The instability growth leading to a liquid sheet breakup

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The instability growth leading to a liquid sheet breakup has been studied with the objective of improving the understanding of the fundamental mechanisms of atomization. A three-dimensional Lagrangian code based on vortex dynamics methods has been implemented to track the air/liquid interfaces treated as inviscid vortex sheets. The results of these numerical simulations indicate a possible explanation for the presence of transverse and longitudinal filaments observed in liquid sheet air-assisted atomization experiments. © 1998 American Institute of Physics.

INTRODUCTION

Spray flows are common in many daily applications, but their presence in combustion environments is probably the aspect that attracts larger attention. Although most air-blast commercial atomizers have an axisymmetric design, two-dimensional geometries where a liquid sheet is surrounded by a coflow of high speed air, are useful in research studies. The basic physical mechanisms that cause the liquid breakup are the same in both configurations, but planar designs are easier to visualize and simulate.

Liquid film disintegration was studied early by Hagerty and Shea in a high aspect ratio sheet experiment, by Taylor in an axisymmetric configuration, and by Dombrowski and collaborators in a fan spray geometry. It has not been until recently, with the works of Rizk and Lefebvre, Arai and Hashimoto, and Stapper and Samuelsen, and Lozano et al. among others that high aspect ratio two-dimensional air-blast designs have been employed to explore liquid sheet breakup phenomena.

With a high speed air flow in the presence of a liquid sheet at lower speed, significant shear develops at the interface between the two fluids, particularly in the vicinity of the nozzle. The shear gives rise to Kelvin-Helmholtz type instability, causing the sheet to oscillate, and as the disturbance amplitude grows, to break. At lower shearing rates, sacklike structures develop and layer upon each other. Stretching and surface tension eventually cause the sheet to tear into ligaments which are nominally oriented in the spanwise direction (parallel to the nozzle slit). As the relative air/liquid velocity ratio increases, the breakup transitions to another regime. Streamwise vorticity becomes of equal or greater importance to the spanwise component, causing the cellular structures. In this breakup regime, ligaments oriented parallel to the flow direction predominate over spanwise ligaments. At sufficiently high air velocities, the streamwise ligaments recede to the nozzle tip and sheet rupture into ligaments is immediate. This rupture mode is identified by Stapper and Samuelsen as “stretched streamwise ligament breakup.” In principle, the breakup mechanisms appear to be to some extent independent of the liquid viscosity and surface tension; however, these properties do affect the ultimate drop size distribution.

Modeling air-blast atomizers has received very little attention. Rangel and Sirignano and Yang have recently analyzed the linear and nonlinear 2-D evolution of infinite liquid sheets considering perturbations in symmetric and antisymmetric modes. However, as noted above, the three-dimensional disturbance growth due to the interaction between spanwise and streamwise vorticity is known to have a definite influence on the breakup. Three-dimensional simulations of mixing layers have been successfully attempted, but in general density jumps or surface tension effects have not been considered. In this paper, it is proposed to model the interfacial instability growth in the flow using three-dimensional vortex dynamics.

THE VORTEX DYNAMICS MODEL

To track the air/liquid interface, a three-dimensional vortex dynamics method has been developed and implemented. Previous experimental studies tend to indicate that the liquid viscosity may have little or no effect on the sheet breakup. The present model exploits the largely inviscid nature of this flow and the fact that for large Reynolds number flows the vorticity is mainly concentrated within very thin layers on both sides of the interface between the two fluids. This vorticity region is then considered to behave as a vortex sheet while the fluids on either side of it may be treated as incompressible and irrotational. If both fluids are incompressible, the volume rate of expansion is zero throughout the flow field, and the velocity field satisfies the following Poisson equation:

$$\nabla^2 \mathbf{u} = -\nabla \times \mathbf{\omega},$$

where $\mathbf{\omega}$ is the vorticity defined as the curl of the velocity field $\mathbf{u}$.

