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J. Bastero
M. Romance

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JOHN'S DECOMPOSITION OF THE IDENTITY IN THE NON-CONVEX CASE

JESÚS BASTERO AND MIGUEL ROMANCE

Abstract. We prove an extension of the classical John's Theorem, that characterises the ellipsoid of maximal volume position inside a convex body by the existence of some kind of decomposition of the identity, obtaining some results for maximal volume position of a compact and connected set inside a convex set with nonempty interior. By using those results we give some estimates for the outer volume ratio of bodies not necessarily convex.

1. Introduction and notation

Throughout this paper, we consider $\mathbb{R}^n$ with the canonical basis $(e_1, \ldots, e_n)$ and its usual Euclidean structure $\langle \cdot, \cdot \rangle$. Let $B_2^n = \{x \in \mathbb{R}^n \,; |x| = \langle x, x\rangle^{1/2} \leq 1\}$ be the euclidean ball on $\mathbb{R}^n$. If $K \subseteq \mathbb{R}^n$, then $\operatorname{int} K$, K^c and ∂K will denote the interior, the complementary and the border of K, respectively; $\operatorname{conv}(K)$ will be the convex hull of K, K^0 will denote the polar of K with respect to the origin, i.e. $K^0 = \{y \in \mathbb{R}^n \,; \langle x, y\rangle \leq 1, \forall x \in K\}$ and $\operatorname{vol}(K)$ represents the Lebesgue measure on $\mathbb{R}^n$ of K.

Following [TJ], if $K_1 \subseteq K_2 \subseteq \mathbb{R}^n$, we call a pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ a *contact pair for* (K_1, K_2) if it satisfies:

(i) $x \in \partial K_1 \cap \partial K_2$,
(ii) $y \in \partial K_2^0$
(iii) $\langle x, y\rangle = 1$.

As it is usual $y \otimes x$ denotes the linear transformation on $\mathbb{R}^n$ defined by $y \otimes x(z) = \langle z, y\rangle\, x$ and I_n will be the identity map on $\mathbb{R}^n$.

John's ellipsoid theorem is a classical tool in the theory of convex bodies; it says how far a convex body is from being an ellipsoid. John showed that each convex body contains a unique ellipsoid of maximal volume and characterised it. The decomposition of the identity associated to this characterisation gives an effective method to introduce an appropriated euclidean structure in finite dimensional normed spaces, when we consider centrally symmetric convex bodies. We can state John's theorem in the following way:

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Theorem 1.1 (([J], [Ba2], [Ba3])). *Let K be a convex body in $\mathbb{R}^n$ and suppose that the euclidean ball B_2^n is contained in K, then the following assertions are equivalent:*

(i) *B_2^n is the ellipsoid of maximal volume contained in K,*

(ii) *there exist $\lambda_1, \ldots, \lambda_m > 0$ and $u_1, \ldots, u_m \in \partial K \cap \partial B_2^n$, with $m \leq n(n+3)/2$ such that $I_n = \sum_{i=1}^{m} u_i \otimes u_i$ and $\sum_{i=1}^{m} \lambda_i u_i = 0$.*

(iii) *B_2^n is the unique ellipsoid of maximal volume contained in K.*

One can consider this situation for any general couple of convex bodies (K_1, K_2) or, even more, for any couple of compact sets in $\mathbb{R}^n$ instead of (B_2^n, K). Suppose that K_1 is a compact set in $\mathbb{R}^n$ with $\operatorname{vol}(K_1) > 0$ and K_2 is another compact set in $\mathbb{R}^n$, with $\operatorname{int} K_2 \neq \emptyset$. A compactness argument shows that there exists an affine position of K_1, namely $\tilde{K}_1$, such that $\tilde{K}_1 \subseteq K_2$ and

$$\operatorname{vol}(\tilde{K}_1) = \max\{\operatorname{vol}(a + T(K_1)); a + T(K_1) \subseteq K_2, a \in \mathbb{R}^n, T \in GL(n)\}.$$

This position $\tilde{K}_1$ is called *maximal volume position of* K_1 *inside* K_2. Very recently, Giannopoulos, Perisinaki and Tsolomitis have considered the convex situation and proved the following

Theorem 1.2 ([G-P-T]). *Let $K_1 \subseteq K_2$ be two smooth enough convex bodies in $\mathbb{R}^n$ such that K_1 is in maximal volume position inside K_2. Then for every point z in the interior of K_1, there exist $\lambda_1, \ldots, \lambda_N > 0$, with $N \leq n^2+n+1$ and contact pairs for $(K_1 - z, K_2 - z)$, $(x_i - z, y_i)$, $(1 \leq i \leq N)$ such that:*

(i) $\sum_{i=1}^{N} \lambda_i y_i = 0$ *and*

(ii) $I_n = \sum_{i=1}^{N} \lambda_i y_i \otimes x_i$.

Furthermore, if we assume the extra assumption for K_1 to be a polytope and K_2 to have $\mathcal{C}^{(2}$ boundary with strictly positive curvature, then a centre z can be chosen in $K_1 \setminus \{\text{vertices of } K_1\}$ for which we have (i), (ii) and also (iii) $\frac{1}{n} \sum_{i=1}^{N} \lambda_i x_i = z$.

The special case of considering K_1 and K_2 centrally symmetric convex bodies was first observed by Milman (see [TJ], Theorem 14.5).

The aim of this paper is to extend this result to the non-convex case. We obtain a general result which is valid for K_1 a compact, connected set in $\mathbb{R}^n$ with $\operatorname{vol}(K_1) > 0$, $K_1 \subseteq K_2$, where K_2 is a compact in $\mathbb{R}^n$ such that $\operatorname{int} \operatorname{conv}(K_2) \neq \emptyset$ and K_1 is in maximal volume position inside $\operatorname{conv}(K_2)$ (no extra assumptions on the boundary of the bodies are used).

