j \leq n$, where L is a constant.

If K is a convex body in $\mathbb{R}^n$ we shall say that K is in *isotropic position* if its centroid is the origin and the measure $\mu = \chi_K(x)dx$ is isotropic, i.e. for some constant L

$$\int_K \langle x, \theta \rangle^2 dx = L^2$$

[1]Partially supported by CICYT Grant (Spain)
[2]Partially supported by CICYT and FPU Grants (Spain)

for all $\theta \in S^{n-1}$. Given a convex body K in $\mathbb{R}^n$ we consider the family of its *positions* $\{t + TK; t \in \mathbb{R}^n, T \in SL(n)\}$, where $SL(n)$ denotes the family of $n \times n$ real matrices with determinant equal to ± 1. It is well known that any convex body has a unique (up to orthogonal transformations) isotropic position.

A. Giannopoulos and V. Milman associated isotropic measures to extremal positions of convex bodies in $\mathbb{R}^n$ as a tool to discuss an isometric approach to the study of the different positions for convex bodies, which have been introduced in the local theory of Banach spaces (see [GM1] and [GM2]).

For instance the isotropic position of a convex body, defined before, is the solution of the extremal problem $\min\left\{\int_K |Tx+t|^2\, dx; T \in SL(n), t \in \mathbb{R}^n\right\}$, (see [MP] for the symmetric case and [Dar] for the non symmetric one).

In the same way, the euclidean ball D_n is the ellipsoid of maximal volume among all ellipsoids contained in a symmetric convex body K (*John's ellipsoid*) if and only if the identity I_n is the solution of the extremal problem

$$\min\{\|T : \ell_2^n \to X_K\|\}$$

and this situation is characterized by the existence of an isotropic measure μ supported on the contact points of K and D_n (X_K represents the normed space $\mathbb{R}^n$ endowed with the gauge of K and D_n, the euclidean ball). In [GPT] the authors characterize the extremal volume position between two convex bodies in terms of decompositions of the identity (see [BR] for a non-convex case). These decompositions of the identity can also be understood as the existence of some *generalized isotropic measures* supported on the contact pairs, which emphasizes the close relation between extremal problems of convex bodies and measures with isotropic type properties.

The minimal surface position is another example of this phenomenon. If K is a convex body, we denote by $\partial(K)$ its surface area. Then K is in minimal surface area position (i.e. $\partial(K) \leq \partial(TK)$, for all $T \in SL(n)$) if and only if the area measure σ_K of K is isotropic. Recall that σ_K is the measure supported on S^{n-1} defined on each borelian A in S^{n-1} as the measure of the set of points in the boundary of K whose outer normal is in A (see [P], [GP]).

The mean width of a convex body $\omega(K)$ is defined by

$$\omega(K) = 2\int_{S^{n-1}} h_K(u)d\sigma(u),$$

where h_K is the support function and $d\sigma$ is the Lebesgue measure on S^{n-1}. In [GM1] the authors also show that a *"smooth enough"* convex body K (that is, h_K is twice continuously differentiable) is in minimal mean width position if and only if the measure $h_K(u)d\sigma(u)$, supported on the unit sphere S^{n-1}, is isotropic.

These last two cases are also the extreme cases which respectively minimize the quermassintegrals of K, $W_i(K)$ for the values $i = n-1$ and $i = 1$. In the same paper [GM1], the authors also consider the remainder cases and they obtain necessary conditions for minimizing the corresponding $W_i(K)$.

The main goal of this paper is to extend these ideas of A. Giannopoulos and V. Milman and to show that a similar situation occurs when we consider the dual mixed volumes,

$\tilde{V}_i(K,L)$, or dual quermassintegrals, $\tilde{W}_i(K)$, introduced by E. Lutwak (see [L1], [L3] and the references therein). If $K \subseteq \mathbb{R}^n$ is a star shaped body and $i \in \mathbb{R}^n$, we consider the dual quermassintegral $\tilde{W}_i(K)$ defined by

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u)\, d\sigma(u).$$

We will consider the following extremal problems:

$$\max\left\{\tilde{W}_i(TK),\ \ T \in SL(n)\right\}, \qquad \text{if } \ i \in (0,n),$$
$$\min\left\{\tilde{W}_i(TK),\ \ T \in SL(n)\right\}, \qquad \text{if } \ i \notin [0,n].$$

In section 2 we study the position of a convex body which is solution of these extremal problems and we show that there is a close relation between these *extremal* positions and properties of isotropic type of some measures. In fact, we prove that if $i \in (-\infty,0) \cap [n+1,+\infty)$, the isotropy of some Borel measures on S^{n-1} is necessary and sufficient for a convex body K (symmetric when $i \geq n+1$) to be in position that minimizes $\tilde{W}_i(TK)$. If $i \in (0,n)$ the phenomenon is not so clear and, in general, we can only ensure that the isotropy of some measures is a necessary condition for a convex body K to optimize $\tilde{W}_i(\cdot)$, while we don't know if this condition is also sufficient. In subsection 2.1, we use Fourier Analysis and Special Functions techniques in order to prove that the isotropic conditions of some measures are also sufficient to ensure that K optimizes $\tilde{W}_i(TK)$ for some particular examples of convex bodies K in the plane.

It is well known that inequalities like Brunn-Minkowski or even its most important consequence the isoperimetric inequality cannot be reversed, as simple examples show. However V. Milman in the very remarkable paper [M] proved that we can reverse Brunn-Minkowski inequality, up to an absolute constant, if we consider different positions for the convex bodies, i.e., there exist positions which are now called M-positions which allow to reverse the inequality of Brunn-Minkowski (see [Pi] for another approach to the problem using interpolation of operators and [BBP] for its extension to the non convex case).

In the same spirit, K. Ball (see [B]) proved that among all the positions of a convex body, John's position lead us to reverse the isoperimetric inequality and so, there is one position of a convex body for which the surface area is less or equal to the one of a cube (in the symmetric case) or a simplex (in the non symmetric one) with the same volume. This situation can be seen as a *reverse inequality* for a particular case of Minkowski's classical inequalities.

In section 3 we consider the same problem for dual quermassintegrals, $\tilde{W}_i(K)$, and we study the corresponding *reverse dual* Minkowski inequalities for them. Apart from the interest of this reverse inequalities as a natural complement of the dual mixed volumes theory on its own, it is also interesting that in these reverse inequalities we come across with the classical hyperplane conjecture. Moreover, theorem 3.1 will allow us to reformulate the hyperplane conjecture in terms of reverse Minkowski inequalities for $-\infty < i < 1$. In this section we also study reverse inequalities for other indexes $i \in (1,+\infty)$ and we find out that they are related with different classical positions of convex bodies such that maximal volume positions or ℓ-positions. In the range $1 \leq i < n$, we can say something else for the balls B_p^n, $1 \leq p \leq \infty$, by using the estimates given in [SZ].

Eventually, section 4 is dedicated to further remarks and other considerations and we prove there that some measures of isotropic type that appear in section 2 characterize the solutions of slight modifications of the original extremal problems and we also show another examples where there is a close relation between the solution of extremal problems involving dual mixed volumes and properties of *isotropic type* for some Borel measures on the sphere.

Next we are going to introduce some notation. As usual we denote by $\|x\|_p = \left(\sum_1^n |x_i|^p\right)^{1/p}$, for $x = (x_1 \ldots, x_n) \in \mathbb{R}^n$ and $0 < p < \infty$. D_n denotes the euclidean unit ball, i.e. $\{x \in \mathbb{R}^n ; \|x\|_2 \leq 1\}$. B_p^n is the unit ball of the norm $\|\cdot\|_p$, so $B_2^n = D_n$. If $A \subseteq \mathbb{R}^n$, $|A|$ will represent the n-dimensional Lebesgue measure in $\mathbb{R}^n$. Notice that $|\cdot|$ may also represent the absolute value of a real number and the euclidean norm of a vector, i.e. $\|\cdot\|_2$, since the context will avoid any confusion.

A set $K \subseteq \mathbb{R}^n$ is star shaped at 0 if $\lambda x \in K$, whenever $0 \leq \lambda \leq 1$ and $x \in K$. If K is nonempty, compact and star shaped at 0, its *radial function* ρ_K is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. This function is homogeneous of degree -1. We say that a body (compact with nonempty interior) K is a *star body at* 0 if it is star shaped at 0, 0 belongs to the interior of K and the restriction of its radial function ρ_K is continuous on the sphere S^{n-1}. Every convex body with 0 in its interior is a star body at 0. In this case

$$\rho_K(x) = \frac{1}{\|x\|_K} = \frac{1}{h_{K^\circ(x)}}$$

where $\|\cdot\|_K$ is the gauge of K, K° denotes the polar set of K and h_{K° its support function (all these notions can also be seen in [Sc]).

We recall the definition of the *isotropy constant* L_K of a convex body K

$$nL_K^2|K|^{2/n} = \inf_{\substack{T \in SL(n) \\ t \in \mathbb{R}^n}} \frac{1}{|K|} \int_K |t + Tx|^2 \, dx.$$

Eventually, throughout this paper, unless otherwise stated, we will use C to denote a positive absolute constant, which can assume different values in different occurrences.

2. EXTREMAL POSITIONS FOR DUAL MIXED VOLUMES

The Brunn-Minkowski theory becomes the natural framework to work with shadows (projections) of convex bodies and when the data concern sections through a fixed point the dual Brunn-Minkowski theory provides a natural setting. In 1975, E. Lutwak (see [L1], [L3] and the references therein) introduced the concept of *dual mixed volumes.* If $K_1, \ldots, K_n \subseteq \mathbb{R}^n$ are star bodies at 0, the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ is defined by

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) \, d\sigma(u),$$

where $\rho_{K_i}(\cdot)$ is the radial function of K_i and σ is the Lebesgue measure on S^{n-1}. General properties of dual mixed volumes can be also found in [Ga].

In the same way than the mixed volumes are a useful tool in order to compute $|\lambda_1 K_1 + \cdots + \lambda_N K_N|$, one can define another addition $\tilde{+}$ such that

$$\left|\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_N K_N\right| = \sum_{1 \le i_1, \ldots i_n \le N} \tilde{V}(K_{i_1}, \ldots, K_{i_n})\, \lambda_{i_1} \cdots \lambda_{i_n}.$$

This new addition $\tilde{+}$ is the *radial addition* (see [L2]) that verifies that for all $K_1, \ldots, K_N$ star bodies at 0 and $\lambda_1, \ldots, \lambda_N \ge 0$,

$$\rho_{\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_N K_N}(\cdot) = \lambda_1 \rho_{K_1}(\cdot) + \cdots + \lambda_N \rho_{K_N}(\cdot). \tag{2.1}$$

We should note that $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_N K_N$ is also a star body at 0. As with mixed volumes, we can use the notation $\tilde{V}(K_1, i_1; \ldots; K_m, i_m)$ to denote the value

$$\begin{aligned} \tilde{V}(K_1, i_1; \ldots; K_m, i_m) &= \tilde{V}(K_1, \overset{i_1)}{\ldots}, K_1, K_2, \ldots, K_m, \overset{i_m)}{\ldots}, K_m) \\ &= \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}^{i_1}(u) \cdots \rho_{K_m}^{i_m}(u)\, d\sigma(u) \end{aligned}$$

where $\sum i_j = n$.

It will be convenient to relax the condition on the numbers i_j, so, for any star bodies at 0 $K, L \subseteq \mathbb{R}^n$ and $i \in \mathbb{R}$ we can define

$$\tilde{V}_i(K, L) = \tilde{V}(K, n-i; L, i) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \rho_L^i(u)\, d\sigma(u). \tag{2.2}$$

By changing variables, it is clear that

$$\tilde{V}_i(K, L) = \tilde{V}_{n-i}(L, K) = |\det T|^{-1} \tilde{V}_i(TK, TL) \tag{2.3}$$

for all $T \in GL(n)$.

It seems that great part of the results stated in Brunn-Minkowski theory, has its analogous in this dual Brunn-Minkowski theory. A clear example of this is the Minkowski inequality. A simple use of Hölder inequality implies that,

$$\tilde{V}_i(K, L) \le \ |K|^{\frac{n-i}{n}} |L|^{\frac{i}{n}} \qquad \text{if } i \in [0, n], \tag{2.4}$$

$$\tilde{V}_i(K, L) \ge \ |K|^{\frac{n-i}{n}} |L|^{\frac{i}{n}} \qquad \text{if } i \notin (0, n), \tag{2.5}$$

which can be understood as a *dual* of the well-known result of Minkowski. These inequalities make us wonder when

$$\tilde{V}_i(K, L) = \max \ \left\{ \tilde{V}_i(TK, L); T \in SL(n) \right\} \qquad \text{if } i \in [0, n], \tag{2.6}$$

or

$$\tilde{V}_i(K, L) = \min \ \left\{ \tilde{V}_i(TK, L); T \in SL(n) \right\} \qquad \text{if } i \notin (0, n). \tag{2.7}$$

We should note that the origin takes now an important role. This theory is not translation invariant, so we should only consider linear positions of convex bodies, i.e. $\{TK; T \in SL(n)\}$.

In this section we are going to study necessary and sufficient conditions for K and L to fulfill the extremal problems stated in (2.6) and (2.7) and we are going to show that the necessary and sufficient conditions for K and L to be solution of these extremal problems are related with the existence of measures with *"isotropic" type properties*, extending the ideas of A. Giannopoulos and V. Milman in [GM1].

Our first result states that the *"isotropy"* of some measures are necessary condition for K and L to be a solution of extremal problem stated in (2.6) or (2.7).

PROPOSITION 2.1. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies having 0 in their interior such that K° and L° are "smooth enough" (that is, h_{K° and h_{L° are twice continuously differentiable). Let $i \neq 0$ and $i \neq n$. Then either*

$$\tilde{V}_i(K,L) = \max\left\{\tilde{V}_i(TK,L); T \in SL(n)\right\}$$

for $i \in (0,n)$ or

$$\tilde{V}_i(K,L) = \min\left\{\tilde{V}_i(TK,L); T \in SL(n)\right\}$$

for $i \in (-\infty,0) \cup (n,\infty)$ imply that

$$\begin{aligned} \frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle \, d\sigma(u) \qquad (2.8)\\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u). \qquad (2.9)\end{aligned}$$

for all $T \in GL(n)$.

Proof. We only prove the case $i \in (0,n)$ since the other case is similar. It is also clear (see (2.3)) that $\tilde{V}_i(TK,L) = \tilde{V}_{n-i}(T^{-1}L,K)$, for all $T \in SL(n)$, so we only need establish (2.8).