A general Green's function solution for $\mathbf{u}$ exists and is commonly known as the Biot–Savart law:

$$
\mathbf{u}(\mathbf{x},t) = -\frac{1}{4\pi} \int_V \frac{(\mathbf{x} - \mathbf{r}) \times \mathbf{\omega}(\mathbf{r},t)}{|\mathbf{x} - \mathbf{r}|^3} \, d\mathbf{r} + \nabla \phi,
$$

where $\mathbf{x}$ is a position vector from the origin to any arbitrary field point, and the potential function $\phi$ is a harmonic function for incompressible flows, chosen to satisfy boundary conditions.
conditions. Application of Eq. (2) is limited because the vorticity field is generally unknown. This formalism, however, is particularly useful for the analysis of those flow fields in which the vorticity is confined to small or thin regions as in the present case.

It is useful to envision a physical model for the interface. A portion of it is depicted in Fig. 1. The lower fluid is moving with velocity \( u_1 \) and the upper fluid with \( u_2 \). The vortex sheet which in the limit contains a density discontinuity is differentially thin with thickness \( \eta \) of the order of negative powers of the Reynolds numbers. A mean vortex sheet velocity is defined to move with the average velocity directly above and below it

\[
\mathbf{u}_i = \frac{1}{2} (\mathbf{u}_1 + \mathbf{u}_2) |_{i} .
\]

To describe interface variables, a coordinate system is defined on its surface. The surface area element is

\[
d s = d\alpha d\beta = (\hat{a} \times \hat{b}) da \cdot db .
\]

The scalars \( da \) and \( db \) are differential arc lengths of the surface coordinates, \( \hat{a} \) and \( \hat{b} \) are unit vectors initially defined to be orthogonal; \( \hat{n} \) is the unit normal to the surface pointing from fluid 1 to fluid 2.

The vortex sheet strength vector is defined as an integral of the vorticity through the sheet:

\[
\hat{n} \times \mathbf{\gamma} = \int_{\eta/2}^{\pi/2} \mathbf{\omega} \, d\mathbf{n} .
\]

This definition implies the following expression:

\[
\mathbf{\gamma} = (\mathbf{u}_2 - \mathbf{u}_1) \, t ,
\]

which defines \( \mathbf{\gamma} \) as the local velocity jump across the interface. If the sheet thickness tends to zero, the vorticity field is nonzero only at the interface surface and can in accordance to Eq. (5) be represented by an integral over the three-dimensional Dirac delta function

\[
\mathbf{\omega}(\mathbf{x}, t) = \int_S \left( \hat{n} \times \mathbf{\gamma} \right) \delta(\mathbf{x} - \mathbf{r}(a, b)) \, ds ,
\]

where \( \mathbf{r}(a, b) \) is a position vector from the origin to an arbitrary interface point. Upon insertion of Eq. (7) in the Biot-Savart integral the following expression similar to that described by Leonard \(^13\) is finally obtained:

\[
\mathbf{u}_i(\mathbf{x}, t) = \frac{1}{4\pi} \int_J \frac{(\mathbf{x} - \mathbf{r}) \times [\mathbf{\gamma} \times (\hat{a} \times \hat{b})]}{|\mathbf{x} - \mathbf{r}|^3} \, da \cdot db .
\]

All variables are given in space coordinates, referenced to a fixed origin. The integration is over all interface points, and thus the 3-D problem has been reduced to a calculation on a two-dimensional manifold. The spatial evolution in time of the interface is integrated in a lagrangian sense and the unit vectors \( \hat{a} \) and \( \hat{b} \) become in general nonorthogonal. The instantaneous interface location is obtained from

\[
\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{u}_i(\mathbf{x}, t') \, dt' .
\]

For an inviscid flow with uniform density, the theorems of Kelvin and Helmholtz state that vortex lines move as material lines with invariant circulation. Exploiting these results, it is customary to interpret vortex sheets as a collection of vortex filaments with constant circulation including a conservation equation to account for the structure of the vorticity in each filament. Examples are given in Leonard \(^13,14\) and Ashurst and Meiburg. \(^16\) The flow field considered here need not have constant density and an equation for the three-dimensional evolution of the vorticity, or equivalently of the vortex sheet strength is required. This equation must describe the advection and redistribution of the vorticity among the mesh points. However, the two-phase nature of the flow enables also the possibility of vorticity creation at the interface due both to the baroclinic term and to the interaction between the gravity field and the density discontinuity (Rayleigh–Taylor instability).