The method we develop to prove our result is different from that in [G-P-T]. We follow the ideas given in [Ba3], with suitable modifications and the main result we achieve is the following

Theorem 1.3. *Let $K_1 \subseteq \mathbb{R}^n$ be a connected, compact set with* $\operatorname{vol}(K_1) > 0$ *and $K_2 \subseteq \mathbb{R}^n$ be a compact set such that $K_1 \subseteq K_2$. If K_1 is in maximal volume* *position inside* $\mathrm{conv}(K_2)$*, for every* $z \in \mathrm{int}\,\mathrm{conv}(K_2)$ *there exist* $N \in \mathbb{N}$*,* $N \leq n^2 + n$*,* (x_i, y_i) *contact pairs for* $(K_1 - z, K_2 - z)$ *and* $\lambda_i > 0$ *for all* $i = 1, \ldots, N$ *such that:*

$$\sum_{k=1}^{N} \lambda_k \, y_k \otimes x_k = \frac{1}{n} I_n$$

$$\sum_{k=1}^{N} \lambda_k \, y_k = 0.$$

It is well known that if there exists a decomposition of the identity in the sense of theorem 1.3 we can not expect that K_1 were the unique maximal volume position inside K_2 even for convex bodies, as it can be shown by considering simpleces or octahedra inscribed in the cube. Furthermore, an equivalence as it appears in John's Theorem is not true in general. We study this problem and as a consequence we obtain

Theorem 1.4. *Let* $K_1 \subseteq \mathbb{R}^n$ *be a connected, compact set with* $\mathrm{vol}\,(K_1) > 0$ *and* $K_2 \subseteq \mathbb{R}^n$ *be a compact set such that* $K_1 \subseteq K_2$*. Let* z *be a fixed point in* $\mathrm{int}\,\mathrm{conv}\,(K_2)$*. Then the following assumptions are equivalents:*

- *(i)* $\mathrm{vol}\,(K_1) = \max\{\mathrm{vol}\,(a + S(K_1)); a \in \mathbb{R}^n, a + S(K_1) \subseteq \mathrm{conv}(K_2)\}$*, where* S *runs over all symmetric positive definite matrices.*
- *(ii) There exist* $N \in \mathbb{N}$*,* $N \leq \frac{n^2+3n}{2}$*,* (x_i, y_i) *contact pairs for* $(K_1 - z, K_2 - z)$ *and* $\lambda_i > 0$ *for all* $i = 1, \ldots, N$ *such that:*

$$\sum_{k=1}^{N} \lambda_k \,(y_k \otimes x_k + x_k \otimes y_k) = \frac{1}{n} I_n \tag{1}$$

$$\sum_{k=1}^{N} \lambda_k y_k = 0.$$

- *(iii)* K_1 *is the unique position of* K_1 *verifying (i).*

In section 3 we extend the upper estimates of the volume ratio proved in [G-P-T] by defining the outer volume ratio of a compact K_1 with respect to a convex body K_2, by considering an appropriate index. We follow the methods that appears in [G-P-T] by using Brascamp-Lieb and reverse Brascamp-Lieb inequalities as the main tools.

2. PROOFS OF MAIN THEOREMS

Throughout this section K_1 will be a connected, compact set in $\mathbb{R}^n$ with $\mathrm{vol}\,(K_1) > 0$ and K_2 will be a compact in $\mathbb{R}^n$ such that $K_1 \subseteq K_2$. Theorem 1.3 gives us a sufficient condition for the existence of *some kind* of John's decomposition of the identity and it follows the spirit of the work of K.M. Ball (see for instance [Ba2] or [Ba3]).

Proof of theorem 1.3: First of all, notice that $\mathrm{int}\,\mathrm{conv}\,(K_2) \neq \emptyset$, since $K_1 \subseteq \mathrm{conv}\,(K_2)$ and $\mathrm{vol}\,(K_1) > 0$. Without loss of generality we can assume that $z = 0$. Furthermore, since $K_1 \subseteq K_2$ and $\mathrm{conv}(K_2)^0 = K_2^0$ a contact pair for

$(K_1, \mathrm{conv}(K_2))$ is also a contact pair for (K_1, K_2), so, we may suppose $\mathrm{conv}(K_2) = K_2$.

Let $\mathcal{A} = \{(y \otimes x, y) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{R}^n\,;\, (x, y) \,\text{is a contact pair for}\, (K_1, K_2)\}$. By using the maximality of the volume of K_1, the convexity of K_2 and since $0 \in \mathrm{int}\, K_2$ it is easy to prove that $\mathcal{A}$ is a non empty subset of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{R}^n$. We will show that $(\frac{1}{n} I_n, 0) \in \mathrm{conv}(\mathcal{A})$, where $\mathrm{conv}(\mathcal{A})$ is the convex hull of $\mathcal{A}$ in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{R}^n$.

Suppose that $(\frac{1}{n} I_n, 0) \notin \mathrm{conv}(\mathcal{A})$. Then by using a separation theorem, there exist $H \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $b \in \mathbb{R}^n$ such that:

$$\langle \frac{1}{n} I_n, H \rangle_{\mathrm{tr}} + \langle 0, b \rangle > \langle y \otimes x, H \rangle_{\mathrm{tr}} + \langle b, y \rangle$$

for all (x, y) contact pair and where $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$ denotes the trace duality on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, i.e. $\langle T, S \rangle_{\mathrm{tr}} = \mathrm{tr}\, ST$.

Thus for every contact pair (x, y)

$$\frac{1}{n}\, \mathrm{tr}\, H > \langle Hx, y \rangle + \langle b, y \rangle. \tag{2}$$

There is no loss of generality to assume tr $H = 0$. Indeed, we can choose $\tilde{H} = H - \frac{\mathrm{tr}\, H}{n} I_n \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, which is a linear operator with trace zero that verifies:

$$\langle \tilde{H} x, y \rangle + \langle b, y \rangle = \langle Hx, y \rangle - \frac{\mathrm{tr}\, H}{n} \langle x, y \rangle + \langle b, y \rangle < 0 = \frac{\mathrm{tr}\, \tilde{H}}{n}$$

for all (x, y) contact pair. By using the linear map defined by the matrix H and $b \in \mathbb{R}^n$ we are going to construct a family of affine maps S_δ's, with $0 < \delta < \delta_1$, such that $|\det S_\delta| \geq 1$ and $S_\delta(K_1) \subseteq \mathrm{int}\, K_2$, which contradicts the maximality of the volume of K_1. We will divide the proof of that fact into 3 steps.

By continuity, there exists a positive number $\delta_0 > 0$ such that $I_n - \delta H$ is invertible for all $0 < \delta < \delta_0$. For each $0 < \delta < \delta_0$ we take $S_\delta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $S_\delta(z) = (I_n - \delta H)^{-1}(z) + \delta (I_n - \delta H)^{-1}(b)$.