If we take $T \in GL(n)$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we can define

$$S_\varepsilon = \frac{I + \varepsilon T}{|\det(I+\varepsilon T)|^{1/n}}.$$

By hypothesis, $\tilde{V}_i(K,L) \geq \tilde{V}_i(S_\varepsilon^{-1}K, L) = \tilde{V}_i(K, S_\varepsilon L)$, that is

$$\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\, d\sigma(u) \geq \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_{S_\varepsilon L}^i(u)\, d\sigma(u),$$

but since $\rho_{\phi(L)}(u) = \rho_L(\phi^{-1}u)$,

$$|\det(I+\varepsilon T)|^{i/n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\, d\sigma(u) \geq \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i((I+\varepsilon T)^{-1}u)\, d\sigma(u).$$

It's easy to prove that if $\|\varepsilon T\| < 1$

$$\begin{aligned} |\det(I+\varepsilon T)|^{i/n} &= 1 + \frac{i\varepsilon(\operatorname{tr} T)}{n} + O(\varepsilon^2), \\ (I+\varepsilon T)^{-1}(u) &= u - \varepsilon T u + O(\varepsilon^2), \end{aligned}$$

and

$$\begin{aligned} \rho_L^i(u - \varepsilon T u + O(\varepsilon^2)) &= \left(\frac{1}{h_{L^\circ}(u) - \varepsilon \langle \nabla h_{L^\circ}(u), Tu\rangle + O(\varepsilon^2)} \right)^i \\ &= \frac{1}{h_{L^\circ}^{i+1}(u)} \left(h_{L^\circ}(u) + i\varepsilon \langle \nabla h_{L^\circ}(u), Tu\rangle + O(\varepsilon^2) \right) \end{aligned}$$

when $\varepsilon \to 0^+$. Hence,

$$\begin{aligned} &\left(1 + \frac{i\varepsilon(\operatorname{tr} T)}{n} + O(\varepsilon^2)\right) \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^i(u)\, d\sigma(u) \\ &\geq \int_{S^{n-1}} \frac{\rho_K^{n-i}(u)\, d\sigma(u)}{\left(h_{L^\circ}(u) - \varepsilon\langle \nabla h_{L^\circ}(u), Tu\rangle + O(\varepsilon^2)\right)^i} \\ &= n\tilde{V}_i(K,L) + i\varepsilon \int_{S^{n-1}} \rho_K^{n-i}(u) \frac{\langle \nabla h_{L^\circ}(u), Tu\rangle}{h_{L^\circ}^{i+1}(u)}\, d\sigma(u) + O(\varepsilon^2). \end{aligned}$$

Then, if $\varepsilon \to 0^+$

$$\begin{aligned} \frac{\operatorname{tr} T}{n} \tilde{V}_i(K,L) &\geq \int_{S^{n-1}} \rho_K^{n-i}(u) \frac{1}{h_{L^\circ}^{i+1}(u)} \langle \nabla h_{L^\circ}(u), Tu\rangle\, d\sigma(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u) \langle \nabla h_{L^\circ}(u), Tu\rangle\, d\sigma(u). \end{aligned}$$

But if we replace T by $-T$ we conclude that

$$\frac{\operatorname{tr} T}{n} \tilde{V}_i(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u) \langle \nabla h_{L^\circ}(u), Tu\rangle\, d\sigma(u).$$

∎

Note that conditions (2.8) and (2.9) can be understood as *non commutative isotropic conditions* for the measures $\rho_K^{n-i}(\cdot)\rho_L^{i+1}(\cdot)\, d\sigma(\cdot)$ and $\rho_K^{n-i+1}(\cdot)\rho_L^{i}(\cdot)\, d\sigma(\cdot)$ respectively.

Next we are going to show that these necessary conditions appearing in the previous proposition 2.1 are also sufficient in some cases, but first of all we shall study relations between these two conditions (2.8) and (2.9). This is stated in the following result.

Proposition 2.2. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies having 0 in their interior and such that K° and L° are "smooth enough". The following assertions are equivalent:*

(i) For every $T \in GL(n)$ symmetric

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L).$$

(ii) For every $T \in GL(n)$ symmetric,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{V}_i(K,L).$$

Proof. It is enough to prove that the following assertions are equivalent:

(i) For every $\theta \in S^{n-1}$,

$$\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), \theta\rangle\langle u,\theta\rangle \, d\sigma(u) = \tilde{V}_i(K,L).$$

(ii) For every $\theta \in S^{n-1}$,

$$\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle \nabla h_{K^\circ}(u), \theta\rangle\langle u,\theta\rangle \, d\sigma(u) = \tilde{V}_i(K,L).$$

Take $\theta \in S^{n-1}$. We shall use the Laplace-Beltrami operator. If we define $F : \mathbb{R}^n\backslash\{0\} \longrightarrow [0,+\infty)$ given by

$$F(x) = \frac{\langle x,\theta\rangle^2}{2|x|^2} \qquad x \in \mathbb{R}^n\backslash\{0\},$$

it is easy to check that for every $u \in S^{n-1}$, $\nabla F(u) = \langle u,\theta\rangle\theta - \langle u,\theta\rangle^2 u$ and $\Delta F(u) = 1 - n\langle u,\theta\rangle^2$.

On the other hand, we define $H : \mathbb{R}^n\backslash\{0\} \longrightarrow [0,+\infty)$ by

$$H(x) = h_{K^\circ}(\frac{x}{|x|})^{i-n} h_{L^\circ}(\frac{x}{|x|})^{-i} \qquad x \in \mathbb{R}^n\backslash\{0\}.$$

Since the support function are 1-homogeneous, it can be proved that for every $u \in S^{n-1}$,

$$\begin{aligned}\nabla H(u) &= (i-n)h_{L^\circ}(u)^{-i}h_{K^\circ}(u)^{i-n-1}\left(\nabla h_{K^\circ}(u) - h_{K^\circ}(u)u\right)\\ &\quad -ih_{L^\circ}(u)^{-i-1}h_{K^\circ}(u)^{i-n}\left(\nabla h_{L^\circ}(u) - h_{L^\circ}(u)u\right).\end{aligned}$$

Now, if we integrate on the sphere and we use the Green's formula for Beltrami operator (see for instance [Gr], pp. 7), we get that

$$\int_{S^{n-1}} H(u)\Delta F(u)\, d\sigma(u) = -\int_{S^{n-1}} \langle \nabla F(u), \nabla H(u)\rangle \, d\sigma(u).$$

Hence we deduce

$$(n-i)\int_{S^{n-1}}\rho_K^{n-i+1}(u)\rho_L^i(u)\langle\nabla h_{K^\circ}(u),\theta\rangle\langle u,\theta\rangle\,d\sigma(u)$$
$$=n\tilde{V}_i(K,L)-i\int_{S^{n-1}}\rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle\nabla h_{L^\circ}(u),\theta\rangle\langle u,\theta\rangle\,d\sigma(u),$$

for all $\theta\in S^{n-1}$ which completes the proof. ∎

We don't know if this result is true for general matrices. We can achieve a complete characterization only in special cases, especially when one of the bodies is the euclidean ball. We also remark that, if $L=D_n$, the condition *(i)* in the last theorem means that the measure $\rho_K^{n-i}(\cdot)\,d\sigma(\cdot)$ is isotropic.

Now we are going to study if the assertions (2.8) or (2.9) are sufficient to ensure that K solves the extremal problem (2.6) or (2.7).

PROPOSITION 2.3. *Let $K,L\subseteq\mathbb{R}^n$ be convex bodies having 0 in their interior and such that K° and L° are "smooth enough". If $i\le -1$ and L is 0-symmetric, then the following assertions are equivalent:*

(i) $\tilde{V}_i(K,L)=\min\left\{\tilde{V}_i(TK,L)\right\}$, *when the minimum runs over all symmetric, positive definite matrices $T\in SL(n)$.*

(ii) For every T symmetric, positive definite matrix in $GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle\nabla h_{L^\circ}(u),Tu\rangle\,d\sigma(u)=\frac{\operatorname{tr}T}{n}\tilde{V}_i(K,L).$$

(iii) K is the unique symmetric positive definite position such that

$$\tilde{V}_i(K,L)=\min\left\{\tilde{V}_i(TK,L);\ T\ \textit{symmetric positive definite}\right\}$$

Furthermore, if $i=-1$ the results holds without any symmetry assumptions on L.

Proof. *(i)⇒(ii)* can be proved by using the same ideas than in proposition 2.1.

(ii)⇒(iii). We shall assume $i<-1$. If we take $T\in SL(n)$, by using (2.3)

$$\begin{aligned}\tilde{V}_i(TK,L) &= \tilde{V}_i(K,T^{-1}L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\rho_{T^{-1}L}^i(u)\,d\sigma(u)\\ &= \frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)h_{T^\star(L^\circ)}^{-i}(u)\,d\sigma(u).\end{aligned}$$

By using Hölder inequality with respect to the measure $\frac{1}{n}\rho_K^{n-i}(\cdot)d\sigma(\cdot)$ we get

$$\begin{aligned}\tilde{V}_i(TK,L) &\ge \left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\rho_L^i(u)d\sigma(u)\right)^{i+1}\\ &\quad\left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\rho_L^{i+1}(u)h_{T^\star(L^\circ)}(u)d\sigma(u)\right)^{-i}.\end{aligned}$$

Since $\langle \nabla h_{L^\circ}(u), Tu\rangle \le h_{T^\star(L^\circ)}(u)$ for all $u \in S^{n-1}$ (see [Sc], pp. 40) and the symmetry of L implies that also

$$|\langle \nabla h_{L^\circ}(u), Tu\rangle| \le h_{T^\star(L^\circ)}(u)$$

we have

$$\begin{aligned}\tilde V_i(TK,L) &\ge \left(\tilde V_i(K,L)\right)^{i+1}\left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle \nabla h_{L^\circ}(u), Tu\rangle\, d\sigma(u)\right)^{-i}\\ &= \left(\tilde V_i(K,L)\right)^{i+1}\left(\frac{\operatorname{tr} T}{n}\tilde V_i(K,L)\right)^{-i}\\ &\ge (\det T)^{-i/n}\,\tilde V_i(K,L) = \tilde V_i(K,L),\end{aligned}$$

so we obtain the result for $i < -1$.

The uniqueness is consequence of the fact that for symmetric definite matrices

$$\frac{\operatorname{tr} T}{n} = (\det T)^{1/n}$$

if and only if T is the identity.

The case $i = -1$ can be proved by analogous methods and we don't need any symmetry property on L. ∎

Remark 1. Since $\tilde V_i(K,L) = \tilde V_{n-i}(L,K)$, by using the last result we can state a similar proposition for $\tilde V_i(K,L)$, with K 0-symmetric and $i \ge n+1$, but now the isotropic type condition is that for every $T \in GL(n)$ symmetric and positive definite

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde V_i(K,L).$$

As we said before we can improve our results if one of the concerned bodies is the euclidean ball D_n. In the sequel

$$\tilde W_i(TK) = \tilde V_i(TK, D_n) = \frac{1}{n}\int_{S^{n-1}} \frac{\rho_K^{n-i}(u)}{|Tu|^i}\, d\sigma(u), \tag{2.10}$$

where $T \in GL(n)$ and $K \subseteq \mathbb{R}^n$ is a convex body.

By using the symmetry properties on D_n, it is easy to check that we only have to consider $T \in SL(n)$ which are symmetric and positive definite in order to optimize the dual quermassintegrals. As an application of dual Minkowski inequalities (2.4), (2.5) and the next lemma we can ensure the existence of extremal positions for the dual quermassintegrals.

LEMMA 1. *Let K be a convex body with 0 in its interior, then*

$$\lim_{\substack{T\in SL(n)\\ \|T\|\to\infty}} \tilde W_i(TK) = \begin{cases} 0 & \text{if } i \in (0,n),\\ +\infty & \text{if } i \in (-\infty,0)\cup(n,\infty).\end{cases} \tag{2.11}$$

Proof. First of all we suppose $T \in SL(n)$ is diagonal, with diagonal elements $d_1, \dots d_n > 0$ such that $\prod_{j=1}^n d_j = 1$.

If $0 < i < n$, by using polar coordinates it is clear that

$$\begin{aligned}
\tilde{W}_i(TK) &= \frac{1}{n}\int_{S^{n-1}} \rho_{TK}^{n-i}(u)d\sigma(u) \\
&= \frac{n-i}{n}\int_K \frac{dx}{|Tx|^i} = \frac{n-i}{n}\int_K \frac{dx}{\left(\sum_{j=1}^n d_j^2 x_j^2\right)^{i/2}} \\
&\leq C(n,i)\int_K \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i}.
\end{aligned}$$

where $C(n,i)$ denotes a constant depending on n and i, which could vary from line to line. If we denote $B_\infty^n = Q_n = \{x \in \mathbb{R}^n; |x_i| \leq 1\}$, there exist $r, R > 0$ such that $rQ_n \subseteq K \subseteq RQ_n$. Therefore, if $d_1 = \max\{d_j : 1 \leq j \leq n\}$, by using Fubini's theorem

$$\begin{aligned}
\tilde{W}_i(TK) &\leq C(n,i)\int_{RQ_n} \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i} \\
&\leq C(n,i)R^{n-i}\int_{Q_n} \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i} \\
&= C(n,i)R^{n-i}\left(\int_0^1\int_0^{d_2}\cdots\int_0^{d_n} \frac{dy_1\dots dy_n}{\sum_{j=1}^n y_j^i}\right. \\
&\qquad\qquad \left. + \int_1^{d_1}\int_0^{d_2}\cdots\int_0^{d_n} \frac{dy_1\dots dy_n}{\sum_{j=1}^n y_j^i}\right).
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_0^1\int_0^{d_2}\cdots\int_0^{d_n} \frac{dy_1\dots dy_n}{\sum_{j=1}^n |y_j|^i} &\leq C(n)\int_0^1 \frac{dy_1}{|y_1|^{i/n}}\left(\prod_{j=2}^n \int_0^{d_j} \frac{dy_j}{|y_j|^{i/n}}\right) \\
&\leq C(n)\left(\prod_{j=2}^n d_j\right)^{\frac{n-i}{n}} = \frac{C(n)}{d_1^{\frac{n-i}{n}}} \longrightarrow 0,
\end{aligned}$$

when $\|T\| \to +\infty$. On the other hand, if $i \neq 1$

$$0 \leq \int_1^{d_1}\int_0^{d_2}\cdots\int_0^{d_n} \frac{dy_1\dots dy_n}{\sum_{j=1}^n |y_j|^i} \leq \left(\prod_{j=2}^n d_j\right)\int_1^{d_1} \frac{dy_1}{y_1^i} = C(i)\left(\frac{1}{d_1^i} - \frac{1}{d_1}\right) \longrightarrow 0,$$

when $\|T\| \to +\infty$ and if $i = 1$

$$0 \leq \int_1^{d_1}\int_0^{d_2}\cdots\int_0^{d_n} \frac{dy_1\dots dy_n}{\sum_{j=1}^n |y_j|} \leq \frac{1}{d_1}\int_1^{d_1} \frac{dy_1}{y_1} = C(i)\frac{\log d_1}{d_1} \longrightarrow 0,$$

when $\|T\| \to +\infty$, therefore $\tilde{W}_i(TK) \longrightarrow 0$, when $\|T\| \to +\infty$.