A possible strategy to obtain the sheet strength evolution equation\(^19\) is to write a momentum balance for each side of the interface, i.e., an Euler equation, since the flow is assumed inviscid, subtract one from the other and then express the velocities on each fluid in terms of the interface quantities, \( \mathbf{\gamma} \) as defined in Eq. (6) and the interface velocity.

Following this procedure, introducing surface tension forces to account for the pressure discontinuity across the interface and enforcing the condition that \( \mathbf{\gamma} \) is by definition always tangent to the interface surface the following equation can be derived:

\[
\frac{D \mathbf{\gamma}}{Dt} = \tilde{\mathbf{P}} \left[ -2A \frac{D \mathbf{u}_i}{Dt} (\mathbf{\gamma} \cdot \nabla) \mathbf{u}_i + 2A g j - A \left( \frac{\mathbf{\gamma} \cdot \nabla}{2} \right) \mathbf{\gamma} - 2\sigma \rho_1 + \rho_2 \right] \nabla H - \left( \mathbf{\gamma} \cdot \frac{D \hat{n}}{Dt} \right) \hat{n} ,
\]

where \( \tilde{\mathbf{P}} \) is the projection tensor defined as

\[
P_{ij} = \delta_{ij} - n_i n_j .
\]

\( H \) is the local mean surface curvature and \( A \) is the Atwood density ratio:

\[
A = (\rho_1 - \rho_2) / (\rho_1 + \rho_2) .
\]
Alternative derivations of Eq. (10) are possible. The present one is valid for any point at the mathematical vorticity-containing interface and for this reason it does not include the contribution due to material-element surface area changes $\gamma(V \cdot u_f)$ which appears in Wu.\textsuperscript{17} This contribution must be included in any equation for the strength associated to a material surface element. On the other hand, Eq. (10) is correct for our purposes, since the interface is modeled as a set of Lagrangian mesh cells, each carrying the circulation of the whole material element at each cell.

Equations (8), (9), and (10) constitute the set of coupled integro-differential equations that has to be solved numerically to calculate the interface evolution. These expressions are in dimensional form. They can be nondimensionalized with $T = L / \gamma_0$ and $L = 2 \pi \lambda$, where $\lambda$ is the wavelength of the initial perturbation in the streamwise direction. The initial value of sheet strength, $\gamma_0$, is chosen to match experimental measurements or linear stability theory. The disturbance wavelength is taken from experimental measurements or linear stability theory. Alternatively, the liquid sheet thickness $\delta$ can be selected as the length scale to obtain the dimensionless equations. Upon inserting these definitions into Eq. (6), two dimensionless parameters appear:

Weber Number: $W_e = (\rho_1 + \rho_2) L \gamma_0^2 / \sigma$, 

Froude Number: $F_r = \gamma_0 / \sqrt{4gL}$, \quad (13)

due to the presence of surface tension and gravity effects. Note however that the liquid sheet break up process does not require the presence of gravity, and that similar results are observed atomizing horizontal\textsuperscript{1} or vertical\textsuperscript{10} liquid sheets, because the perturbation wavelength is usually quite small (on the order of 1 mm). This considered, gravity effects have been neglected in the simulations presented in this work.

THE NUMERICAL IMPLEMENTATION

Discretization

The above set of Eqs. (8), (9), and (10) has to be discretized and solved numerically. In this implementation, a McCormack type algorithm has been used to discretize the vortex sheet strength evolution equation. A two-step explicit predictor-corrector scheme has been used to temporally advance the surface mesh and a nonregular nonorthogonal mesh has been used to solve the integral in Eq. (8) approximated by a double summation over all surface elements.

It is well known\textsuperscript{20-22} that when solving these types of unstable flows inclusion of some type of stabilization mechanism such as viscosity or surface tension is required. Otherwise, the solution becomes numerically unstable. Following the form first suggested by Rosenhead\textsuperscript{20} a `$8$'-parameter has been introduced in the denominator of the Biot–Savart integral. Without the smoothing of this parameter, small wavelength disturbances, such as those induced by the discretization of the sheet and round-off errors, grow unbounded at a rate proportional to their wave number. The value of $\delta$ has been dynamically adjusted to be twice the larger side of each cell. As shown in previous studies (see Triggvason et al.\textsuperscript{21}) and confirmed in this work, a larger $\delta$ value produces a damping of the small scale roll-up of the vortical structures (although maintaining the large scale evolution). However, the program produced nonphysical results due to the growth of spurious perturbations whenever $\delta$ was assigned a value below the size of a cell.