Step 1: *There exists* $0 < \delta_1 \leq \delta_0$ *such that* $S_\delta(K_1) \cap \partial K_2 = \emptyset$, *for all* $0 < \delta < \delta_1$.

Consider

$$M = \left\{ x \in \partial K_2\,;\, \exists\, y \in K_2^0 \,\text{such that}\, \langle x, y \rangle = 1 \text{ and } \langle Hx, y \rangle + \langle b, y \rangle \geq 0 \right\}.$$

It is easy to check that M is a compact subset of ∂K_2 and also $M \cap \partial K_1 = \emptyset$. If there exists an $x \in M \cap \partial K_1 \subseteq \partial K_2 \cap \partial K_1$, there would exist $y \in K_2^0$ such that $\langle x, y \rangle = 1$ (so (x, y) is a contact pair) and $\langle Hx, y \rangle + \langle b, y \rangle \geq 0 = \mathrm{tr}\, H$ which would contradict (2). Therefore $M \subseteq K_1^c$; by compactness of M and by continuity, there exists $0 < \delta_1 \,(\leq \delta_0)$ such that $(I_n - \delta H)(M) - \delta b \subseteq K_1^c$, for all $0 < \delta < \delta_1$, and so $S_\delta(K_1) \cap M = \emptyset$.

Now let $x \in \partial K_2$. We will prove that $x \notin S_\delta(K_1)$. We only have to consider the case $x \notin M$. Then

$$\langle Hx, y \rangle + \langle b, y \rangle < 0$$

for all $y \in K_2^0$ such that $\langle x, y \rangle = 1$. Since $0 \in \mathrm{int}\, K_2$ and K_2 is a convex body there exists $y_0 \in K_2^0$ such that $\langle x, y_0 \rangle = 1$, so we have that

$$\langle x - \delta Hx - \delta b, y_0 \rangle = 1 - \delta \left(\langle Hx, y_0 \rangle + \langle b, y_0 \rangle \right) > 1$$

for all $\delta > 0$ and in particular for all $0 < \delta < \delta_1$. Hence $x - \delta Hx - \delta b \notin K_1$, or equivalently $x \notin S_\delta(K_1)$.

Step 2: *For every* $0<\delta<\delta_1$ *there exists* $\lambda_\delta > 1$ *such that* $\lambda_\delta\, S_\delta(K_1)\subseteq \operatorname{int} K_2$.

Note that $S_\delta(K_1)$ is connected and $S_\delta(K_1)\cap\partial K_2=\emptyset$, therefore either $S_\delta(K_1)\subseteq \operatorname{int} K_2$ or $S_\delta(K_1)\subseteq K_2^c$. Fix $x\in K_1\cap \operatorname{int} K_2$, it exists since $\operatorname{vol}(K_1)\neq 0$, and take

$$C_x=\{S_\delta(x)\,;\,0\le\delta<\delta_1\}\,.$$

It is easy to check that C_x is connected, $C_x\cap\partial K_2=\emptyset$, $C_x\cap \operatorname{int} K_2\neq\emptyset$ and $C_x\subseteq \operatorname{int} K_2$. Therefore $S_\delta(K_1)\cap \operatorname{int} K_2\neq\emptyset$ and by connectedness of K_1 we conclude that $S_\delta(K_1)\subseteq \operatorname{int} K_2$, for all $0<\delta<\delta_1$. Now, by a compactness argument and the fact that $S_\delta(K_1)\subseteq \operatorname{int} K_2$ we conclude that for every $0<\delta<\delta_1$ there exists $\lambda_\delta>1$ such that $\lambda_\delta\, S_\delta(K_1)\subseteq \operatorname{int} K_2$.

Step 3: $\operatorname{vol}(\lambda_\delta\, S_\delta(K_1))>\operatorname{vol}(K_1)$, *for all* $0<\delta<\delta_1$.

Indeed,

$$\operatorname{vol}(\lambda_\delta\, S_\delta(K_1))=\frac{\lambda_\delta^n \operatorname{vol}(K_1)}{|\det(I_n-\delta H)|}.$$

Now, by using the inequality between arithmetic mean and geometric mean (denoted briefly *AM-GM inequality*) we obtain that $|\det(I_n-\delta H)|^{\frac{1}{n}}\le\frac{1}{n}\operatorname{tr}(I_n-\delta H)=1$, and so

$$\operatorname{vol}(\lambda_\delta\, S_\delta(K_1))\ge\lambda_\delta^n\operatorname{vol}(K_1)>\operatorname{vol}(K_1).$$

Therefore, if $(\frac{1}{n}I_n,0)\notin\operatorname{conv}\mathcal{A}$, then K_1 is not in *maximal volume position* inside K_2.

Note that the fact that $N\le n^2+n$ is deduced from the classical Caratheodory's theorem since $\{(y\otimes x-\frac{1}{n}I_n,y)\,;(x,y)\text{ is a contact pair}\}$ is contained in a (n^2+n-1) dimensional vector space.

□

Remark 2.1.

1. For every $K\subseteq\mathbb{R}^n$ with $\operatorname{vol}(K)>0$ it is easy to check that K is in maximal volume position inside $\operatorname{conv}(K)$.
2. We note that $K_1\subseteq K_2$ and K_1 is in maximal volume position inside $\operatorname{conv}(K_2)$ implies that K_1 is in maximal volume position inside K_2. The converse is not true, as the following example shows. Consider

$$\begin{aligned}K_1&=\{x\in\mathbb{R}^n;\|x\|_\infty=\max_{1\le i\le n}|x_i|\le 1\}\\ K_2&=K_1\cup 2\partial K_1.\end{aligned}$$

   It is trivial to see that K_1 is in maximal volume position inside K_2, K_1 is not in maximal volume position inside $\operatorname{conv}(K_2)$ and there is no decomposition of the identity as before, since there are no contact pairs for (K_1,K_2).