If $-\infty < i < 0$ the proof is almost the same, but the case $i > n$ is different. Following the same ideas than before we get that

$$\tilde{W}_i(TK) \geq \frac{C(n,i)}{R^{n-i}} \int_{Q_n^c} \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i}$$

where Q_n^c is the complementary set of Q_n. If $d_1 = \min\{d_j : 1 \leq j \leq n\}$, we have

$$\begin{aligned}
\tilde{W}_i(TK) &\geq C(n,i) \int_{Q_n^c} \frac{dx}{\sum_{j=1}^n d_j^i |x_j|^i} \\
&\geq C(n,i) \int_1^2 dx_1 \int_0^{d_1/d_2} \cdots \int_0^{d_1/d_n} \frac{dx_2 \dots dx_n}{d_1^i 2^i + \sum_{j=2}^n d_1^i} \\
&= C(n,i) \frac{1}{d_1^i\,(2^i + n - 1)} \prod_{j=2}^n \frac{d_1}{d_j} \\
&= C(n,i) d_1^{n-i} \longrightarrow \infty
\end{aligned}$$

when $\|T\| \to +\infty$.

If $T \in SL(n)$ a symmetric, positive definite matrix. There exist an orthogonal matrix $V \in O(n)$ and a diagonal matrix D with diagonal elements $d_1, \dots, d_n > 0$ such that $\prod_{j=1}^n d_j = 1$ and $T = V^* D V$. Henceforth if we assume $rD_n \subseteq K \subseteq RD_n$

$$\begin{aligned}
\tilde{W}_i(TK) &= \frac{1}{n} \int_{S^{n-1}} \frac{\rho_K^{n-i}(u)}{|Tu|^i} d\sigma(u) \\
&\simeq \int_{S^{n-1}} \frac{1}{|DVu|^i} d\sigma(u) \\
&\simeq \int_{S^{n-1}} \frac{1}{|Dv|^i} d\sigma(v)
\end{aligned}$$

where $A \simeq B$ means here that the quotient A/B is bounded from above and from below for constants depending only on n, i, R and r. Hence

$$\lim_{\substack{T \in SL(n) \\ \|T\| \to \infty}} \tilde{W}_i(TK) = \lim_{\substack{D \in SL(n) \\ D \text{ diagonal} \\ \|D\| \to \infty}} \tilde{W}_i(DK) = \begin{cases} 0 & \text{si } i \in (0,n), \\ +\infty & \text{si } i \in (-\infty, 0) \cup (n, \infty). \end{cases}$$

∎

The isotropy of some measure characterizes exactly when K optimizes the dual quermassintegrals in the range $i \in (-\infty, 0)$, as it is shown in the following result.

THEOREM 2.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body having 0 in its interior. Suppose that K° is "smooth enough". Let $i \in (-\infty, 0)$. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K) = \min\left\{\tilde{W}_i(TK); T \in SL(n)\right\}$.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iii) For every $T \in GL(n)$ *symmetric,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iv) The measure given by $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ *is isotropic in* S^{n-1}

(v) For every $\alpha > 0$ *and for every* $T \in SL(n)$

$$\int_K \left(\frac{|Tx|}{|x|}\right)^{\alpha} |x|^{-i}\, dx \geq \int_K |x|^{-i}\, dx.$$

(vi) There exists $\alpha_0 > 0$ *and for every* $T \in SL(n)$

$$\int_K \left(\frac{|Tx|}{|x|}\right)^{\alpha_0} |x|^{-i}\, dx \geq \int_K |x|^{-i}\, dx.$$

(vii) K *is the unique position, up to orthogonal transformation, such that minimizes* $\tilde{W}_i(TK)$*, that is, if* $T \in SL(n)$ *such that*

$$\tilde{W}_i(TK) = \min\left\{\tilde{W}_i(TK); T \in SL(n)\right\},$$

then $T \in O(n)$.

Proof. *(i)⇒(ii)* is consequence of proposition 2.1.

(ii)⇒(iii) is trivial.

(iii)⇒(iv) is deduced from proposition 2.2, since for $L = D_n$ the condition *(i)* in that theorem is just the isotropy of the measure $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$.

(iv)⇒(v) is consequence of the following lemma 2 applied to the measure $d\mu(\cdot) = \rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ and the use of polar coordinates.

(v)⇒(vi) is trivial.

(vi)⇒(vii) is also a consequence of the following lemma 2 applied to $d\mu(\cdot) = \rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ for $\alpha_0 = -i$ and the use of polar coordinates. ∎

LEMMA 2. *Let* μ *be a positive and finite Borel measure on* S^{n-1}*. The following assertions are equivalent:*

(i) μ *is isotropic on* S^{n-1}.

(ii) For every $T \in SL(n)$ and for every $\alpha > 0$

$$\int_{S^{n-1}} |Tu|^\alpha \, d\mu(u) \geq \int_{S^{n-1}} d\mu(u). \tag{2.12}$$

(iii) There exists $\alpha_0 > 0$ such that for every $T \in SL(n)$

$$\int_{S^{n-1}} |Tu|^{\alpha_0} \, d\mu(u) \geq \int_{S^{n-1}} d\mu(u). \tag{2.13}$$

Proof.

(i)⇒(ii) We first prove that (2.12) holds for every $T \in SL(n)$ diagonal, with diagonal elements $d_1, \dots d_n > 0$ such that $\prod d_i = 1$.

If $\alpha \in (0, 2]$, then $f(x) = x^{\alpha/2}$ is concave in $[0, +\infty)$ and since $\sum u_j^2 = 1$

$$\begin{aligned}
\int_{S^{n-1}} |T(u)|^\alpha \, d\mu(u) &= \int_{S^{n-1}} \left(\sum_{j=1}^n d_j^2 u_j^2 \right)^{\alpha/2} d\mu(u) \geq \int_{S^{n-1}} \sum_{j=1}^n d_j^\alpha u_j^2 \, d\mu(u) \\
&= \sum_{j=1}^n d_j^\alpha \int_{S^{n-1}} u_j^2 \, d\mu(u) = \sum_{j=1}^n d_j^\alpha \frac{1}{n} \int_{S^{n-1}} |u|^2 \, d\mu(u) \\
&\geq \left(\prod_{j=1}^n d_j^\alpha \right)^{1/n} \int_{S^{n-1}} |u|^2 \, d\mu(u) = \int_{S^{n-1}} d\mu(u).
\end{aligned}$$

If $\alpha \in (2, +\infty)$, let us consider $p = \frac{\alpha}{2} \in (1, +\infty)$ and if $\frac{1}{p} + \frac{1}{q} = 1$, by using Hölder's inequality we get that

$$\begin{aligned}
\int_{S^{n-1}} |T(u)|^2 \, d\mu(u) &= \int_{S^{n-1}} \sum_{j=1}^n d_j^2 u_j^2 \, d\mu(u) \\
&\leq \left(\int_{S^{n-1}} \left(\sum_{j=1}^n d_j^2 u_j^2 \right)^{\alpha/2} d\mu(u) \right)^{\frac{2}{\alpha}} \cdot \left(\int_{S^{n-1}} d\mu(u) \right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore ,

$$\int_{S^{n-1}} \left(\sum_{j=1}^n d_j^2 u_j^2 \right)^{\alpha/2} d\mu(u) \geq \left(\int_{S^{n-1}} \sum_{j=1}^n d_j^2 u_j^2 \, d\mu(u) \right)^p \left(\int_{S^{n-1}} d\mu(u) \right)^{\frac{-p}{q}}.$$

But, notice that

$$\begin{aligned}\int_{S^{n-1}}\sum_{j=1}^{n} d_j^2 u_j^2\, d\mu(u) &= \sum_{j=1}^{n} d_j^2\left(\int_{S^{n-1}} u_j^2\, d\mu(u)\right) = \sum_{j=1}^{n} d_j^2\left(\frac{1}{n}\int_{S^{n-1}} d\mu(u)\right)\\ &\geq \left(\prod_{j=1}^{n} d_j^2\right)^{\frac{1}{n}}\int_{S^{n-1}} d\mu(u) = \int_{S^{n-1}} d\mu(u).\end{aligned}$$

So

$$\int_{S^{n-1}} |T(u)|^\alpha\, d\mu(u) \geq \left(\int_{S^{n-1}} d\mu(u)\right)^{p-p/q} = \int_{S^{n-1}} d\mu(u).$$

Now, if $T \in SL(n)$, there exist orthogonal matrices $V, W \in O(n)$ and diagonal matrix D with diagonal elements $d_1, \ldots, d_n > 0$ such that $\prod d_j = 1$ and $T = WDV$ (in this case we cannot restrict to symmetric, positive definite matrices). Then,

$$\begin{aligned}\int_{S^{n-1}} |T(u)|^\alpha\, d\mu(u) &= \int_{S^{n-1}} |WDV(u)|^\alpha\, d\mu(u) = \int_{S^{n-1}} |DV(u)|^\alpha\, d\mu(u)\\ &= \int_{S^{n-1}} |D(u)|^\alpha\, dV(\mu)(u),\end{aligned}$$

where $V(\mu)$ denotes the *image measure* of μ by V. It is easy to check that if μ is a Borel isotropic measure in S^{n-1}, then for every orthogonal transformation $V \in O(n)$, $V(\mu)$ is also a Borel isotropic measure in S^{n-1} and $\mu(S^{n-1}) = V(\mu)(S^{n-1})$. Hence,

$$\begin{aligned}\int_{S^{n-1}} |T(u)|^\alpha\, d\mu(u) &= \int_{S^{n-1}} |D(u)|^\alpha\, dV(\mu)(u)\\ &\geq \int_{S^{n-1}} dV(\mu)(u) = \int_{S^{n-1}} d\mu(u).\end{aligned}$$

(ii)⇒*(iii)* is trivial.

In order to prove *(iii)*⇒*(i)*, it is enough to show that for every $T \in GL(n)$

$$\int_{S^{n-1}} \langle Tu, u\rangle\, d\mu(u) = \frac{\operatorname{tr} T}{n}\int_{S^{n-1}} d\mu(u). \tag{2.14}$$

If we take $T \in GL(n)$, we consider for every $0 < \varepsilon < \varepsilon_0$

$$S_\varepsilon = \frac{I + \varepsilon T}{|\det(I + \varepsilon T)|^{1/n}}.$$

It can be shown that by using the variational technique stated in proposition 2.1, we obtain (2.14) . ∎

Remark 2. If K has its centroid at 0 and $i = -2$, taking $\alpha = 2$, the last result ensures that $\rho_K^{n+2}(\cdot)\, d\sigma(\cdot)$ is an isotropic measure on S^{n-1} if and only if for every $T \in SL(n)$

$$\int_K |Tx|^2\, dx \geq \int_K |x|^2\, dx,$$

that is, K is in isotropic position. Notice that if $-2 \neq i \in (-\infty, 0)$, the isotropy of the measure $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ is equivalent to the fact that for every $T \in SL(n)$

$$\int_K |Tx|^{-i}\, dx \geq \int_K |x|^{-i}\, dx,$$

which can be understood as a "$(-i)$-*isotropic type position*" of K.

The results we gather for the range $i \in [n+1, \infty)$ are not so complete as the preceding ones and are consequence of remark 1.

Corollary 2.2. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body with 0 in its interior. Suppose that K° is "smooth enough". Let $i \in [n+1, \infty)$. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K) = \min\left\{\tilde{W}_i(TK); T \in SL(n)\right\}$.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iii) For every $T \in GL(n)$ symmetric,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iv) K is the unique position, up to orthogonal transformation, that minimizes $\tilde{W}_i(TK)$.

Furthermore any of the previous assertions implies

(v) The measure given by $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ is isotropic in S^{n-1}.

Proof. *(i)⇒(ii)* is consequence of proposition 2.1.
(ii)⇒(iii) and *(iv)⇒(i)* are trivial.
(iii)⇒(iv) is deduced from remark 1.
The fact that any of the assertions *(i)*, *(ii)*, *(iii)* or *(iv)* implies *(v)* is deducted from proposition 2.2. ∎

We want to point out that simply by using remark 1 we could obtain theorem 2.1, but only in the range $i \in (-\infty, -1]$. Eventually, We are going to state s result for other values of i.

Corollary 2.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body with 0 in its interior. Suppose that K° is "smooth enough". Let $i \in (0, n+1)$. Consider the following assertions:*

(i) $\tilde{W}_i(K) =$ optimizes the values of $\left\{\tilde{W}_i(TK); T \in SL(n)\right\}$.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iii) For every $T \in GL(n)$ symmetric,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iv) The measure given by $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ is isotropic in S^{n-1},

then (i)⇒(ii)⇒(iii)⇒(iv).

2.1. About necessary conditions in the plane

Following the philosophy of theorem 2.1 and corollary 2.2, it would be interesting to study, at least in the range $0 < j < n$, when the condition *(iv)* appearing in the last corollary 2.3 is also a sufficient conditions for a convex body K to verify that

$$\tilde{W}_j(K) = \max\left\{\tilde{W}_j(TK);\ T \in SL(n)\right\}.$$

We do not know how to solve this question in general and it seems to us a very difficult problem. If we center the very special case of the plane (i.e. $n = 2$) we can give an affirmative answer to this problem for some particular examples, by using trigonometric series and special functions methods, since the problem is related with potential functions.

If $n = 2$ and $K \subseteq \mathbb{R}^2$ is a convex body, the fact that $\tilde{W}_j(K)$ is extremal ($j \in (0,2)$) can be rewritten as follows: For every $a > 0$ and $\alpha \in [0, 2\pi]$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho_K^{2-j}(\theta)\, d\theta}{\left(a^2\cos^2(\theta+\alpha) + a^{-2}\sin^2(\theta+\alpha)\right)^{j/2}} \leq \frac{1}{2\pi}\int_0^{2\pi} \rho_K^{2-j}(\theta) d\theta. \tag{2.15}$$

Therefore it would be enough to prove that the isotropy of the measure $\rho_K^{2-j}(\theta)d\theta$ on $S^1 = \mathbb{T}$ guarantees that condition (2.15) holds for all $a > 0$ and $\alpha \in [0, 2\pi]$.