Additionally, an artificial viscosity of the form
\begin{equation}
 w_i = -c(\partial a)^2 \left[ \begin{array}{c} \partial y_i \\ \partial a \\ \partial b \end{array} - c(\partial b)^2 \right] \left[ \begin{array}{c} \partial y_i \\ \partial a \\ \partial b \end{array} \right] \left[ \begin{array}{c} \partial y_i \\ \partial a \\ \partial b \end{array} \right], \quad (14)
\end{equation}

where $c$ is a constant, has been included in the vortex sheet strength evolution equation to smooth the unstable characteristics arising from the advective terms. The value of $c$ lies usually in the range 0.05 < $c$ < 2.0 (Potter\textsuperscript{23}). In this study, variations of $c$ between 0.25 and 1 have resulted in essentially identical results, and a value of 0.5 has been selected for the calculations.

All the parameters have been calculated in space coordinates. In order to do it, a generic vector $\mathbf{k}$ on the surface can be expressed as (see, for example, Aris\textsuperscript{24}):
\begin{equation}
 \mathbf{k} = \frac{1}{(\partial \times \mathbf{b})^2} \left[ [(\mathbf{k} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{k} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{b})] \mathbf{a} \\
 + [(\mathbf{k} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{k} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b})] \mathbf{b} \right], \quad (15)
\end{equation}

and, hence, terms involving spatial gradients of the form $(\mathbf{\gamma} \cdot \mathbf{\nabla}) \varphi$ can be written as
\begin{equation}
 (\mathbf{\gamma} \cdot \mathbf{\nabla}) \varphi = \frac{1}{(\mathbf{a} \times \mathbf{b})^2} \left[ [(\mathbf{\gamma} \cdot \mathbf{a}) - (\mathbf{\gamma} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{b})] \frac{\partial \varphi}{\partial a} \\
 + [(\mathbf{\gamma} \cdot \mathbf{b}) - (\mathbf{\gamma} \cdot \mathbf{a})(\mathbf{a} \cdot \mathbf{b})] \frac{\partial \varphi}{\partial b} \right]. \quad (16)
\end{equation}

Boundary conditions and mesh size

In accordance with Eq. (8), the domain of the Biot–Savart integral extends to the entire fluid interface. This implies that to exactly model the planar atomizer under study, boundary conditions should be included at the nozzle with the proper initial flow conditions at this section. Moreover, in the downstream direction the integration domain should extend to infinity. Some examples appear in the literature where the nozzle exit is simulated by releasing vortex lines or rings from an upstream boundary at fixed intervals of time, tracking them until they reach a lower boundary. Leonard\textsuperscript{13} used this method for the planar Kelvin–Helmholtz problem. A simpler approach adopted in the present case is to assume a periodic flow. A finite sheet surface is extended ideally to infinity by adding periodic images to its edges. The evolution of the reduced surface is calculated but including the influence on its mesh points originated from all the points in the replicated images, thus increasing accuracy while minimizing computer cost. An example of this approach is provided by Ashurst and Meiburg\textsuperscript{16} for the planar Kelvin–Helmholtz problem. The first approach is arguably a better physical model. However, its application is complicated by the uncertainty in the treatment of both upstream and downstream boundaries.
The geometry simulated in the present work consists of two parallel vortex sheets representing the two air/water interfaces using $21 \times 21$ grid points on each surface, extended by six images in each one of the upstream, downstream, and transverse directions yielding a domain of $260 \times 260$ grid points. Initially the two air/water interfaces are separated by a distance corresponding to four mesh points. To evaluate the influence of grid size, a test case was also calculated with grids of $11 \times 11$ and $31 \times 31$ points. For the three grid sizes the same temporal evolution of the interfaces was obtained, varying only in spatial resolution.

To consider the influence of edge effects, a case was also studied with a modified geometry. The two interfaces were connected forming a flattened cylinder of $21 \times 21$ surface points joined in their transverse edges by circular sections containing nine mesh points each, replicated with six images both in the upstream and downstream directions.