Corollary 2.2. *Let* $K_1\subseteq K_2\subseteq\mathbb{R}^n$ *be as in the theorem 1.3 If* $\operatorname{conv}(K_1)$ *is a polytope,* $\operatorname{conv}(K_2)$ *has* $\mathcal{C}^{(2)}$ *boundary with strictly positive curvature and* K_1 *is maximal volume position inside* $\operatorname{conv}(K_2)$*, then there exist* $z\in\operatorname{conv}(K_1)$*,* $N\in\mathbb{N}$*,* $N\le n^2+n$*,* (x_k,y_k) *contact pairs of* (K_1-z,K_2-z) *and* $\lambda_k>0$ *for all*

$k=1,\dots,N$ *such that*

$$\sum_{k=1}^{N} \lambda_k \, y_k \otimes x_k = \frac{1}{n} I_n$$

$$\sum_{k=1}^{N} \lambda_k \, y_k = \sum_{k=1}^{N} \lambda_k \, x_k = 0.$$

Proof. It is easy to prove that the fact that K_1 is in maximal volume position inside $\mathrm{conv}\,(K_2)$ implies that $\mathrm{conv}\,(K_1)$ is in maximal volume position inside $\mathrm{conv}\,(K_2)$ and $\partial K_1 \cap \partial(\mathrm{conv}\,(K_2)) = \partial(\mathrm{conv}\,(K_1)) \cap \partial(\mathrm{conv}\,(K_2))$. Therefore we can assume that K_1 is a polytope and K_2 is a convex which has a $\mathcal{C}^{(2)}$ boundary with strictly positive curvature. But notice that this case was studied by A. Giannopoulos, I. Perissinaki and A. Tsolomitis (see [G-P-T]) concluding the result needed. $\square$

Now we can ask if the existence of some kind of decomposition of the identity in $\mathbb{R}^n$ would imply that K_1 were the unique maximal volume position inside K_2, as it happens in the classical John's Theorem. It is well known that we can't expect such a thing, simply by considering simplices or octahedra inscribed in the cube. Theorem 1.4 shows, loosely speaking, that not only the existence of a "modified" *John's decomposition of the identity* for a pair (K_1, K_2) implies that K_1 is the unique "pseudo" maximal volume position inside K_2, but also that this "pseudo" *maximality* implies the existence of a "modified" *decomposition of the identity* too.

Proof of Theorem 1.4: As before, we can show that $\mathrm{int\,conv}\,(K_2) \neq \emptyset$. We can also assume $z=0$ and K_2 convex.

$(i) \Rightarrow (ii)$ Let $\mathcal{B} = \{(\frac{1}{2}(y\otimes x + x\otimes y), y) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\times\mathbb{R}^n \,;\, (x,y) \text{ is a contact pair}\}$. By using the maximality of the volume of K_1, the convexity of K_2 and since $0 \in \mathrm{int}\, K_2$ it is easy to prove that $\mathcal{B}$ is a non empty subset of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\times\mathbb{R}^n$. As in the proof of theorem 1.3, we will show that $(\frac{1}{n}I_n, 0) \in \mathrm{conv}\,(\mathcal{B})$.

Suppose, on the contrary, that $(\frac{1}{n}I_n, 0) \notin \mathrm{conv}\,(\mathcal{B})$. Then by using a separation theorem, there exist $H \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $\theta \in \mathbb{R}^n$ such that:

$$\langle \frac{1}{n} I_n, H\rangle_{\mathrm{tr}} + \langle 0, \theta\rangle > \frac{1}{2}(\langle y\otimes x, H\rangle_{\mathrm{tr}} + \langle x\otimes y, H\rangle_{\mathrm{tr}}) + \langle \theta, y\rangle$$

for all (x,y) contact pair. Therefore

$$\frac{1}{n}\,\mathrm{tr}\, H > \frac{1}{2}(\langle Hx, y\rangle + \langle x, Hy\rangle) + \langle \theta, y\rangle.$$

There is no loss of generality to assume that:

1. H is a symmetric matrix because in other case we could take $\tilde{H} = \frac{1}{2}(H + H^\star)$ which is a symmetric matrix that verifies that $\langle \tilde{H}x, y\rangle + \langle x, \tilde{H}y\rangle = \langle Hx, y\rangle + \langle x, Hy\rangle$ and therefore for every contact pair (x,y)

$$\frac{1}{n}\,\mathrm{tr}\, H > \langle Hx, y\rangle + \langle \theta, y\rangle.$$

(2) tr $H=0$ since in other case we can choose $\tilde{H} = H - \frac{\operatorname{tr} H}{n} I_n \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ which is a linear operator with trace zero that verifies:

$$\langle \tilde{H}x, y\rangle + \langle \theta, y\rangle = \langle Hx, y\rangle - \frac{\operatorname{tr} H}{n}\langle x, y\rangle + \langle \theta, y\rangle < 0 = \frac{\operatorname{tr} \tilde{H}}{n}$$

for all (x, y) contact pair.

Therefore there exist a symmetric matrix $H \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{tr} H = 0$ and $\theta \in \mathbb{R}^n$ such that:

$$0 > \langle Hx, y\rangle + \langle \theta, y\rangle$$

for all (x, y) contact pair. By using the linear map defined by the matrix H and $\theta \in \mathbb{R}^n$ we are going to construct a family of affine maps $T_\delta(\cdot) = S_\delta(\cdot) + b_\delta$, with S_δ symmetric positive definite matrix for all $0 < \delta < \delta_1$, such that $|\det S_\delta| \geq 1$ and $T_\delta(K_1) \subseteq \operatorname{int} K_2$, which contradicts the maximality of K_1.

By continuity, there exists a positive number $\delta_0 > 0$ such that $I_n - \delta H$ is invertible and symmetric positive definite for all $0 < \delta < \delta_0$. For each $0 < \delta < \delta_0$ we take $T_\delta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T_\delta(z) = (I_n - \delta H)^{-1}(z) + \delta(I_n - \delta H)^{-1}(\theta)$. By the same methods as in the proof of Theorem 1.3 , we can show that:

1. There exists $0 < \delta_1 \leq \delta_0$ such that $T_\delta(K_1) \cap \partial K_2 = \emptyset$, for all $0 < \delta < \delta_1$.
2. For every $0 < \delta < \delta_1$ there exists $\lambda_\delta > 1$ such that $\lambda_\delta T_\delta(K_1) \subseteq \operatorname{int} K_2$.
3. vol $(\lambda_\delta T_\delta(K_1)) > \operatorname{vol}(K_1)$, for all $0 < \delta < \delta_1$.

Note that these assertions contradict the maximality of K_1.