For general Borel measures on $\mathbb{T}$, the isotropy of a measure μ is not enough to ensure that condition (2.15) is verified. Indeed, if μ a finite positive Borel measure on $\mathbb{T}$, the isotropy condition is equivalent to the fact that the Fourier coefficients $\hat{\mu}(\pm 2) = 0$, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\pm 2i\theta} d\mu(\theta) = 0.$$

For example, if we consider the particular measure

$$\mu = \delta_0 + \delta_{\pi/2} + \delta_\pi + \delta_{3\pi/2},$$

where δ_0, $\delta_{\pi/2}$, δ_π and $\delta_{3\pi/2}$ are the corresponding Dirac mass, then

$$\hat{\mu}(\pm 2k) = \frac{1}{2\pi}\left(1 + (-1)^k\right)$$

and so, μ is isotropic. But on the other hand

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{d\mu(\theta)}{\left(a^2\cos^2\theta + a^{-2}\sin^2\theta\right)^{j/2}} &= \frac{1}{\pi}\left(a^{-j} + a^j\right) \\ &> \frac{1}{2\pi}\int_{\mathbb{T}} d\mu(\theta)\end{aligned}$$

if $a \neq 1$ and hence (2.15) is not satified for $a \neq 1$ and $\alpha = 0$.

Consequently, one could think that we only should consider absolutely continuous measures $d\mu(\theta) = f(\theta)d\theta$, but a straightforward approximation argument says that for general $C^{(\infty}$ positive functions f or even for measures of the form $d\mu(\theta) = \rho_L^{n-i}(\theta)d\theta$, with L a general star body, the result is not true, so we have to restrict ourselves to a very particular case of absolutely continuous measures on $\mathbb{T}$.

Let us introduce some notation. If $a > 0$ and $j \in (0, 2)$ we define $g_a(\theta)$ for every $\theta \in \mathbb{T}$ by

$$g_a(\theta) = \frac{1}{\left(a^2 \cos^2 \theta + a^{-2} \sin^2 \theta\right)^{j/2}}.$$

The following lemma let us study the inequality (2.15) in terms of Fourier coefficients.

Lemma 3. *Let g_a be defined as before. If we denote by*

$$\ell = \frac{a^2 - 1}{a^2 + 1} \in (-1, 1) \tag{2.16}$$

then

$$g_a(\theta) = \sum_{k=0}^{\infty} A_{2k} \cos(2k\theta)$$

where

$$A_0 = \left(1 - \ell^2\right)^{j/2} \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}^2 \tag{2.17}$$

$$A_{2k} = 2\ell^k \left(1 - \ell^2\right)^{j/2} \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m} \binom{-j/2}{m+k} \tag{2.18}$$

and the trigonometric series converges absolutely and uniformly in θ. Furthermore $A_0 < 1$ whenever $a \neq 1$ and

(i) $\{A_{2k}\}_{k=0}^{\infty}$ is a non increasing convergent to 0 sequence if $a < 1$,

(ii) $\{(-1)^k A_{2k}\}_{k=0}^{\infty}$ is a non increasing convergent to 0 sequence otherwise.

Proof. It is very easy to see that for every $\theta \in \mathbb{T}$

$$\begin{aligned}
\frac{1}{a^2 \cos^2 \theta + a^{-2} \sin^2 \theta} &= \frac{1 - \ell^2}{(1 + \ell)^2 \cos^2 \theta + (1 - \ell)^2 \sin^2 \theta} \\
&= \frac{1 - \ell^2}{|e^{i\theta} + \ell e^{-i\theta}|^2} \\
&= \frac{1 - \ell^2}{\left(1 + \ell e^{-2i\theta}\right)\left(1 + \ell e^{2i\theta}\right)}.
\end{aligned}$$

So for every $\theta \in \mathbb{T}$

$$\begin{aligned}
\left(1-\ell^2\right)^{-j/2} g_a(\theta) &= \sum_{n,m=0}^{\infty} \binom{-j/2}{n}\binom{-j/2}{m} \ell^{n+m} e^{2i\theta(n-m)} \\
&= \sum_{m=0}^{\infty}\sum_{k=-m}^{\infty} \binom{-j/2}{m}\binom{-j/2}{m+k} \ell^{2m+k} e^{2i\theta k} \\
&= \sum_{k=0}^{\infty} \ell^k e^{2i\theta k} \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}\binom{-j/2}{m+k} \\
&\quad + \sum_{k=-\infty}^{-1} \ell^k e^{2i\theta k} \sum_{m=-k}^{\infty} \ell^{2m} \binom{-j/2}{m}\binom{-j/2}{m+k} \\
&= \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}^2 \\
&\quad +2\sum_{k=1}^{\infty} \ell^k \cos(2k\theta) \sum_{m=0}^{\infty} \ell^{2m} \binom{-j/2}{m}\binom{-j/2}{m+k}
\end{aligned}$$

and we get (2.17) and (2.18).

Since $0 < j < 2$, we get that $\dfrac{\frac{j}{2}+m+k}{m+k+1} < 1$ and

$$\left|\binom{-j/2}{m}\binom{-j/2}{m+k}\right| = (-1)^{2m+k}\binom{-j/2}{m}\binom{-j/2}{m+k}.$$

Hence

$$\left|\binom{-j/2}{m}\binom{-j/2}{m+k}\right| \geq \left|\binom{-j/2}{m}\binom{-j/2}{m+k+1}\right|,$$

what implies the monotonic character stated in *(i)* and *(ii)*.

Eventually, since the function $h(t) = t^{j/2}$ is concave in $[0, +\infty)$, we get that whenever $a \neq 1$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} g_a(\theta)\, d\theta < \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a^j \cos^2\theta + a^{-j}\sin^2\theta} = 1$$

∎

Observe that the A_{4k} are hipergeometric functions. Indeed, given $a, b, c \in \mathbb{R}$ and $z \in \mathbb{C}$ the *hypergeometric function* $F(a, b; c; z)$ is defined by

$$F(a,b;c;z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!},$$

where $(a)_m = a(a+1)\cdots(a+m-1)$ (see [AS]). It can be checked that

$$\begin{aligned}
A_0 &= (1-\ell^2)^{j/2} F(\frac{j}{2}, \frac{j}{2}; 1; \ell^2), \\
A_{4k} &= 2\ell^{2k}(1-\ell^2)^{j/2}\binom{-j/2}{2k} F(\frac{j}{2}, \frac{j}{2}+2k; 2k+1; \ell^2),
\end{aligned}$$

for $k \geq 1$. In order to get some upper estimates for A_{4k} that will be useful later, we are going to give some upper estimates for some hypergeometric functions.

LEMMA 4. *Let $\alpha \in (0,1)$ and $k \in \mathbb{N} \cup \{0\}$. For every $x \in (-1,1)$*

$$\begin{aligned} F(\alpha,\alpha;1;x) &\leq e^{\alpha-1}(1-x)^{-\alpha} + (1-e^{\alpha-1}) + x(\alpha^2 - \alpha e^{\alpha-1}) \\ &\leq e^{\alpha-1}(1-x)^{-\alpha} + (1-e^{\alpha-1}), \end{aligned}$$

$$\begin{aligned} &F(\alpha,\alpha+2k;2k+1;x) \\ &\quad \leq \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}} \left[(-1)^k \binom{-\alpha}{k} e^{\frac{\alpha-1}{2}} - (1-x)^{\alpha} \left((-1)^k \binom{-\alpha}{k} e^{\frac{\alpha-1}{2}} - \binom{-\alpha}{2k} \right) \right] \\ &\quad \leq \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}} (-1)^k \binom{-\alpha}{k} e^{\frac{\alpha-1}{2}}. \end{aligned}$$

Proof. First of all, we study $F(\alpha,\alpha+2k;2k+1;x)$. If $m \geq 0$ and $k \geq 1$, on the one hand

$$0 \leq \frac{\alpha \cdots (\alpha+m-1)}{m!} = (-1)^m \binom{-\alpha}{m}.$$

On the other hand

$$\begin{aligned} \frac{(\alpha+2k)_m}{(2k+1)_m} &= \frac{(\alpha+2k)\cdots(\alpha+2k+m-1)}{(2k+1)\cdots(2k+m)} \\ &= \frac{(2k)!}{(\alpha)\cdots(\alpha+2k-1)} \frac{\alpha(\alpha+1)\cdots(\alpha+2k+m-1)}{(2k+m)!} \\ &= \binom{-\alpha}{2k}^{-1} \frac{\alpha\cdots(\alpha+k-1)}{k!} \frac{(\alpha+k)\cdots(\alpha+2k+m-1)}{(k+1)\cdots(2k+m)} \\ &= (-1)^k \binom{-\alpha}{k} \binom{-\alpha}{2k}^{-1} \prod_{n=k+1}^{2k+m} \frac{\alpha+n-1}{n}. \end{aligned}$$

Hence, since $\alpha < 1$, we get that

$$\begin{aligned} \frac{(\alpha+2k)_m}{(2k+1)_m} &\leq (-1)^k \binom{-\alpha}{k} \binom{-\alpha}{2k}^{-1} \left(\frac{\alpha+2k+m-1}{2k+m} \right)^{m+k} \\ &\leq (-1)^k \binom{-\alpha}{k} \binom{-\alpha}{2k}^{-1} e^{\frac{\alpha-1}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} F(\alpha,\alpha+2k;2k+1;x) &= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\alpha+2k)_m}{m!(2k+1)_m} x^m \\ &\leq (-1)^k \binom{-\alpha}{k} \binom{-\alpha}{2k}^{-1} e^{\frac{\alpha-1}{2}} \sum_{m=0}^{\infty} \binom{-\alpha}{m} (-x)^m \\ &= \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}} (-1)^k \binom{-\alpha}{k} e^{\frac{\alpha-1}{2}}. \end{aligned}$$

If we want to get a sharper estimate, by using the same idea we get that

$$
\begin{aligned}
F(\alpha, \alpha+2k; 2k+1; x) &= 1 + \sum_{m=1}^{\infty} \frac{(\alpha)_m (\alpha+2k)_m}{m!(2k+1)_m} x^m \\
&\le 1 + (-1)^k \binom{-\alpha}{k} \binom{-\alpha}{2k}^{-1} e^{\frac{\alpha-1}{2}} \sum_{m=1}^{\infty} \binom{-\alpha}{m} (-x)^m \\
&= \frac{(1-x)^{-\alpha}}{\binom{-\alpha}{2k}} \left[(-1)^k \binom{-\alpha}{k} e^{\frac{\alpha-1}{2}} - (1-x)^{\alpha} \left((-1)^k \binom{-\alpha}{k} e^{\frac{\alpha-1}{2}} - \binom{-\alpha}{2k} \right) \right].
\end{aligned}
$$

In order to get upper estimates for $F(\alpha, \alpha; 1; x)$, we compute $\dfrac{(\alpha)_m^2}{(1)_m m!}$. If $m \ge 0$

$$
\begin{aligned}
\frac{(\alpha)_m^2}{(1)_m m!} &= (-1)^m \binom{\alpha}{m} \prod_{n=1}^{m} \left(\frac{\alpha+n-1}{n} \right) \\
&\le (-1)^m \binom{\alpha}{m} \left(\frac{\alpha+m-1}{m} \right)^m = (-1)^m \binom{\alpha}{m} e^{\alpha-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
F(\alpha, \alpha; 1; x) &\le 1 + e^{\alpha-1} \sum_{m=1}^{\infty} \binom{-\alpha}{m} (-x)^m \\
&= e^{\alpha-1}(1-x)^{-\alpha} + (1 - e^{\alpha-1}).
\end{aligned}
$$

The other estimate for $F(\alpha, \alpha; 1; x)$ can be obtained by the same technique. ∎

Corollary 2.4. *We have*

$$
\begin{aligned}
A_0 &\le e^{j/2-1} + \left(1-\ell^2\right)^{j/2} \left[1 - e^{j/2-1} - l^2 \frac{j}{2} \left(e^{j/2-1} - \frac{j}{2} \right) \right] \\
&\le e^{j/2-1} + \left(1-\ell^2\right)^{j/2} \left(1 - e^{j/2-1} \right) \\
A_4 &\le 2\ell^2 \left[\frac{j}{2} e^{\frac{j-2}{4}} - (1-\ell^2)^{j/2} \left(\frac{j}{2} e^{\frac{j-2}{4}} - \binom{-j/2}{2} \right) \right] \\
&\le 2\ell^2 \frac{j}{2} e^{\frac{j-2}{4}} \\
A_{4k} &\le 2 e^{\frac{j-2}{4}} \ell^{2k} (-1)^k \binom{-j/2}{k}
\end{aligned}
$$

for $k \ge 1$.

If we want to prove (2.15) for some $K \subseteq \mathbb{R}^2$, by using lemma 3 it would be enough to show that

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho_K^{2-j}(\theta)\,d\theta}{\big(a^2\cos^2(\theta+\alpha)+a^{-2}\sin^2(\theta+\alpha)\big)^{j/2}}$$

$$= \sum_{k=0}^{\infty} A_{2k}\frac{1}{2\pi}\int_0^{2\pi}\rho_K^{2-j}(\theta)\cos 2k(\theta+\alpha)\,d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi}\rho_K^{2-j}(\theta)\,d\theta.$$

In the sequel, let f be a positive continuous function on $\mathbb{T}$. We will suppose f has the following symmetry conditions $f(\theta)=f(2\pi-\theta)=f(\pi-\theta)=f(\frac{\pi}{2}-\theta)$, for all $\theta\in[0,2\pi]$ and that f is smooth enough on the interval $[0,\frac{\pi}{4}]$, (for instance f is the $(2-j)$-power of the radial function of a smooth enough star body, symmetric with respect to the coordinate axes and to the bisectors of the quadrants). If we denote by

$$B_{2k}=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos(2k\theta)d\theta$$

for $k\ge 0$, it is quite easy to compute that $B_{2k}=0$ if k is odd and

$$B_{2k}=\frac{4}{\pi}\int_0^{\pi/4} f(\theta)\cos(2k\theta)d\theta=\frac{-2}{\pi k}\int_0^{\pi/4} f'(\theta)\sin(2k\theta)d\theta$$

otherwise. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)g_a(\theta+\alpha)d\theta=\sum_{k=0}^{\infty}A_{4k}B_{4k}\cos(4\alpha)$$

and the inequality we study (2.15) can be expressed as

$$\sum_{k=0}^{\infty}A_{4k}B_{4k}\cos(4\alpha)\le B_0 \tag{2.19}$$

for all $a\in(0,\infty)$ and $\alpha\in\mathbb{R}$ (recall that A_k's depend on a).