**Initial conditions**

To trigger the instability growth, the flow was initially perturbed. From the multiple possibilities for this initial perturbation, an asymmetric sinusoidal perturbation applied to the spatial field in the streamwise direction was chosen. The amplitude was taken to be one-half of the sheet thickness, corresponding to $10\%$ of the perturbation wavelength. Cases with an amplitude ten times smaller have also been calculated. A second sinusoidal spatial perturbation with half the amplitude and the wavelength was introduced in the spanwise direction. It has to be noted that if the liquid sheet is considered to be infinite, the only relevant spatial scales are the perturbation wavelengths and amplitudes and the sheet thickness. Our wavelength to thickness ratio choice would correspond to the wave number of maximum growth rate for a Weber number based on the sheet thickness of 1000, as predicted by linear perturbation theory. To ensure that the results were not imposed by the initial conditions, other amplitudes were also tested. In particular, cases were calculated starting with one-half and one-tenth of the original amplitude of the transverse perturbation, maintaining the initial amplitude of the longitudinal wave.

In the second geometry described in the previous section, starting with an initial mesh of equally sized cells, the limited number of grid points yielded a sheet aspect ratio of 14, quite thicker than the sheets used in reported experiments. To reproduce the aspect ratio of these experiments, maintaining equally sized cells and a number of nodes at the sheet edges capable of yielding enough resolution would have implied an excessive computation time. The contribution of edge effects can, however, be observed even for this thickness.

The initial vortex sheet strength was normalized to 1, and was defined to have only downstream component, being always constrained to be tangent to the air/water interface. Defining an initial geometry and vortex sheet strength is enough to start the calculation of the interface evolution. Time has been made nondimensional with a factor $T = \lambda / \gamma_0$.

**RESULTS AND DISCUSSION**

**Longitudinal perturbation**

Figure 2 shows the initial and final $(t = 3.0)$ stages of a calculation when the flow is initially perturbed only in the streamwise direction. As in the rest of the figures that will be presented, the image corresponds to the computation domain replicated twice in each direction (60 $\times$ 60 grid points). The value of the Atwood number is $A = 0.99$ corresponding to the water/air case, and surface tension has been neglected. It can be seen that the model reproduces correctly the expected evolution of the Kelvin–Helmholtz instability as described in previous reported 2D simulations. When entering the nonlinear deformation regime, points far away from the axis of symmetry, which will be denoted as maxima, accelerate following the surrounding air, while points close to the axis, which will be denoted as minima, move more slowly. At the same time, there is vorticity advection from minima to maxima, resulting in the generation of rollers in the maxima points that cause the sheet to convolute. As time evolves, the vortex centers assume a sawtooth configuration. As the vortices rotate, the sheet grows thinner at the initial minima locations. In the final stage, the thinning tends to a limit where the upper and lower interfaces finally touch. As the sheet thickness tends to zero in these points, any perturbation in a real case (e.g., acoustic noise, pressure wave, etc.) would cause the sheet to tear. The tear would generate a hole with regions of high curvature, where the effects of surface tension would be very intense. This mechanism explains the generation of spanwise ligaments, oriented parallel to the nozzle.

**Transverse perturbation**

The presence of longitudinal filaments cannot be explained if a transverse perturbation is not included. To start
the growth of these structures, the sheet was initially altered by symmetric and antisymmetric sinusoidal perturbations, with an amplitude of 25% of the sheet thickness, corresponding to 5% of the wavelength. Figure 3 presents the initial and final ($t=3.0$) stages of a case with antisymmetric perturbations both in the longitudinal and transverse directions, where surface tension has been included. In Fig. 4 the initial transverse perturbation is symmetric. The evolution of the longitudinal perturbation is essentially similar to that described in the previous section. Both air/liquid interfaces end up touching each other. However, in this case, the first contact does not occur simultaneously on a whole line transverse to the sheet, but in single points of this line. Figure 5 shows the evolution of a plane perpendicular to the liquid sheet at the location where contact between the interfaces first happens for the simulations presented in Fig. 3 (antisymmetric transversal perturbation) and Fig. 4 (symmetric perturbation). In both cases, the upper surface has been notably flattened due to the higher stretching. The cross sections share similar characteristics, although the contact points are displaced half a wavelength. Tearing at the touching points can be expected causing tubular sections with pointed edges. The high curvature of these edges will be quickly smoothed by surface tension forces, originating the cylindrical streamwise filaments observed in the experiments. It has to be pointed out that according to this theory streamwise filaments would appear if there is a transverse wave, either if it is symmetrical or antisymmetrical. Its wavelength and amplitude would determine the filament spacing and diameter. Note also that the surface undulation causes the two interfaces to touch earlier, and at points closer to the roller crests.