(ii) $\Rightarrow$ *(iii)* Let $T(\cdot) = S(\cdot) + a$ be such that $T(K_1) \subseteq K_2$, $a \in \mathbb{R}^n$ and S is a symmetric positive definite matrix. It is well known that we can find an orthogonal matrix $U \in O(n)$ and a diagonal matrix D with diagonal elements $\alpha_1, \ldots, \alpha_n > 0$ such that $S = U^\star D\, U$ and therefore

$$\operatorname{vol}(T(K_1)) = |\det(U^\star D\, U)| \operatorname{vol}(K_1) = \left(\prod_{k=1}^{n} \alpha_k\right) \operatorname{vol}(K_1).$$

Hence we have to estimate $\prod \alpha_k$. On the one hand, we obtain that

$$\langle U^\star D\, Ux, y\rangle = \sum_{j=1}^{n} \alpha_j \langle U^\star e_j, x\rangle\langle U^\star e_j, y\rangle$$

for all $x, y \in \mathbb{R}^n$, by straightforward computation.

On the other hand, if (x, y) is a contact pair then $\langle Tx, y\rangle \leq 1$ and therefore

$$\begin{aligned} 1 &= \sum_{k=1}^{N} \lambda_k \geq \sum_{k=1}^{N} \lambda_k \langle Tx_k, y_k\rangle = \sum_{k=1}^{N} \lambda_k \langle U^\star D\, Ux_k, y_k\rangle \\ &= \sum_{k=1}^{N} \lambda_k \sum_{j=1}^{n} \alpha_j \langle U^\star e_j, x_k\rangle\langle U^\star e_j, y_k\rangle = \sum_{j=1}^{n}\left(\alpha_j \sum_{k=1}^{N} \lambda_k \langle U^\star e_j, x_k\rangle \langle U^\star e_j, y_k\rangle\right) \\ &= \frac{1}{n}\sum_{j=1}^{n} \alpha_j \langle U^\star e_j, U^\star e_j\rangle = \frac{1}{n}\sum_{j=1}^{n} \alpha_j. \end{aligned}$$

Now by using the AM-GM inequality, we conclude that $1 \geq \frac{1}{n}\sum \alpha_j \geq (\prod \alpha_j)^{\frac{1}{n}}$, which implies that in (2) we obtain $\operatorname{vol}(T(K_1)) \leq \operatorname{vol}(K_1)$.

In addition to this, note that if T is such that $\operatorname{vol}(T(K_1)) = \operatorname{vol}(K_1)$, then, by the equality case in the AM-GM inequality we would have that $\alpha_1 = \ldots = \alpha_n = 1$, so $T = I_n + a$. Therefore we would obtain that

$$1 \geq \langle Tx, y\rangle = \langle x + a, y\rangle = 1 + \langle a, y\rangle$$

for all (x, y) contact pair and, in particular, $\langle a, y_k\rangle \leq 0$ for all (x_k, y_k) contact pair that appears in the *decomposition of the identity*. But we also would have that:

$$\sum_{k=1}^{N} \lambda_k \langle a, y_k\rangle = \sum_{k=1}^{N} \langle a, \lambda_k\, y_k\rangle = 0$$

which would imply that, $\langle a, y_k\rangle = 0$ for all (x_k, y_k) and then we would conclude that

$$\frac{1}{n}\,\langle a, a\rangle = \sum_{k=1}^{N} \lambda_k \langle a, y_k\rangle \langle a, x_k\rangle = 0.$$

Hence $T = I_n$. ☐

Corollary 2.3. *Let $K_1 \subseteq K_2$ be as in theorem 1.3 Fix $z \in \operatorname{int}\operatorname{conv}(K_2)$. Then the following assumptions are equivalents:*

(i) $\operatorname{vol}(K_1) = \max\{\operatorname{vol}(a + S(K_1)); a \in \mathbb{R}^n, a + S(K_1) \subseteq \operatorname{conv}(K_2)\}$*, where S runs over all symmetric positive definite matrices.*

(ii) For every $S \in GL(n)$ symmetric matrix and every $\theta \in \mathbb{R}^n$ there exists a contact pair (x, y) for $(K_1 - z, K_2 - z)$ such that

$$\frac{\operatorname{tr} S}{n} \leq \langle Sx, y\rangle + \langle \theta, y\rangle.$$

Proof. As before, we can show that $\operatorname{int}\operatorname{conv}(K_2) \neq \emptyset$. We can also assume $z = 0$ and K_2 convex.

(i) ⇒ *(ii)* By Theorem 1.4 there exist (x_i, y_i) contact pairs for $(K_1 - z, K_2 - z)$ and $\lambda_i > 0$ for all $i = 1, \ldots, N$ such that:

$$\sum_{k=1}^{N} \lambda_k\,(y_k \otimes x_k + x_k \otimes y_k) = \frac{1}{n} I_n \quad \text{and} \quad \sum_{k=1}^{N} \lambda_k y_k = 0.$$

Suppose that there would exist $S \in GL(n)$ symmetric matrix and $\theta \in \mathbb{R}^n$ such that for every (x, y) contact pair

$$\frac{\operatorname{tr} S}{n} > \langle Sx, y\rangle + \langle \theta, y\rangle.$$

Therefore

$$\begin{aligned}
\frac{\operatorname{tr} S}{n} &= \frac{\operatorname{tr} S}{n} + \langle \theta, \sum_{i=1}^{N} \lambda_i y_i\rangle = \langle (S, \theta), (\frac{1}{n} I_n, \sum_{i=1}^{N} \lambda_k y_k)\rangle = \\
&= \sum_{i=1}^{N} \lambda_i \langle (S, \theta), (\frac{1}{2}(y_i \otimes x_i + x_i \otimes y_i), y_i)\rangle = \sum_{i=1}^{N} \lambda_i (\langle S x_i, y_i\rangle + \langle \theta, y_k\rangle) \\
&< \sum_{i=1}^{N} \lambda_i \frac{\operatorname{tr} S}{n} = \frac{\operatorname{tr} S}{n}
\end{aligned}$$

which leads us to a contradiction.

$(ii) \Rightarrow (i)$ By using the hypothesis, for every $H \in GL(n)$, $\theta \in \mathbb{R}^n$, there exists a contact pair such that

$$\frac{1}{n}\operatorname{tr} H > \frac{1}{2}(\langle Hx, y\rangle + \langle x, Hy\rangle) + \langle \theta, y\rangle$$

which make that $(\frac{1}{n}, 0) \in \operatorname{conv}(\{(\frac{1}{2}(y\otimes x + x\otimes y), y) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\times\mathbb{R}^n\,;$ where (x, y) is a contact pair $\})$ $\square$

Remark 2.4.