PROPOSITION 2.4. *Let f be as before a derivable, positive function with "enough symmetries". Suppose that f' is non positive and non decreasing on $[0,\frac{\pi}{4}]$ then $B_{4k}\ge 0$, for all $k\ge 0$.*

Proof. If $k\ge 1$, we have

$$\int_0^{\pi/4} f'(\theta)\sin(4k\theta)d\theta = \int_0^{\pi/4k} f'(\theta)\sin(4k\theta)d\theta+\int_{\pi/4k}^{2\pi/4k} f'(\theta)\sin(4k\theta)d\theta$$

$$+\cdots+\int_{(k-1)\pi/4k}^{k\pi/4k} f'(\theta)\sin(4k\theta)d\theta.$$

Since f' is non decreasing on $[0, \frac{\pi}{4}]$ we have

$$\begin{aligned}\int_0^{\pi/4k} f'(\theta)\sin(4k\theta)d\theta &+ \int_{\pi/4k}^{2\pi/4k} f'(\theta)\sin(4k\theta)d\theta \\ &= \int_0^{\pi/4k} \left(f'(\theta) - f'\left(\theta + \frac{\pi}{4k}\right)\right)\sin(4k\theta)d\theta \leq 0.\end{aligned}$$

In fact, by using the same idea, for every $i = 1, \ldots, [\frac{k}{2}]$ we get that

$$\int_{(2i-2)\pi/4k}^{(2i-1)\pi/4k} f'(\theta)\sin(4k\theta)d\theta + \int_{(2i-1)\pi/4k}^{2i\pi/4k} f'(\theta)\sin(4k\theta)d\theta \leq 0,$$

hence

$$\int_0^{\pi/4} f'(\theta)\sin(4k\theta)d\theta \leq 0,$$

if k is even. In other case, the last summand is also negative since $f' \leq 0$ and $\sin(4k\theta) \geq 0$ in that interval. Eventually, we get that $B_{4k} \geq 0$ for all $k \geq 1$ and therefore the result holds. ∎

Corollary 2.5. *The Fourier coefficients B_{4k} ($k \geq 0$), corresponding to the function*

$$f(\theta) = \rho_{B_1^2}(\theta)^{2-j} \qquad \theta \in \mathbb{T}$$

for $0 < j < 2$, verifies that $B_{4k} \geq 0$.

Proof. We only should note that

$$\rho_{B_1^2}(\theta)^{2-j} = \frac{1}{(|\cos\theta| + |\sin\theta|)^{2-j}}$$

for $\theta \in \mathbb{T}$, verifies the conditions appearing in the preceding proposition. ∎

Lemma 5. *Let f be as before a continuous positive function with "enough symmetries". Suppose that $F(\theta) = f(\frac{\pi}{4} - \theta)\cos(2\theta)$ satisfies $F'(0) = 0$ and the function $F''(\theta)$ is non positive and non increasing in $[0, \frac{\pi}{4}]$, then $B_{4k} \geq B_{4k+4}$, for all $k \geq 0$.*

Proof. By changing variables we get that

$$\begin{aligned}B_{4k} - B_{4k+4} &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\left[\cos(4k\theta) - \cos\left((4k+4)\theta\right)\right] d\theta \\ &= \frac{(-1)^k}{2\pi}\int_0^{2\pi} f\left(\frac{\pi}{4} - \theta\right)\left[\cos(4k\theta) + \cos\left((4k+4)\theta\right)\right] d\theta \\ &= \frac{(-1)^k}{\pi}\int_0^{2\pi} f\left(\frac{\pi}{4} - \theta\right)\cos\left((4k+2)\theta\right)\cos(2\theta)\, d\theta \\ &= (-1)^k\frac{8}{\pi}\int_0^{\pi/4} F(\theta)\cos\left((4k+2)\theta\right)\, d\theta.\end{aligned}$$

Now, by integrating twice by parts,

$$B_{4k} - B_{4k+4} = -\frac{8(-1)^k}{\pi(4k+2)^2} \int_0^{\pi/4} F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta$$

Let $k = 2m$ be an even number. Then

$$\begin{aligned} B_{4k} - B_{4k+4} &= \frac{8}{\pi(4k+2)^2} \left(\int_0^{\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \right. \\ &+ \int_{\pi/2(4k+2)}^{5\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \\ &+ \cdots \\ &+ \left. \int_{(4m-3)\pi/2(4k+2)}^{(4m+1)\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \right). \end{aligned}$$

Since $-F'' \geq 0$ and $-F''$ is non decreasing on $[0, \frac{\pi}{4}]$ we obtain that

$$\int_0^{\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \geq 0$$

and for every $i = 1, \ldots, m$

$$\int_{(4i-3)\pi/2(4k+2)}^{(4i+1)\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \geq 0$$

which ensures that $B_{4k} \geq B_{4k+4}$.

Let now $k = 2m+1$ be an odd number. As before,

$$\begin{aligned} B_{4k} - B_{4k+4} &= -\frac{8}{\pi(4k+2)^2} \left(\int_0^{3\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \right. \\ &+ \int_{3\pi/2(4k+2)}^{7\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \\ &+ \cdots \\ &+ \left. \int_{(4m-1)\pi/2(4k+2)}^{(4m+3)\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \right). \end{aligned}$$

By similar reasons as before we get that

$$\int_0^{3\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \leq 0$$

and for every $i = 1, \ldots, m$

$$\int_{(4i-1)\pi/2(4k+2)}^{(4i+3)\pi/2(4k+2)} -F''(\theta)\cos\left((4k+2)\theta\right)\, d\theta \leq 0$$

and so the conclusion of the lemma holds. ∎

If we take $f(\cdot)=\rho_{B^2_\infty}^{2-j}(\cdot)$ we obtain the following result.

COROLLARY 2.6. *Let* $j\in(0,x_0]$ *(*$x_0=-\frac{5}{3}+\sqrt{73}/3\simeq 1.18$*). Then the Fourier coefficients* B_{4k} *of* $f(\cdot)=\rho_{B^2_\infty}^{2-j}(\cdot)$ *verify that* $B_0\geq B_4\geq\cdots\geq B_{4k}\geq\cdots\geq 0$.

Hence, if we are in the conditions of preceding lemmas we have

$$\begin{aligned}\sum_{k=0}^{\infty}A_{4k}B_{4k} &\leq B_0\left(A_0+\frac{B_4}{B_0}\sum_{k=1}^{\infty}A_{4k}\right)\\ &\leq B_0\left(A_0+2e^{\frac{j-2}{4}}\frac{B_4}{B_0}\sum_{k=1}^{\infty}\ell^{2k}(-1)^k\binom{-j/2}{k}\right)\\ &= B_0\left(A_0+2e^{\frac{j-2}{4}}\frac{B_4}{B_0}\left((1-\ell^2)^{-j/2}-1\right)\right)\\ &\leq B_0\left[e^{j/2-1}-2e^{(j-2)/4}\frac{B_4}{B_0}+(1-\ell^2)^{j/2}\left(1-e^{j/2-1}\right)\right.\\ &\qquad\left.+(1-\ell^2)^{-j/2}2e^{(j-2)/4}\frac{B_4}{B_0}\right]. \qquad (2.20)\end{aligned}$$

Hence, if $\frac{B_4}{B_0}$ is *"small enough"*

$$\frac{1}{2\pi}\int_0^{2\pi}f(\theta)g_a(\theta+\alpha)\,d\theta\leq B_0=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta,$$

for all a *" not close to"* 0 or $+\infty$. For a sharper inequality we notice that

$$\begin{aligned}\sum_{k=0}^{\infty}A_{4k}B_{4k} &\leq B_0\left(A_0+\frac{B_4}{B_0}A_4+\frac{B_8}{B_0}\sum_{k=2}^{\infty}A_{4k}\right)\\ &\leq B_0\left(A_0+\frac{B_4}{B_0}A_4+2e^{\frac{j-2}{4}}\frac{B_8}{B_0}\sum_{k=2}^{\infty}\ell^{2k}(-1)^k\binom{-j/2}{k}\right)\\ &= B_0\left(A_0+\frac{B_4}{B_0}A_4+2e^{\frac{j-2}{4}}\frac{B_8}{B_0}\left((1-\ell^2)^{-j/2}-1-\frac{j}{2}\ell^2\right)\right)\end{aligned}$$

and then

$$\begin{aligned}\sum_{k=0}^{\infty}A_{4k}B_{4k} &\leq B_0\left[e^{\frac{j}{2}-1}-2e^{(j-2)/4}\frac{B_8}{B_0}+(1-\ell^2)^{\frac{j}{2}}\left(1-e^{\frac{j}{2}-1}\right)\right.\\ &\quad+\ell^2je^{(j-2)/4}\left(\frac{B_4}{B_0}-\frac{B_8}{B_0}\right)\\ &\quad-\ell^2(1-\ell^2)^{\frac{j}{2}}\frac{j}{2}\left(\left(e^{\frac{j}{2}-1}-\frac{j}{2}\right)+2\frac{B_4}{B_0}\left(e^{(j-2)/4}-\frac{j+2}{4}\right)\right)\\ &\quad\left.+(1-\ell^2)^{-\frac{j}{2}}2e^{(j-2)/4}\frac{B_8}{B_0}\right]. \qquad (2.21)\end{aligned}$$

If $\frac{B_4}{B_0}$ and $\frac{B_8}{B_0}$ are *"small enough"* and ℓ^2 *"not close to"* 1 ($B_8 \neq 0$) we get that

$$\sum_{k=0}^{\infty} A_{4k} B_{4k} \leq B_0.$$

In order to study the problem when a is close to 0 or $+\infty$ we study the behavior of

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta) g_a(\theta+\alpha)\, d\theta,$$

when $a \to +\infty$.

LEMMA 6. *Let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be as before, then for every $j \in [1, 2]$ and every $a \in (1, +\infty)$*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)\, d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} \leq \|f\|_\infty \frac{a^j \log(2a^2)}{\sqrt{a^4-1}}.$$

Proof. By using the symmetry of f we get that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)\, d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} \leq \frac{2}{\pi}\int_0^{\pi/2} \frac{\|f\|_\infty\, d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}}.$$

On the other hand, by changing variables

$$\begin{aligned}
\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} &= \frac{2a^j}{\pi}\int_0^{\pi/2} \frac{d\theta}{((a^4-1)\sin^2\theta + 1)^{j/2}} \\
&\leq \frac{2a^j}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{(a^4-1)\sin^2\theta + 1}} \\
&\leq \frac{2a^j}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\left(\frac{2\sqrt{a^4-1}}{\pi}\theta\right)^2 + 1}} \\
&= \frac{a^j}{\sqrt{a^4-1}} \operatorname{arcsinh}\left(\sqrt{a^4-1}\right),
\end{aligned}$$

but $\operatorname{arcsinh} x = \log\left(x + \sqrt{x^2+1}\right)$ and therefore

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \frac{f(\theta)\, d\theta}{(a^2\cos^2\theta + a^{-2}\sin^2\theta)^{j/2}} &\leq \frac{a^j\|f\|_\infty}{\sqrt{a^4-1}} \log(a^2 + \sqrt{a^4-1}) \\
&\leq \frac{a^j\|f\|_\infty}{\sqrt{a^4-1}} \log(2a^2).
\end{aligned}$$

∎

By combining the estimates (2.20), (2.21) and lemma 6, we get (2.19) for some f with *"enough symmetries"*. Let's see an example of this phenomenon.

EXAMPLE 1. For $j = 1$ and $f(\theta) = \rho_{B_1^2}(\theta)$ we have $B_0 = 0.793515\ldots$, $B_4 = 0.055311\ldots$ and $B_8 = 0.017445\ldots$, therefore

$$\begin{aligned}\sum_{k=0}^{\infty} A_{4k}B_{4k} \quad &\leq \quad B_0\Big((0.572287\ldots) + (0.393469\ldots)(1-\ell^2)^{1/2} \\ &+(0.037163\ldots)\ell^2 - \ell^2(1-\ell^2)^{1/2}(0.055272\ldots) \\ &+(1-\ell^2)^{-1/2}(0.034243\ldots)\Big) \leq (0.985776\ldots)B_0 < B_0.\end{aligned}$$

whenever $0 \leq \ell^2 \leq 0.995$, since this function is non increasing when $\ell^2 < 0.9767\ldots$ and then is non decreasing (the numerical computations have been performed with `Maple` processer). We achieve the conclusion since for the remainder values of ℓ the result is true due to lemma 6, since for all $\ell \in (0,1)$

$$\sum_{k=0}^{\infty} A_{4k}B_{4k} \leq \sqrt{\frac{1-\ell^2}{4\ell}}\log\left(\frac{2+2\ell}{1-\ell}\right),$$

and if $\ell \geq 0.94$

$$\sum_{k=0}^{\infty} A_{4k}B_{4k} \leq 0.733567\ldots < 0.793515\ldots = B_0,$$

therefore, we get that for every $a > 0$ and every $\alpha \in [0, 2\pi]$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho_{B_1^2}(\theta)\, d\theta}{\left(a^2\cos^2(\theta+\alpha) + a^{-2}\sin^2(\theta+\alpha)\right)^{1/2}} \leq \frac{1}{2\pi}\int_0^{2\pi} \rho_{B_1^2}(\theta) d\theta$$

and hence $\tilde{W}_1(B_1^2) = \min\left\{\tilde{W}_1(TB_1^2);\ T \in SL(2)\right\}$.

Remark 3. This technique could be also applied to other *"symmetric enough"* convex bodies K, simply by considering the estimates given in corollary 2.4 properly improved. If we want to use a similar method to obtain results in $\mathbb{R}^n$ $(n > 2)$, we should use spherical harmonics.

3. REVERSE ISOPERIMETRIC INEQUALITIES

In this section we study reverse inequalities for Minkowski dual inequalities associated to dual quermassintegrals. Let K be a convex body with 0 in its interior. According to (2.4) and (2.5) we have

$$\tilde{W}_i(K) \leq \quad |K|^{\frac{n-i}{n}}|D_n|^{\frac{i}{n}} \qquad \text{if } i \in [0,n], \tag{3.22}$$

$$\tilde{W}_i(K) \geq \quad |K|^{\frac{n-i}{n}}|D_n|^{\frac{i}{n}} \qquad \text{if } i \notin (0,n). \tag{3.23}$$

It is well known that we cannot reverse this inequalities since this would imply that K is homothetic to D_n. We want to reverse the inequalities by using different affine positions of K, as it was done by V. Milman and K. Ball in other situations (see introduction). This problem is closely related to that of the previous section. Indeed, we can define the function

$$\psi_{i,K}(t,T) = \tilde{W}_i(t+TK)$$

where $t \in \mathbb{R}^n$ and T varies on $SL(n)$ in such a way that 0 is in the interior of $t+TK$. Since $\psi_{i,K}(t,T)$ is bounded (see (3.22) or (3.23)) and it has a suitable behavior in the boundary of $SL(n)$ (cf. 2) we know that the function $\psi_{i,K}(t,T)$ attains its extreme value for fixed t. In section 2, we obtained necessary and/or sufficient conditions for a position to be extreme. What we shall do now is to estimate how close are the universal bounds given in (3.22) or(3.23) from the corresponding extreme values of the function $\psi_{i,K}(t+T)$. The results we get depend on the range of i's and, as before, they are sharp for the interval $i \in (-\infty, 1)$.