Different values for the initial spanwise perturbation amplitude were tested ($1/2$ and $1/10$ of the nominal case). It is observed that the transverse perturbation grows at a slower rate than the longitudinal one whose initial growth is exponential. This is due to the fact that both air and water velocities are initially oriented in the longitudinal direction. By itself, the transverse perturbation growth rate might be insuf-
sufficient to generate longitudinal filaments starting from an
infinitesimal initial perturbation. This could not be a problem if
there is a mechanism that can generate finite initial trans-
verse perturbations, for example inside the nozzle. However,
even if this is not the case, higher growth rates have been
observed when the initial perturbation is not orthogonal to
the initial vortex sheet strength. Figure 6 illustrates this
point, when the initial vortex sheet strength forms an angle
of ±45° with two perturbations, whose initial amplitudes in
this case were 1% of the wavelength (5% of the sheet thick-
ness). This case results in a final situation where the trans-
verse section shows a sinusoidal undulation of finite ampli-
tude. If this same initial condition is combined with an in-
phase longitudinal perturbation parallel to the vortex sheet
strength, i.e., wave numbers \( k=(\pi, \pi, 0) \), \( k=(\pi, -\pi, 0) \),
and \( k=(2\pi, 0, 0) \), the end result is similar to that described
in our nominal case (Fig. 3) but with an enhanced growth
of the transverse wave. Figure 7 presents a final stage
\((t=3.0)\) of this calculation. Figure 8 depicts the temporal
evolution of a transverse section \((t=0, 1.5, 3.0, \text{and } 4.0)\). It
is to be compared with Fig. 5, considering that now the ini-
tial amplitude is five times smaller. This suggests that in a
real situation, with formation of streamwise filaments, the
triggering perturbation could be a superposition of different
waves with more than just a longitudinal and a transverse
components. The presence of this oblique waves will con-

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**FIG. 6.** Initial and final stages \((t=3.0)\) of a liquid sheet simulation with
antisymmetric perturbations oriented at ±45° with the flow direction. Note
that in this figure flow direction is towards the viewer.

**FIG. 7.** Initial and final stages \((t=3.0)\) of a liquid sheet simulation with
same initial perturbation of Fig. 6 plus an in-phase antisymmetric longitu-
dinal perturbation.

**FIG. 8.** Evolution of a section of the liquid sheet presented in Fig. 7 perpendicular to the plane where contact between the two air/water interfaces first occurs.
Cuts correspond to times \(t=0, t=1.5, t=3, \text{and } t=4\).
FIG. 9. Final stages (t = 3.0) of a liquid sheet simulation with same initial conditions as those in Fig. 3 except that the initial separation between the two air/water interfaces has been multiplied by a factor of (a) 0.25 (upper), (b) 0.5 (middle), and (c) 2.0 (lower), respectively.

FIG. 10. Final stage of a nominal case simulation with a surface tension coefficient multiplied by a factor of 1000.

FIG. 11. Initial and final stages of a liquid sheet simulation with closed edges and antisymmetric spanwise and streamwise sinusoidal perturbations.

tribute to the evolution of a finite transverse wave starting from infinitesimal perturbations, in agreement with the mechanism proposed by Wu and Stewart.25

Figure 9 compares the perturbation evolution for different initial separations between the two air/water interfaces. Initial separations in Figs. 9(a), 9(b), and 9(c) were 0.25, 0.5, and 2 times larger than that depicted in Fig. 2. As the longitudinal perturbation was spanning the same number of grid points in all cases, a thinner sheet is equivalent to a longer wavelength. As predicted by linear perturbation theory, shorter wavelength perturbations grow faster.