(1) If K_1 is in maximal volume position inside $\operatorname{conv}(K_2)$, then K_1 is unique if we only consider affine transformations given by symmetric, positive definite matrices. Indeed, this is due to the fact that $\frac{1}{n}I_n = \sum_{k=1}^N \lambda_k\, y_k\otimes x_k$ implies that $\frac{1}{n}I_n = \sum_{k=1}^N \lambda_k\, x_k\otimes y_k$.

(2) If we suppose either $K_1 = B_2^n$ or $\operatorname{conv}(K_2) = B_2^n$ in the last theorem, we obtain a stronger conclusion, since the existence of contact pairs (x_k, y_k) and $\lambda_k > 0$ such that

$$\sum_{k=1}^N \frac{\lambda_k}{2}\,(y_k\otimes x_k + x_k\otimes y_k) = \frac{1}{n}I_n \quad\text{and}\quad \sum_{k=1}^N \lambda_k\, y_k = 0$$

is equivalent to the fact that $\operatorname{vol}(K_1) = \max\{\operatorname{vol}(a + T(K_1))$ such that $a + T(K_1) \subseteq \operatorname{conv}(K_2)$, $a \in \mathbb{R}^n$ and $T \in GL(n)\}$ and this maximum is only attained at K_1, up to orthogonal transformation (i.e. if $\operatorname{vol}(T(K_1)) = \operatorname{vol}(K_1)$, then T is an orthogonal transformation). This is the classical John's result. Let's see it briefly.

Suppose that there exists a *decomposition of the identity* (in the sense of (1)). If we take $\operatorname{conv}(K_2) = B_2^n$ and T is an affine transformation such that $T(K_1) \subseteq B_2^n$, then there exist orthogonal matrices U, V, a diagonal matrix D with diagonal elements $\alpha_1, \ldots, \alpha_n > 0$ and $a \in \mathbb{R}^n$ such that $T(\cdot) = VDU(\cdot) + a$. Now if we choose $\tilde{T}(\cdot) = U^\star DU(\cdot) + (VU)^\star(a)$ then it is easy to check that this map verifies:

(a) $U^\star DU$ is a symmetric positive definite matrix.
(b) $\tilde{T}(K_1) \subseteq (VU)^\star(B_2^n) = B_2^n$ (since $\tilde{T}(\cdot) = (VU)^\star T(\cdot)$).
(c) $\operatorname{vol}(\tilde{T}(K_1)) = \operatorname{vol}(T(K_1))$.

Therefore by using $(ii) \Rightarrow (iii)$ in theorem 1.4 and since $\tilde{T}$ satisfies (a) and (b) we conclude that

$$\operatorname{vol}(T(K_1)) = \operatorname{vol}(\tilde{T}(K_1)) \leq \operatorname{vol}(K_1)$$

and the equality is only attained if $\tilde{T} = I_n$, and so T is an orthogonal transformation.

Note that a similar reasoning can be applied to the case $K_1 = B_2^n$.

3. Some estimates for the outer volume ratio of compact sets

We can extend the notion of *volume ratio* to a pair $(K_1, K_2) \subseteq \mathbb{R}^n \times \mathbb{R}^n$, where K_2 is a convex body and K_1 is a compact set with $\mathrm{vol}\,(K_1) > 0$, simply by

Definition 3.1. Let $K_1 \subseteq \mathbb{R}^n$ be compact set with $\mathrm{vol}\,(K_1) > 0$ and $K_2 \subseteq \mathbb{R}^n$ be a convex body. We define *outer volume ratio* as

$$vr(K_2; K_1) = \inf \left\{ \frac{\mathrm{vol}\,(K_2)^{\frac{1}{n}}}{\mathrm{vol}\,(T(K_1))^{\frac{1}{n}}}; T\,\text{affine transformation with}\,T(K_1) \subseteq K_2 \right\}.$$

It is quite easy to show that we cannot expect any upper estimate without assuming extra assumptions. We are going to introduce an index for compact sets with positive volume in order to get general bounds, depending only on the dimension and on the index, for the outer volume ratio with respect to a convex body.

We recall that a set $K \subseteq \mathbb{R}^n$ is p-convex, $(0 < p \leq 1)$ if $\lambda x + \mu y \in K$, for every $x, y \in K$ and for every $\lambda, \mu \geq 0$ such that $\lambda^p + \mu^p = 1$. The p-convex hull of a set K, which we denote by $p - \mathrm{conv}\,(K)$, is defined as the intersection of all p-convex sets that contain K. It is easy to see that $0 \in \overline{p - \mathrm{conv}\,(K)}$.

Definition 3.2. Let $K \subseteq \mathbb{R}^n$ a compact set. We define $p(K)$ as

$$p(K) = \begin{cases} \sup\{p \in (0,1]\,;\ \exists a \in \mathbb{R}^n \ \text{ with } \ p - \mathrm{conv}\{(\mathrm{ext}K) - a\} \subseteq K - a\} & if it exists \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{ext}K$ denotes the set of extreme points of K.

Remark 3.3.

1. If $p \in (0,1]$ verifies that there exist an $a \in \mathbb{R}^n$ such that $p - \mathrm{conv}\{(\mathrm{ext}K) - a\} \subseteq K - a$ then $a \in K$, since 0 is inside the closure of $p - \mathrm{conv}\{(\mathrm{ext}K) - a\}$, which is embedded in $K - a$ and so $a \in K$.
2. $p(K)$ is an affine invariant of K, i.e. if $T = a + S$ is an affine transformation on $\mathbb{R}^n$ with $a \in \mathbb{R}^n$ and $S \in GL(n)$ then $p(T(K)) = p(K)$.
3. The supremo in the last definition can be replaced by maximum, simply by using compactness and continuity arguments.
4. If K is a p-convex body with $0 < p \leq 1$ then $p(K) \geq p$, but if $0 < p < 1$ then there are compact sets K with $p(K) \geq p$ which are not p-convex. Notice that $p(K) = 1$ if and only if K is convex, simply by using Krein-Milman's theorem.