THEOREM 3.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body and let $i \in (-\infty,1)$, $i \neq 0$. Then, there exists an affine position of K, $t+TK$, with $t \in \mathbb{R}^n$ and $T \in SL(n)$ such that 0 belongs to the interior of $t+TK$,*

$$C_1^{|i|} \leq \frac{\tilde{W}_i(t+TK)}{L_K^{-i}|K|^{\frac{n-i}{n}}|D_n|^{\frac{i}{n}}} \leq (C_2|i|)^{|i|+1},$$

for $-\infty < i < 0$ and

$$C_1 \leq \frac{\tilde{W}_i(t+TK)}{L_K^{-i}|K|^{\frac{n-i}{n}}|D_n|^{\frac{i}{n}}} \leq \frac{C_2}{(1-i)^i},$$

for $0 < i < 1$, where C_1, C_2 are absolute constants and L_K is the isotropy constant of K.

Proof. There exists $t \in \mathbb{R}^n$ and $T \in SL(n)$ such that $t+TK$ is in isotropic position (see [MP] and [Dar]). Then the origin is the centroid of $t+TK$ and

$$\tilde{W}_i(t+TK) = \frac{1}{n}\int_{S^{n-1}} \rho_{t+TK}^{n-i}(u)\,d\sigma(u) = \frac{n-i}{n}\int_{t+TK}\frac{dx}{|x|^i}.$$

Since $i \in (-\infty,1)$, by using well known results about equivalence of moments of order $-i \in (-1,+\infty)$ of a norm on any convex body (see, for instance [MS], [MP], [La], [Gue]) we obtain that for some absolute constant $C>0$ we have

$$\frac{1+\min\{-i,0\}}{C}\left(\frac{1}{|K|}\int_{t+TK}|x|^2\,dx\right)^{1/2} \leq \left(\frac{1}{|K|}\int_{t+TK}|x|^{-i}\,dx\right)^{-1/i}$$
$$\leq C\max\{2,-i\}\left(\frac{1}{|K|}\int_{t+TK}|x|^2\,dx\right)^{1/2}.$$

Since for $n \geq 2$ and $i < 1$

$$\frac{1}{2} \leq \frac{n-i}{n} \leq \max\{1, 1-i\}$$

and

$$nL_K^2|K|^{2/n} = \frac{1}{|K|}\int_{t+TK}|x|^2dx$$

we conclude the result. ∎

The estimate we obtained is sharp in the following sense. Suppose $i=-1$. Our result says that for any convex body K of volume equal to 1, we can find a position such that

$$C_1 L_K\,|D_n|^{-1/n} \le \tilde{W}_{-1}(t+TK) \le C_2 L_K\,|D_n|^{-1/n}.$$

Furthermore, to prove that "for any convex body K of volume equal to 1 there exists a position such that

$$C_1 |D_n|^{-1/n} \le \tilde{W}_{-1}(t+TK) \le C_2 |D_n|^{-1/n}\text{"}$$

is a reformulation of the hyperplane conjecture (see [MP]). Note that the case $i=-2$ would be exactly the hyperplane conjecture. Now we know that we can reformulate the hyperplane conjecture in terms of sharp estimates for the dual quermassintegrals of the convex bodies in the range $i \in (-\infty,0)\cup(0,1)$.

Apart from this reformulation of hyperplane conjecture, if we consider $i \to 0$ in the last theorem, we get that

$$C_1 L_K \left(\frac{|K|_n}{|B_2^n|_n}\right)^{1/n} \le \exp\left(\frac{1}{|t+TK|_n}\int_{s+T'K} \log|x|\,dx\right) \le C\left(\frac{|K|_n}{|B_2^n|_n}\right)^{1/n}.$$

Moreover, if we could prove that there exists an absolute constant C such that for every dimension n and every convex body K there exist an affine position $s+T'K$ such that

$$\exp\left(\frac{1}{|K|_n}\frac{1}{n}\int_{S^{n-1}} \rho^n_{s+T'K}(u)\log\rho_{s+T'K}(u)\,\sigma(u)\right) \le C\left(\frac{|K|_n}{|B_2^n|_n}\right)^{1/n}, \qquad (3.24)$$

then we would have proved the hyperplane conjecture. Notice that inequality (3.24) can be understood as a reverse of an inequality proved by V. Milman and A. Pajor (see [MP], pp. 76-77).

Next we consider the case $i \in (1,n)$.

THEOREM 3.2. *Let $K \subseteq \mathbb{R}^n$ be a convex body and let $i \in [1,n]$. There exists an affine position of K, $t+T(K)$, such that 0 belongs to the interior of $t+T(K)$ and*

$$\tilde{W}_i(t+TK) \ge \max\left\{\left(\frac{C}{L_K}\right)^i\frac{n-i}{n}, \left(\frac{c}{\sqrt{n}}\right)^{\min\{i,n-i\}}\right\}|K|^{\frac{n-i}{n}}|D_n|^{\frac{i}{n}}.$$

Proof. First of all, we use the isotropy position of the convex body K, as in proposition 3.1, and we get that

$$\tilde{W}_i(t+TK) = \frac{n-i}{n}\int_{t+TK}\frac{dx}{|x|^i} \ge \frac{n-i}{n}|K|\left(\frac{1}{|K|}\int_{t+TK}|x|^2\,dx\right)^{-i/2},$$

simply by using Jensen inequality. Hence

$$\tilde{W}_i(t+TK) \ge C^i\frac{n-i}{n}L_K^{-i}|K|^{\frac{n-i}{n}}|D_n|^{\frac{i}{n}}.$$

Now we consider John's position. Henceforth $D_n \subseteq \lambda(t+TK)$, for some $t \in \mathbb{R}^n$, $T \in SL(n)$ and $\lambda > 0$ and D_n is the ellipsoid of maximal volume contained in $t+TK$. We can use an estimate given by Ball (cf. [B])

$$\frac{\lambda |K|^{1/n}}{D_n^{1/n}} \leq C\sqrt{n}$$

and since

$$1 \leq \rho_{\lambda(t+TK)}(u) = \lambda \rho_{(t+TK)}(u)$$

for all $u \in S^{n-1}$, we obtain that

$$\tilde{W}_i(t+TK) = \frac{1}{n}\int_{S^{n-1}} \rho^{n-i}_{(t+TK)}(u)d\sigma(u) \geq \frac{|D_n|}{\lambda^{n-i}} \geq \left(\frac{C}{\sqrt{n}}\right)^{n-i} |K|^{\frac{n-i}{n}} |D_n|^{\frac{i}{n}}.$$

Eventually if we consider the position $t+TK$ such that D_n is the ellipsoid of minimal volume containing $t+TK$ then we get that

$$\tilde{W}_i(t+TK) \geq \left(\frac{C}{\sqrt{n}}\right)^{i} |K|^{\frac{n-i}{n}} |D_n|^{\frac{i}{n}}.$$

Hence the result follows. ∎

If $K = B_p^n$ we can give a sharper estimate for $\tilde{W}_i(B_p^n)$ with $0 < i < n$ and $1 \leq p \leq \infty$, as it is shown in the next result.

Proposition 3.1. *Let $1 \leq p \leq \infty$. There exists $C_p > 0$ such that for every $n \in \mathbb{N}$ and every $0 < i < n$*

$$\tilde{W}_i(B_p^n) \geq C_p^{\min\{i,n-i\}} |D_n|^{\frac{i}{n}} \, |B_p^n|^{\frac{n-i}{n}}.$$

Proof. It is easy to check that for every star body $K \subseteq \mathbb{R}^n$, if $0 < i < n$

$$\begin{aligned}
\tilde{W}_i(K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i}\, d\sigma(u) = \frac{n-i}{n}\int_0^{+\infty} \left|\{x \in K;\ \frac{1}{|x|^i} > t\}\right| dt \\
&= \frac{n-i}{n}\int_0^{+\infty} \left|\{x \in K;\ \frac{1}{t^{1/i}} > |x|\}\right| dt = \frac{n-i}{n}\int_0^{+\infty} |K \cap \frac{1}{t^{1/i}} D_n|\, dt \\
&= \frac{(n-i)i}{n}\int_0^{+\infty} |K \cap sD_n| \frac{ds}{s^{i+1}}.
\end{aligned}$$

So we only have to give lower estimates for $|B_p^n \cap sD_n|$.

If $1 \leq p \leq 2$,

$$
\begin{aligned}
\tilde{W}_i(B_p^n) &= \frac{(n-i)i}{n}\left[\int_0^{n^{\frac{1}{2}-\frac{1}{p}}} |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}} \right.\\
&\left. + \int_{n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}} + \int_1^{+\infty} |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}}\right]\\
&\geq \frac{(n-i)i}{n}\left[\int_0^{n^{\frac{1}{2}-\frac{1}{p}}} |D_n| \, s^{n-i-1}\, ds + \int_{n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}}\right]\\
&= \frac{(n-i)i}{n}\left[\frac{1}{n-i}|D_n|\left(\frac{1}{n^{1/p-1/2}}\right)^{n-i} + \int_{n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}}\right].
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\int_{n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}} &\geq \int_{2n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}}\\
&= \int_{2n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n| \left(1 - \frac{|\{x \in B_p^n;\ |x| > s\}|}{|B_p^n|}\right) \frac{ds}{s^{i+1}},
\end{aligned}
$$

hence, by using the estimates of the volume of the intersection of two ℓ_p^n balls (see [SZ]) we get that

$$
\begin{aligned}
\int_{n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n \cap sD_n| \, \frac{ds}{s^{i+1}} &\geq \int_{2n^{\frac{1}{2}-\frac{1}{p}}}^1 |B_p^n| \left(1 - \frac{\exp(-cs^p n)}{|B_p^n|}\right) \frac{ds}{s^{i+1}}\\
&\geq C_p |B_p^n| \int_{2n^{\frac{1}{2}-\frac{1}{p}}}^1 \frac{ds}{s^{i+1}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{W}_i(B_p^n) &\geq \frac{(n-i)i}{n}\left[\frac{1}{n-i}|D_n|\left(\frac{1}{n^{1/p-1/2}}\right)^{n-i} + \frac{C_p}{i2^i}|B_p^n| n^{i/p-i/2}\right]\\
&= |D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} \left[\frac{i}{n}\left(\frac{1}{n^{1/p-1/2}}\right)^{n-i}\left(\frac{|D_n|}{|B_p^n|}\right)^{\frac{n-i}{n}}\right.\\
&\left. + C_p \frac{n-i}{n}\left(\frac{|B_p^n|}{|D_n|}\right)^{i/n} \frac{n^{\frac{i}{p}-\frac{i}{2}}}{2^i}\right]\\
&\geq |D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} \left[\frac{i}{n} C_p^{n-i} + \frac{n-i}{n} C_p^i\right].
\end{aligned}
$$

Now it is easy to check that if $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and $0 < x < 1$, then

$$
x^{\min\{\alpha,\beta\}} \geq \alpha x^\beta + \beta x^\alpha \geq \frac{1}{2} x^{\min\{\alpha,\beta\}},
$$

hence

$$\tilde{W}_i(B_p^n) \geq |D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} \left[\frac{i}{n} C_p^{n-i} + \frac{n-i}{n} C_p^i\right] \geq \frac{1}{2}|D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} C_p^{\min\{i,n-i\}}.$$

On the other hand, if $2 \leq p < \infty$,

$$\begin{aligned}
\tilde{W}_i(B_p^n) &= \frac{(n-i)i}{n}\left[\int_0^1 |B_p^n \cap sD_n| \frac{ds}{s^{i+1}}\right. \\
&\quad \left. + \int_1^{n^{1/2-1/p}} |B_p^n \cap sD_n| \frac{ds}{s^{i+1}} + \int_{n^{1/2-1/p}}^{+\infty} |B_p^n \cap sD_n| \frac{ds}{s^{i+1}}\right] \\
&\geq \frac{(n-i)i}{n}\left[\int_1^{n^{1/2-1/p}} |B_p^n \cap sD_n| \frac{ds}{s^{i+1}} + \int_{n^{1/2-1/p}}^{+\infty} |B_p^n| \frac{ds}{s^{i+1}}\right] \\
&= \frac{(n-i)i}{n}\left[\int_1^{n^{1/2-1/p}} |B_p^n \cap sD_n| \frac{ds}{s^{i+1}} + \frac{1}{i}|B_p^n| \left(\frac{1}{n^{1/2-1/p}}\right)^i\right].
\end{aligned}$$

Now, since

$$\begin{aligned}
|B_p^n \cap sD_n| &= |\{x \in sD_n;\ \|x\|_p \leq 1\}| \\
&= s^n |D_n| \left(1 - \frac{\left|\{y \in D_n;\ \|y\|_p > \frac{1}{s}\}\right|}{|D_n|}\right),
\end{aligned}$$

by using the estimates of the intersection of two ℓ_p^n balls given by G.Schechtman and J.Zinn (see [SZ]) we can assert that if n is big enough

$$\begin{aligned}
\int_1^{n^{\frac{1}{2}-\frac{1}{p}}} |B_p^n \cap sD_n| \frac{ds}{s^{i+1}} &\geq \int_{\frac{1}{2}n^{\frac{1}{2}-\frac{1}{p}}}^{n^{\frac{1}{2}-\frac{1}{p}}} s^{n-i-1}|D_n| \left(1 - \frac{\exp(-c\frac{n}{s^2})}{|D_n|}\right) ds \\
&\geq C|D_n| \int_{\frac{1}{2}n^{\frac{1}{2}-\frac{1}{p}}}^{n^{\frac{1}{2}-\frac{1}{p}}} s^{n-i-1}\, ds.
\end{aligned}$$

Hence

$$\begin{aligned}
\tilde{W}_i(B_p^n) &\geq \frac{(n-i)i}{n}\left[\frac{C_p}{n-i}|D_n| \left(n^{\frac{1}{2}-\frac{1}{p}}\right)^{n-i} + \frac{1}{i}|B_p^n| \left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right)^i\right] \\
&= |D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} \left[C_p \frac{i}{n} \left(\frac{|D_n|}{|B_p^n|}\right)^{\frac{n-i}{n}} (n^{\frac{1}{2}-\frac{1}{p}})^{n-i}\right. \\
&\qquad \left. + \frac{n-i}{n}\left(\frac{|B_p^n|}{|D_n|}\right)^{\frac{i}{n}} \frac{1}{n^{\frac{i}{2}-\frac{i}{p}}}\right] \\
&\geq |D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} \left[\frac{n-i}{n} C_p^i + \frac{i}{n} C_p^{n-i}\right] \\
&\geq \frac{1}{2}|D_n|^{\frac{i}{n}} |B_p^n|^{\frac{n-i}{n}} C_p^{\min\{i,n-i\}}.
\end{aligned}$$

Note that if n is not *"big enough"* (i.e. $1 \le n \le n_0$), we can obtain the same inequality as before simply by adjusting the constant C_p. The case $p = \infty$ can be proved as before but by considering $\frac{1}{p} = 0$. ∎

Next we shall study the remainder case $i > n$.

THEOREM 3.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body. There exists an affine position of K, $t + TK$, such that 0 belongs to the interior of $t + TK$ and for every $\alpha > 0$*

(i) $\tilde{W}_{n+\alpha}(t+TK) \le C(\alpha)\log(n)^\alpha |K|^{\frac{n-i}{n}} |D_n|^{\frac{i}{n}}$, *if K is symmetric.*

(ii) $\tilde{W}_{n+\alpha}(t+TK) \le C(\alpha) n^{\alpha/3} \log(n)^\alpha |K|^{\frac{n-i}{n}} |D_n|^{\frac{i}{n}}$, *for every convex body K.*

where $C(\alpha)$ is a constant which only depends on α.

Proof. Suppose K is symmetric with respect to the origin (0-symmetric). Let

$$M(K) = \frac{1}{\sigma(S^{n-1})}\int_{S^{n-1}} \|u\|_K \, d\sigma(u),$$

where $\|\cdot\|_K$ is the norm on $\mathbb{R}^n$ whose unit ball is K. We use the well known MM^*-estimate and so there exists a $T \in SL(n)$ such that

$$M(TK)M((TK)^\circ) \le C \log n,$$

for some absolute constant $C > 0$ (this position is known as *ℓ-position* or mean width position, see for instance [Pi] and [GM1]). Since

$$M(TK) = |D_n|^{-1}\tilde{W}_{n+1}(TK)$$
$$M((TK)^\circ) = |D_n|^{-1}\tilde{W}_{n+1}((TK)^\circ)$$

and by using (2.4) and Blaschke-Santaló inequality, we get that

$$\begin{aligned}\tilde{W}_{n+1}((TK)^\circ) &\ge |(TK)^\circ|^{\frac{-1}{n}} |D_n|^{\frac{n+1}{n}} \\ &\ge |K|^{\frac{1}{n}} |D_n|^{1-\frac{1}{n}},\end{aligned}$$

so we obtain

$$\tilde{W}_{n+1}(TK) \le C \log n |K|^{-\frac{1}{n}} |D_n|^{1+\frac{1}{n}}. \tag{3.25}$$

Consider now the general case. Let $\alpha > 0$. We use the same ℓ-position and so

$$\begin{aligned}\tilde{W}_{n+\alpha}(TK) &= \frac{1}{n}\int_{S^{n-1}} \|u\|_{TK}^\alpha \, d\sigma(u) \\ &\simeq C(\alpha)\frac{|D_n|}{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}\int_{\mathbb{R}^n} \|x\|_{TK}^\alpha \, d\gamma_n(x),\end{aligned}$$

where $A \simeq B$ means here that the quotient A/B is bounded from above and from below for absolute constants, $C(\alpha)$ is a constant depending on α and $d\gamma_n(x)$ is the canonical Gaussian probability on $\mathbb{R}^n$. Indeed, by using polar coordinates

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|_{TK}^\alpha \, d\gamma_n(x) &= \frac{2^{\frac{\alpha-2}{2}}\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)|D_n|}\int_{S^{n-1}} \|u\|_{TK}^\alpha \, d\sigma(u) \\ &\simeq C\frac{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}{e^{\frac{\alpha-2}{2}}\sigma(S^{n-1})}\left(\frac{n+\alpha-2}{n}\right)^{\frac{n}{2}}\int_{S^{n-1}} \|u\|_{TK}^\alpha \, d\sigma(u) \\ &\simeq C(\alpha)\frac{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}{\sigma(S^{n-1})}\int_{S^{n-1}} \|u\|_{TK}^\alpha \, d\sigma(u).\end{aligned}$$

It is well known that the canonical Gaussian probability is log-concave and the moments of order $\alpha \in (0,\infty)$ of a norm with respect to log-concave measures are equivalent up to an absolute constant (see [La]), i.e. there exists an absolute constant $C>0$ such that

$$C^{-1}\int_{\mathbb{R}^n} \|x\| \, d\gamma_n(x) \leq \left(\int_{\mathbb{R}^n} \|x\|^\alpha \, d\gamma_n(x)\right)^{1/\alpha} \leq C\max\{1,\alpha\}\int_{\mathbb{R}^n} \|x\| \, d\gamma_n(x).$$

Then

$$\begin{aligned}\tilde{W}_{n+\alpha}(TK) &\leq C(\alpha)\frac{|D_n|}{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}\left(\int_{\mathbb{R}^n} \|x\|_{TK} \, d\gamma_n(x)\right)^\alpha \\ &\leq C(\alpha)\frac{|D_n|}{\sqrt{n}(n+\alpha-2)^{\frac{\alpha-1}{2}}}\left(\frac{\sqrt{n}}{\sigma(S^{n-1})}\int_{S^{n-1}} \|u\|_{TK} \, d\sigma(u)\right)^\alpha \\ &\leq C(\alpha)\left(\frac{n}{n+\alpha-2}\right)^{\frac{\alpha-1}{2}}|D_n|^{1-\alpha}\left(\tilde{W}_{n+1}(TK)\right)^\alpha \\ &\leq C(\alpha)\left(\frac{n}{n+\alpha-2}\right)^{\frac{\alpha-1}{2}}(\log n)^\alpha\,|K|^{-\alpha/n}|D_n|^{1+\alpha/n} \\ &\leq C(\alpha)\,(\log n)^\alpha\,|K|^{-\alpha/n}|D_n|^{1+\alpha/n}.\end{aligned}$$

For a general convex body K we use the same method but considering the $MM^\star$-estimate given by M. Rudelson (see [R]) for general convex bodies that ensures that there exists an affine position $t+TK$ of K such that

$$M(t+TK)M((t+TK)^\circ) \leq Cn^{1/3}\log^a(n),$$

instead of the classic $MM^\star$ well known estimate for symmetric convex bodies. ∎

Notice that if K is a symmetric convex body in the last result, we only have to consider a linear transformation while if K is general it is necessary to consider an affine transformation related with its Santalo point.

We finish this section by proving a result for general couples of convex bodies. We consider reverse inequalities associated to dual mixed volumes $\tilde{V}_i(K,L)$.

THEOREM 3.4. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies and $i \in \mathbb{R}$. There exist $T \in SL(n)$ and $t, s \in \mathbb{R}^n$ such that 0 is in the interior of $t + TK$ and $s + L$ and*

$$\left(C\sqrt{n}\log n\right)^{\min\{i,n-i\}} \tilde{V}_i(t+TK, s+L) \geq |K|^{\frac{n-i}{n}}|L|^{\frac{i}{n}}, \quad \text{if } i \in (0,n),$$
$$\left(C\sqrt{n}\log n\right)^{\max\{i,n-i\}} \tilde{V}_i(t+TK, s+L) \leq |K|^{\frac{n-i}{n}}|L|^{\frac{i}{n}}, \quad \text{if } i \in (-\infty,0)\cup(n,\infty).$$

Proof. Let suppose $i \in (0, n)$. Since

$$\tilde{V}_j(t+TK, s+L) = \tilde{V}_{n-j}(T^{-1}(s+L), T^{-1}(t)+K),$$

for all $T \in SL(n)$, it is enough to prove

$$\left(C\sqrt{n}\log n\right)^{n-i} \tilde{V}_i(t+TK, s+L) \geq |K|^{\frac{n-i}{n}}|L|^{\frac{i}{n}}.$$

It is easy to check that there exist $\lambda > 0$, $T \in SL(n)$ and $t, s \in \mathbb{R}^n$ such that 0 is in the interior of $t + TK$ and $s + L$ and $s + L \subseteq \lambda(t + TK)$ is in maximal volume position. Hence, for every $u \in S^{n-1}$

$$\rho_{s+L}(u) \leq \rho_{\lambda(t+TK)}(u) = \lambda\rho_{t+TK}(u),$$

therefore,

$$\begin{aligned} \tilde{V}_i(t+TK, s+L) &= \frac{1}{n}\int_{S^{n-1}} \rho_{t+TK}^{n-i}(u)\rho_{s+L}^{i}(u)\,d\sigma(u) \\ &\geq \frac{1}{n\lambda^{n-i}}\int_{S^{n-1}} \rho_{s+L}^{n}(u) = \frac{1}{\lambda^{n-i}}|L|. \end{aligned}$$

Now we use a recent result by A. Giannopoulos and M. Hartzoulaki (see [GH])

$$\frac{\lambda|K|^{1/n}}{|L|^{1/n}} = vr(K;L) \leq C\sqrt{n}\log n,$$

where $vr(K; L)$ denotes the *volume ratio* of the pair K, L and $C > 0$ is an absolute constant. By using this fact we conclude that

$$\left(C\sqrt{n}\log n\right)^{n-i} \tilde{V}_i(TK, L) \geq |K|^{\frac{n-i}{n}}|L|^{\frac{i}{n}}.$$

The other cases are similar. ∎

4. RELATED QUESTIONS

In this final section we are going to center in two different questions. First of all, we are going to show that the *"isotropic"* conditions appearing in section 2 characterize the solution of extremal problems for dual mixed volumes slightly different from the ones stated in section 2. We apply this technique to show that *"isotropic"* conditions also characterizes extremal problems in the Brunn-Minkowski theory, following the ideas of A. Giannopoulos and V. Milman (see [GM1]). The second question studied is related with other problems involving dual mixed volumes that are close to the existence of some *"isotropic"* type measures on the sphere.

If we want to show that *"isotropic"* type measures characterizes the solutions of some extremal problem involving dual mixed volumes we need to introduce some notation.

Let $K_1, K_2, K_3 \subseteq \mathbb{R}^n$ be star-shaped bodies at 0 and $i_1, i_2, i_3 \in \mathbb{R}$. We denote by $\tilde{V}_{i_1,i_2,i_3}(K_1, K_2, K_3)$ the value

$$\tilde{V}_{i_1,i_2,i_3}(K_1, K_2, K_3) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}^{i_1}(u) \rho_{K_2}^{i_2}(u) \rho_{K_3}^{i_3}(u)\, d\sigma(u).$$

Following this notation, we can state the following result:

PROPOSITION 4.1. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies such that 0 belongs to their interior. Suppose that L is "smooth enough" and take $i \in \mathbb{R}$. Then the following assertions are equivalent:*

(i) $\tilde{V}_i(K, L) = \min \left\{ \tilde{V}_{n-i,-1,i+1}(K, TL, L) \right\}$, *where the minimum runs over all positive definite, symmetric matrices $T \in SL(n)$.*

(ii) For every $T \in GL(n)$ symmetric

$$\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \rho_L^{i+1}(u) \langle \nabla h_{L^\circ}(u), Tu \rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n} \tilde{V}_i(K, L).$$

(iii) K is the only symmetric positive definite position that minimizes $\tilde{V}_i(K, L)$, that is, if $T_0 \in SL(n)$ is symmetric and positive definite, such that

$$\tilde{V}_i(T_0 K, L) = \min \left\{ \tilde{V}_{n-i,-1,i+1}(K, TL, L);\ \ T \in SL(n),\ \ \textit{positive def.} \right\},$$

then $T_0 = I_n$.

Proof. (i)⇒(ii) If we take $T \in GL(n)$ symmetric, there exists an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ it is well defined

$$S_\varepsilon = \frac{I + \varepsilon T}{|\det(I + \varepsilon T)|^{1/n}}$$

and such that S_ε is positive definite. Then

$$\tilde{V}_{n-i,-1,i+1}(K, (I + \varepsilon T)(L), L) \geq |\det(I + \varepsilon T)|^{-1/n} \tilde{V}_i(K, L).$$

Hence, by following the ideas in the proof of proposition 2.1 we get *(ii)*.

(ii) $\Rightarrow$ *(iii)* If we take a positive definite symmetric matrix $T \in SL(n)$,

$$\begin{aligned}\tilde{V}_{n-i,-1,i+1}(K,T^{-1}L,L) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_{T^{-1}L}^{-1}(u)\rho_L^{i+1}(u)\,d\sigma(u)\\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)h_{L^\circ}(Tu)\rho_L^{i+1}(u)\,d\sigma(u)\\ &\geq \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\rho_L^{i+1}(u)\langle\nabla h_{L^\circ}(u),Tu\rangle\,d\sigma(u)\\ &= \frac{\operatorname{tr}T}{n}\tilde{V}_i(K,L) \geq (\det T)^{1/n}\,\tilde{V}_i(K,L)\\ &= \tilde{V}_i(K,L).\end{aligned}$$

The uniqueness follows from the fact that if $T \in SL(n)$ is symmetric positive definite and $\frac{\operatorname{tr}T}{n} = \det T$, then $T = I_n$. ∎

Notice that since $\tilde{V}_i(K,L) = \tilde{V}_{n-i}(L,K)$, we also obtain the following result:

COROLLARY 4.1. *Let $K,L \subseteq \mathbb{R}^n$ be convex bodies such that K is "smooth enough" and take $i \in \mathbb{R}$. Then the following assertions are equivalents:*

(i) $\tilde{V}_i(K,L) = \min\left\{\tilde{V}_{n-i+1,-1,i}(K,TK,L)\right\}$, *where the minimun runs over all positive definite, symmetric matrices $T \in SL(n)$.*

(ii) *For every $T \in GL(n)$ symmetric,*

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^{i}(u)\langle\nabla h_{K^\circ}(u),Tu\rangle\,d\sigma(u) = \frac{\operatorname{tr}T}{n}\tilde{V}_i(K,L).$$

(iii) *K is the only symmetric positive definite position that minimizes $\tilde{V}_i(K,L)$.*

Remark 4. If $L = D_n$ in the last two results, by using theorem 2.1, corollary 2.2, proposition 4.1 and corollary 4.1, we can ensure that the following assertions are equivalent:

(i) The measure $\rho_K^{n-i}(u)d\sigma(u)$ is isotropic.