Now we are going to state and prove some upper estimates for the volume ratio of a pair (K_1, K_2) where K_1 is a compact set with $\mathrm{vol}\,(K_1) > 0$ and $p(K_1) > 0$, and K_2 is a convex body. We can assume that K_1 is in maximal volume position inside K_2, since in other case, there would exist an affine transformation T such that $T(K_1)$ would be in maximal volume position inside K_2 and therefore K_1 would work with the pair $(T(K_1), K_2)$. Hence if $p(K_1) = p$,

$$vr(K_2; K_1) = \frac{\mathrm{vol}\,(K_2)^{\frac{1}{n}}}{\mathrm{vol}\,(K_1)^{\frac{1}{n}}} \leq \frac{\mathrm{vol}\,(K_2)^{\frac{1}{n}}}{\mathrm{vol}\,(p - \mathrm{conv}\,\{(\mathrm{ext}K_1) - a\})^{\frac{1}{n}}}$$

for some $a \in K_1$. Therefore

$$vr(K_2;K_1) \leq \frac{\mathrm{vol}\,(K_2 - a)^{\frac{1}{n}}}{\mathrm{vol}\,(\mathrm{conv}\,\{(\mathrm{ext}K_1) - a\})^{\frac{1}{n}}} \frac{\mathrm{vol}\,(\mathrm{conv}\,\{(\mathrm{ext}K_1) - a\})^{\frac{1}{n}}}{\mathrm{vol}\,(p - \mathrm{conv}\,\{(\mathrm{ext}K_1) - a\})^{\frac{1}{n}}}.$$

It can be shown that $\mathrm{conv}\,\{(\mathrm{ext}K_1) - a\} = \mathrm{conv}\,(K_1 - a)$ and since $\mathrm{conv}\,(K_1 - a)$ is in maximal volume position inside $K_2 - a$ we get

$$vr(K_2;K_1) \leq vr(K_2; \mathrm{conv}\,(K_1)) \frac{\mathrm{vol}\,(\mathrm{conv}\,\{(\mathrm{ext}K_1) - a\})^{\frac{1}{n}}}{\mathrm{vol}\,(p - \mathrm{conv}\,\{(\mathrm{ext}K_1) - a\})^{\frac{1}{n}}}.$$

It is easy to check that $\mathrm{conv}\,\{(\mathrm{ext}K_1) - a\} \subseteq n^{\frac{1}{p}-1}\,(p - \mathrm{conv}\,\{(\mathrm{ext}K) - a\})$. Indeed, since $a \in K_1$ then

$$\mathrm{conv}\,\{(\mathrm{ext}K_1) - a\} = \mathrm{conv}\,(K_1 - a) = \mathrm{conv}\,\{\cup_{x \in K_1} [0, x - a]\}$$

and we can use a stronger version of Caratheodory's theorem appearing in [E] that asserts that for every $x \in \mathrm{conv}\,\{(\mathrm{ext}K_1) - a\}$ there exist $x_i \in (\mathrm{ext}K) - a$ and $\alpha_i \geq 0$, $i = 1, \ldots, n$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$ and $\sum_{i=1}^{n} \alpha_i = 1$. Therefore

$$\left(\sum_{i=1}^{n} \alpha_i^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-1} \sum_{i=1}^{n} \alpha_i,$$

which implies that $x \in n^{\frac{1}{p}-1} p - \mathrm{conv}\,\{(\mathrm{ext}K_1) - a\}$. On the other hand a result of Giannopoulos, Perissinaki and Tsolomitis (see [G-P-T]) shows that $vr(K_2; \mathrm{conv}\,\{(\mathrm{ext}K_1) - a\}) \leq n$ and thus we sumarize all these things in the following result

Proposition 3.4. *Let $K_1, K_2 \subseteq \mathbb{R}^n$ be such that K_1 is a compact set with* $\mathrm{vol}\,(K_1) > 0$*, $p(K_1) = p > 0$ and K_2 a convex body. Then*

$$vr(K_2;K_1) \leq n^{\frac{1}{p}}.$$

Next we are going to prove that if K_1 or K_2 has some kind of symmetry properties then this general estimate can be slightly improved by using decompositions of the identity in the sense of theorem 1.3, following the spirit of K.M. Ball (see [Ba1]) and A. Giannopoulos, I. Perisinaki, A. Tsolomitis ([G-P-T]). We start with a result which can be found in [G-P-T] and whose proof involves Cauchy-Binet formula.

Lemma 3.5. *Let $\lambda_1, \ldots, \lambda_N > 0$. Let $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ be vectors in $\mathbb{R}^n$ satisfying $\langle x_k, y_k \rangle = 1$, for all $k = 1, \ldots, N$ and $\displaystyle\sum_{k=1}^{N} \lambda_k\, y_k \otimes x_k = I_n$. Then $D_x D_y \geq 1$, where D_x and D_y are defined by*

$$D_x = \inf \left\{ \frac{\det(\sum_{k=1}^{N} \lambda_k \alpha_k x_k \otimes x_k)}{\prod_{k=1}^{N} \alpha_k^{\lambda_k}} ; \alpha_k > 0, k = 1, \ldots, N \right\} \tag{3}$$

$$D_y = \inf \left\{ \frac{\det(\sum_{k=1}^{N} \lambda_k \alpha_k y_k \otimes y_k)}{\prod_{k=1}^{N} \alpha_k^{\lambda_k}} ; \alpha_k > 0, k = 1, \ldots, N \right\}. \tag{4}$$

Proposition 3.6. *Let* $K_1, K_2 \subseteq \mathbb{R}^n$ *be such that* K_1 *is a symmetric compact set with* $\mathrm{vol}\,(K_1) > 0$, $p(K_1) = p > 0$ *and* K_2 *is a symmetric convex body. Then*

$$vr(K_2; K_1) \le n!^{\frac{1}{n}} n^{\frac{1}{p}-1}.$$

Proof. First of all it is easy to check that we can assume that K_1 and K_2 are centrally symmetric and so it is $\mathrm{ext}\,K_1$. By using the same arguments than before we conclude that

$$vr(K_2; K_1) \le vr(K_2; \mathrm{conv}\,(K_1)) n^{1/p-1}.$$

Next we are going to give an upper estimate for $vr(K_2; L)$, where K_2 and $L = \mathrm{conv}\,(K_1)$ are centrally symmetric convex bodies and L is in maximal volume position inside K_2.

By using theorem 1.3, we can find contact pairs (x_i, y_i) and $\lambda_i > 0$, for all $i = 1, \ldots, N$, $N \le n^2 + n$, such that

$$\sum_{k=1}^{N} \lambda_k \, y_k \otimes x_k = I_n \qquad \text{and} \qquad \sum_{k=1}^{N} \lambda_k \, y_k = 0.$$

If we take $X = \mathrm{conv}\,\{\pm x_1, \ldots, \pm x_N\} \subseteq L$ and $Y = \{y \in \mathbb{R}^n;\ |\langle y, y_k\rangle| \le 1 \ k = 1, \ldots, N\}$ $K_2 \subseteq Y$, we obtain that

$$vr(K_2; L) = \frac{\mathrm{vol}\,(K_2)^{\frac{1}{n}}}{\mathrm{vol}\,(L)^{\frac{1}{n}}} \le \frac{\mathrm{vol}\,(Y)^{\frac{1}{n}}}{\mathrm{vol}\,(X)^{\frac{1}{n}}}.$$

Therefore if we find some upper estimate for $\mathrm{vol}\,(Y)$ and lower estimate for $\mathrm{vol}\,(X)$ we will obtain some upper estimates for $vr(K_2; K_1)$.

Claim 1: $\mathrm{vol}\,(Y) \le \frac{2^n}{\sqrt{D_y}}$

Consider $g_j : \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, \ldots, N$, defined by $g_j(t) = \chi_{[-1,1]}(t)$. By using the Brascamp-Liev inequality (see [Bar]) we obtain that

$$\int_{\mathbb{R}^n} \prod_{k=1}^{N} (g_k(\langle x, y_k\rangle))^{\lambda_k} \, dx \le \frac{1}{\sqrt{D_y}} \prod_{k=1}^{N} \left(\int_{\mathbb{R}} g_k(t) \, dt \right)^{\lambda_k} = \frac{1}{\sqrt{D_y}} \left(\int_{-1}^{1} dt \right)^{\sum \lambda_k}$$

where D_y was defined in (4). On the other hand, we conclude that

$$\int_{\mathbb{R}^n} \prod_{k=1}^{N} (g_k(\langle x, y_k\rangle))^{\lambda_k} \, dx = \int_{\mathbb{R}^n} \chi_Y(x) \, dx = \mathrm{vol}\,(Y).$$

Therefore $\mathrm{vol}\,(Y) \le \frac{2^n}{\sqrt{D_y}}$

Claim 2: $\mathrm{vol}\,(X) \ge 2^n \frac{\sqrt{D_x}}{n!}$

We define for every $x \in \mathbb{R}^n$

$$N(x) = \inf \left\{ \sum_{k=1}^{N} |\alpha_k|; \quad x = \sum_{k=1}^{N} \alpha_k \, x_k \right\}$$

which is an integrable function that verifies

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-N(x)}\,dx &= \int_{\mathbb{R}^n} \sup\left\{\prod_{k=1}^N e^{-\alpha_k^p}\,;\ \alpha_k\geq 0,\ \ x=\sum_{k=1}^N \alpha_k x_k\right\}dx\\ &= \int_{\mathbb{R}^n} \sup\left\{\prod_{k=1}^N f_k(\theta_k)^{\lambda_k}\,;\ \ x=\sum_{k=1}^N \lambda_k\theta_k x_k\right\}dx\end{aligned}$$

where $f_k:\mathbb{R}\longrightarrow\mathbb{R}$ is defined by $f_k(t)=e^{-|t|}$. Now, if we use the reverse of the Brascamp-Liev inequality (see [Bar]) we can assert that

$$\begin{aligned}\int_{\mathbb{R}^n} \sup\left\{\prod_{k=1}^N f_k(\theta_k)^{\lambda_k}\,;\ \ x=\sum_{k=1}^N \lambda_k\theta_k x_k\right\}dx &\geq \sqrt{D_x}\prod_{k=1}^N\left(\int_{\mathbb{R}} f_k(t)\,dt\right)^{\lambda_k}\\ &= \sqrt{D_x}\prod_{k=1}^N 2^{\lambda_k}=\sqrt{D_x}2^n \qquad (5)\end{aligned}$$

where D_x was defined in (3).

On the other hand, we can compute directly the integral of $e^{-N(x)}$ by

$$\int_{\mathbb{R}^n} e^{-N(x)}\,dx=\int_{\mathbb{R}^n}\int_{N(x)}^{+\infty} e^{-t}\,dt\,dx=\int_0^{+\infty} e^{-t}\int_{\{N(x)\leq t\}} dxdt.$$

It is easy to check that $\{x\in\mathbb{R}^n;\ N(x)\leq t\}=tX$, for all $t>0$, and hence

$$\int_{\mathbb{R}^n} e^{-N(x)}\,dx=\int_0^{+\infty} e^{-t}\,t^n\,\mathrm{vol}\,(X)\,dt=n!\,\mathrm{vol}\,(X).$$

So, combining (5) and (3) we conclude the desired lower estimate for $\mathrm{vol}\,(X)$ and by using Claim 1, Claim 2 and lemma 3.4 we obtain that

$$vr(K_2;L)\leq n!^{\frac{1}{n}}$$

and hence, the result holds. □

By using similar arguments we can prove the following result

Proposition 3.7. *Let $K_1,K_2\subseteq\mathbb{R}^n$ are such that K_1 is a compact set with* $\mathrm{vol}\,(K_1)>0$*, $p(K_1)=p>0$ and K_2 is a convex body, then:*

1. *If K_1 is symmetric, $vr(K_2;K_1)\leq\frac{e}{2}\,(n!)^{\frac{1}{n}}\,n^{\frac{1}{p}-1}$.*
2. *If K_2 is symmetric,$vr(K_2;K_1)\leq 2\,(n!)^{\frac{1}{n}}\,n^{\frac{1}{p}-1}$.*

Proof. (1) Take $\tilde{g}_j(t)=e^t\chi_{(\infty,1]}(t)$ instead of $g_j(t)$ in the proof of proposition 3.5. (2) Take $\tilde{f}_j(t)=e^{-t}\chi_{[0,+\infty)}(t)$ instead of $f_j(t)$ in the proof of proposition 3.5 and substitute $N(x)$ by

$$\tilde{N}(x)=\begin{cases}\inf\left\{\sum_{k=1}^N\alpha_k;\ \alpha_k\geq 0,\ \ x=\sum_{k=1}^N\alpha_k\,x_k\right\} & \text{if it exists}\\ +\infty & \text{otherwise.}\end{cases}$$

□

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address, (Jesús Bastero): `bastero@posta.unizar.es`

E-mail address, (Miguel Romance): `mromance@posta.unizar.es`