(ii) $\tilde{W}_i(K) = \min\left\{\tilde{V}_{n-i,-1,i+1}(K,TD_n,D_n);\; T \in SL(n)\right\}$.

(iii) $\tilde{W}_i(K) = \min\left\{\tilde{V}_{n-i+1,-1,i}(K,TK,D_n);\; T \in SL(n)\right\}$.

Moreover, if $i \in (-\infty,0)\cup\{n+1\}$ and K is a convex body, any of the previous assertions is equivalent to the following:

(iv) $\tilde{W}_i(K) = \min\left\{\tilde{W}_i(TK);\; T \in SL(n)\right\}$.

If $i \in (-\infty,0)\cup[n+1,\infty)$ and K is a symmetric convex body, then any of the previous assertion are also equivalent. These results might be added to those of section 2.

The same philosophy can be applied to find other extremal problems related to some of the results given by A. Giannopoulos and V. Milman in [GM1]. There they proved the following result:

PROPOSITION 4.2. *Let $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body. If*

$$W_i(K) = \min\{W_i(TK);\ T \in SL(n)\},$$

where $W_i(K) = V(K, \overset{n-i)}{\ldots}, K, D_n, \overset{i)}{\ldots}, D_n)$, then:

(i) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \langle \nabla h_K(u), T(u)\rangle \, dS_i(K)(u) = \frac{\operatorname{tr} T}{n} W_i(K),$$

where the measure $S_i(K)(\cdot)$ denotes $dS_i(K)(u) = dS_i(K, D_n)$.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \langle u, T(u)\rangle \, dS_{i-1}(K)(u) = \frac{\operatorname{tr} T}{n} W_i(K).$$

It is stated in [GM1] that it would be interesting to determine not only necessary but also sufficient conditions for the positions minimizing W_i. Following this idea, we have realized that these necessary conditions that appear in proposition 4.2 are also sufficient for a slightly different extremal position involving classical mixed volumes:

PROPOSITION 4.3. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies with K "smooth enough" and take $0 < i < n$. Then the following assertions are equivalents:*

(i) $V_i(K, L) = \min\left\{V(TK, K, \ldots, K, L, \overset{i)}{\ldots}, L)\right\}$, where the minimum runs over all positive definite symmetric matrices $T \in SL(n)$ and we denote

$$V_i(K, L) = V(K, \ldots, K, L, \overset{i)}{\ldots}, L).$$

(ii) For every $T \in GL(n)$ symmetric,

$$\frac{1}{n}\int_{S^{n-1}} \langle \nabla h_K(u), T(u)\rangle \, dS_i(K, L)(u) = \frac{\operatorname{tr} T}{n} V_i(K, L).$$

Remark 5. Since $V_i(K, L) = V_{n-i}(L, K)$, if L is *"smooth enough"* and $0 < i < n$ we ensure that the following assertions are equivalents:

(i) $V_i(K, L) = \min\left\{V(K, K, \overset{n-i)}{\ldots}, K, TL, L, \ldots, L)\right\}$, where the minimum runs over all positive definite symmetric matrices $T \in SL(n)$.

(ii) For every $T \in GL(n)$ symmetric,

$$\frac{1}{n}\int_{S^{n-1}} h_L(u)\langle \nabla h_L(u), T(u)\rangle \, dS_{i-1}(K, L)(u) = \frac{\operatorname{tr} T}{n} V_i(K, L).$$

Now if we put $L = D_n$ we get the following results:

COROLLARY 4.2. *Let $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body and $0 < i < n$. Then the following assertions are equivalents:*

(i) $W_i(K) = \min\{V(T(K), K, \overset{n-i-1)}{\ldots}, K, D_n, \overset{i)}{\ldots}, D_n);\ T \in SL(n)\}$.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \langle \nabla h_K(u), T(u)\rangle \, dS_{n-i-1}(K,u) = \frac{\operatorname{tr} T}{n} W_i(K).$$

COROLLARY 4.3. *Let $K \subseteq \mathbb{R}^n$ be a convex body and take $0 < i < n$. Then the following assertions are equivalents:*

(i) $W_i(K) = \min\{V(K, \overset{n-i)}{\ldots}, K, TD_n, D_n, \overset{i-1)}{\ldots}, D_n);\ T \in SL(n)\}$.

(ii) $S_{n-i}(\cdot)$ *is isotropic.*

Remark 6. By using the Alexandrov-Frenchel inequality it's easy to check that

$$\min\{W_i(TK);\, T \in SL(n)\} \leq \min\left\{V(K, \ldots, TD_n, D_n, \overset{(i-1}{\ldots}, D_n);\, T \in SL(n)\right\},$$
$$\min\{W_i(TK);\, T \in SL(n)\} \leq \min\left\{V(TK, K, \ldots, K, D_n, \overset{(i}{\ldots}, D_n);\, T \in SL(n)\right\}.$$

But we don't known if these inequalities are also equalities for another K different from D_n.

Eventually, we want to show another example of the close relation between the solution of extremal problems and properties of "*isotropic*" type of some measures on the sphere. In [GM1] A. Giannopoulos and V. Milman asked for necessary and sufficient conditions for a convex body $K \subseteq \mathbb{R}^n$ to verify

$$M(K)M^\star(K) = \min\{M(TK)M^\star(TK);\ T \in GL(n)\},$$

where

$$M(K) = \frac{1}{n|D_n|}\int_{S^{n-1}} \|x\|_K \, d\sigma(x)$$

and $M^\star(K) = M(K^\circ)$. Since

$$M(K) = \frac{1}{|D_n|}\tilde{W}_{n+1}(K),$$

it is natural to look for necessary and sufficient conditions for a convex body K to verify

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ);\ T \in GL(n)\right\},$$

for some $i \notin [0, n]$. In fact, it can be checked that for every convex body K there exist a position $T(K)$ that is the solution of the last extremal problem, simply by using the same methods that we used in lemma 1.

By using a standard variational technique the following result can be proved:

THEOREM 4.4. *Let $n \in \mathbb{N}$, $i \notin [0,n]$ and $K \subseteq \mathbb{R}^n$ a "smooth enough" convex body. If*

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ);\ \ T \in GL(n)\right\},$$

then

(i) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, T^\star u\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u).$$

(ii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), T^\star u\rangle\, d\sigma(u)$$
$$= \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), Tu\rangle\, d\sigma(u).$$

Remark 7. If $i \in (0,n)$, conditions *(i)* and *(ii)* are necessary for a smooth enough star-body $K \subseteq \mathbb{R}^n$ to verify

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ);\ \ T \in GL(n)\right\},$$

simply by using the same variational techniques.

According to the last result we wonder if there is any relation between conditions *(i)* and *(ii)* in the last result and if conditions *(i)* and *(ii)* sufficient conditions for a convex body to have extremal $\tilde{W}_i(K)\tilde{W}_i(K^\circ)$. The answer for the first question can be easily given, since by using Laplace-Beltrami operator techniques that we just have used in the proof of proposition 2.2, we can obtain the next result:

PROPOSITION 4.4. *Let $n \in \mathbb{N}$, $i \in \mathbb{R}$ and $K \subset \mathbb{R}^n$ be a "smooth enough" convex body . The following assertions are equivalents:*

(i) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, T^\star u\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u).$$

(ii) For every $T \in GL(n)$ symmetric

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u).$$

(iii) For every $T \in GL(n)$ symmetric

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u)$$
$$= \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle \nabla h_K(u), Tu\rangle\, d\sigma(u).$$

If we avoid the symmetry conditions on K we don't know if the result is still true for a general $i \in \mathbb{N}$ and a general convex body K and we only can ensure that *(iii)⇒(ii)*.

If $i \leq -1$, we can say furthermore and we prove that conditions *(i)* and *(ii)* are not only necessary, but also sufficient for K to be in extremal $\tilde{W}_i(K)\tilde{W}_i(K^\circ)$ position.

THEOREM 4.5. *Let $i \leq -1$, $n \in \mathbb{N}$ and $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body. Then the following assertions are equivalent:*

(i) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, T^\star u\rangle \, d\sigma(u) = \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u).$$

(ii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), T^\star u\rangle \, d\sigma(u) = \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle \nabla h_K(u), Tu\rangle \, d\sigma(u).$$

(iii) For every $T \in GL(n)$ symmetric

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u) = \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u).$$

(iv) $\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ);\ T \in GL(n)\right\}$ and the minimum is unique up to orthogonal transformation.

Proof. We only have to prove *(i)⇒(iv)*.

It is enough to prove that for every diagonal operator $T \in GL(n)$ with diagonal elements $d_1, \ldots, d_n > 0$ and such that $\prod d_i = 1$ it is verified that

$$\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ).$$

By using Hölder's inequality it follows that

$$\begin{aligned} \tilde{W}_i(TK) &= \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) h_{T^\star(D_n)}^{-i}(u) \, d\sigma(u) \\ &\geq \tilde{W}_i(K)^{i+1} \left(\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) h_{T^\star(D_n)}(u) \, d\sigma(u) \right)^{-i}. \end{aligned}$$

and since $0 \leq \langle u, Tu\rangle \leq h_{T^\star(D_n)}(u)$ (see [Sc], pp. 40) we get that

$$\tilde{W}_i(TK) \geq \tilde{W}_i(K) \left(\frac{\tilde{W}_i(K)}{\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u)} \right)^i.$$

If we use the same philosophy with $\tilde{W}_i(TK)^\circ$, we obtain that

$$\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ)\cdot$$
$$\cdot\left(\frac{\tilde{W}_i(K)\tilde{W}_i(K^\circ)}{\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,Tu\rangle\,d\sigma(u)\frac{1}{n}\int_{S^{n-1}}\rho_{K^\circ}^{n-i}(u)\langle u,T^{-1}u\rangle\,d\sigma(u)}\right)^i.$$

By hypothesis we get that

$$\frac{\tilde{W}_i(K)}{\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,T^{-1}u\rangle\,d\sigma(u)} = \frac{\tilde{W}_i(K^\circ)}{\int_{S^{n-1}}\rho_{K^\circ}^{n-i}(u)\langle u,T^{-1}u\rangle\,d\sigma(u)},$$

hence, since $i<0$, it is enough to prove that

$$\tilde{W}_i(K)^2 \leq \frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,Tu\rangle\,d\sigma(u)\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,T^{-1}u\rangle\,d\sigma(u).$$

For every $u\in S^{n-1}$

$$\begin{aligned}\langle u,Tu\rangle\,\langle u,T^{-1}u\rangle &= \left(\sum_{i=1}^n d_iu_i^2\right)\left(\sum_{i=1}^n d_i^{-1}u_i^2\right)\\ &\geq \left(\prod_{i=1}^n d_i^{u_i^2}\right)\left(\prod_{i=1}^n d_i^{-u_i^2}\right)=1.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{W}_i(K) &\leq \frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)(\langle u,Tu\rangle)^{1/2}(\langle u,T^{-1}u\rangle)^{1/2}\,d\sigma(u)\\ &\leq \left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,Tu\rangle\,d\sigma(u)\right)^{1/2}\\ &\qquad\left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,T^{-1}u\rangle\,d\sigma(u)\right)^{1/2}\end{aligned}$$

∎

REFERENCES

[AS] M. Abramowitz y I.A. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables*, Dover, Nueva York (1968).

[B] K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. (2) **44** (1991), pp. 351-359.

[BBP] J. Bastero, J. Bernués and A. Peña, *An extension of Milman's reverse Brunn-Minkowski inequality*, Geom. Funct. Anal. **5** (1995), pp. 572-581.

[BR] J. Bastero and M. Romance, *John's decomposition of the identity in the non-convex case*, Positivity **6** (2002), pp. 1-16.

[Dar] S. Dar, *On the isotropic constant of non-symmetric convex bodies*, Israel J. Math. **97** (1997), pp. 151-156..

[Ga] R.J. Gardner, *Geometric Tomography*, Encyclopedia of Math. and its applications **58**, Cambridge Univ. Press. Cambridge (1995).

[GH] A. Giannopoulos and M. Hartzoulaki, *On the volume ratio of two convex bodies*, preprint.

[GM1] A. Giannopoulos and V. Milman, *Extremal problems and isotropic positions of convex bodies*, Israel J. Math. **117** (2000), pp. 29-60.

[GM2] A. Giannopoulos and V. Milman, *Euclidean structure in finite dimensional normed spaces*, in Handbook of the Geometry of Banach Spaces (Vol. 1), Johnson-Lindenstrauss eds., Elsevier Sciene (2001), pp. 707-780.

[GP] A. Giannopoulos and M. Papadimitrakis, *Isotropic surface area measures*, Mathematika **46** (1999), pp. 1-13.

[GPT] A. Giannopoulos, I. Perissinaki and A. Tsolomitis, *John's theorem for an arbitrary pair of convex bodies*, Geometriae Dedicata **84** (2001), pp.163-179.

[Gue] O. Guédon, *Kahane Khinchine Inequalities for negative exponent*, Mathematika **46** (1999), pp. 165-173.

[Gr] H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics*, Encyclopedia of Math. and its applications, **61**, Cambridge Univ. Press. Cambridge (1996).

[La] R. Latała, *On the equivalence between geometric and arithmetic means for log-concave measures*, Convex Geometric Analysis, MSRI, Publications, **34** (1998), pp. 123-128.

[L1] E. Lutwak, *Dual Mixed Volumes*, Pacific Journal of Math. **58** (1975), pp. 531-538.

[L2] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. in Math. **71** (1988), pp. 232-261.

[L3] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. (3) **60** (1990), pp. 365-391.

[M] V. Milman, *Inegalité de Brunn-Minkowski inverse et applications à la theorie locale des espaces normés*, C. R. Acad. Sc. Paris **302** (1986), pp. 25-28.

[MP] V. Milman and A. Pajor, *Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, GAFA Seminar 87-89, Springer Lecture Notes in Math. **1376** (1989), pp. 64-104.

[MS] V. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Springer Lecture Notes in Math. **1200**, (1986).

[P] C. M. Petty, *Surface area of a convex body under affine transformations*, Proc. Amer. Math. Soc. **12** (1961), pp. 824-828.

[Pi] G. Pisier, *The Volume of Convex bodies and Banach Space Geometry*, Cambridge Tracts in Mathematics **94** (1989).

[R] M. Rudelson, *Distances between non-symetric convex bodies and the $MM^\star$-estimate*, Positivity (4) **2** (2000), pp. 161-178.

[SZ] G. Schechtman and J. Zinn, *On the volume of the intersection of two L_p^n balls*, Proc. of the Amer. Math. Soc. (110) **1** (1990), pp. 217-224.

[Sc] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Math. and its applications **44**, Cambridge Univ. Press. Cambridge (1993).