Surface tension effects

It may be of interest to include some comments about surface tension effects. Regarding the Kelvin–Helmholtz instability, inclusion of surface tension tends to damp its growth. It has to be noted, however, that the term involving surface tension in Eq. (10) is only important in high curvature regions. For a perturbation wavelength \( \lambda \) of 1 mm and an initial value of sheet strength, \( \gamma_0 \) of 40 m/s the Weber number for the air/water case results to be on the order \( O(10^5) \) while the remaining terms in the nondimensional form of Eq. (10), except the gravity term which is smaller, are of order \( O(1) \). As the surface tension term is divided by the Weber number, high values of the mean curvature gradient \( \nabla H \) are required for surface tension effects to be important. The absence of the surface tension term produces results that are almost identical to the cases that include it. To show its damping effect, Fig. 10 presents the final stage of a calculation where the surface tension coefficient has been multiplied by 1000.

Despite this, surface tension is the main mechanism causing filament formation once holes have formed in the liquid sheet due to thinning. After the sheet has been broken into filaments, capillary instabilities contribute to break them into droplets finishing the primary atomization process.

Edge effects

The results presented in previous sections have been calculated for a liquid sheet infinite both in the cross-stream and downstream directions. To study the influence of the sheet edges some simulations were performed for an infinite liquid slab surrounded by an air co-flow. As can be seen in Fig. 11 the edges tend to bend toward the sheet causing the sacklike structures that are observed in experiments for low air/water velocity ratios.7−10 For high ratios sheet breakup occurs too close to the nozzle for these structures to be observed clearly. For an example of these structures, see for example Fig. 14 of Mansour and Chigier.7 This is not only a surface tension effect, as it appears even when the inverse of the Weber number is zero. Surface tension, however, causes real sheets to collapse in the spanwise direction into a jet, an effect that is more easily observed in the absence of air co-flow.7
CONCLUSIONS

A three-dimensional model based on the assumption that the interface behaves as an inviscid vortex sheet has been developed to study the instability growth leading to the breakup of a water sheet surrounded by an air co-flow. Transverse filaments observed in experiments can be explained with a 2D Kelvin–Helmholtz instability mechanism. Longitudinal filaments which are experimentally observed for most air/water velocity ratios leading to efficient atomization require a 3D analysis. Simulations with initial longitudinal and transverse sinusoidal perturbations indicate that the sheet eventually collapses at discrete points in a transverse cross section. Tearing along these points helped by surface tension effects would explain the formation of the longitudinal filaments. Initial symmetric or antisymmetric transverse perturbation result in very similar final configurations. The presence of in-phase oblique waves may explain the growth of the transverse wave starting from an infinitesimal perturbation. Edge effects have also been studied, to explain the sacklike structures that can be observed for low air/water velocity ratios.

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APPENDIX: DISCRETIZED EQUATIONS

Notice the combination of a Mac-Cormack type algorithm for the hyperbolic part of the $\gamma$ equation and a Predictor-Corrector for its $(du/dt)$ and $(dn/dt)$ sources. $\mathbf{a}$ and $\mathbf{b}$ are unit vectors along the non-orthogonal coordinates $\mathbf{a}$, $\mathbf{b}$ defined as the distances on the deformed mesh, initially cartesian, on the surface layer. $\mathbf{e}_3$ is the unit vector in the direction of the gravity field. $\Delta_x a$ is the distance increment between two adjacent mesh points in the increasing $\mathbf{a}$ direction, and:

$\mathbf{D}_{i,j}^{m-1/2} = \mathbf{D}_{i,j}^{m} - e_3 \cdot \mathbf{D}_{i,j}^{m} \sin \gamma_{ij}$

and:

$\gamma^{**}_{ij} = \gamma^{**}_{ij} - 0.5A\Delta t(f_{ij}(\gamma^{**}_{i+1,j} - \gamma^{**}_{ij}) + g_{ij}(\gamma^{**}_{i+1,j} - \gamma^{**}_{ij}))$

$-c_0.5\Delta t\mathbf{a}^2\left(\begin{array}{ccc} \gamma^{**}_{i+1,j} - \gamma^{**}_{ij} & \gamma^{**}_{i+1,j} - \gamma^{**}_{ij} & \gamma^{**}_{i+1,j} - \gamma^{**}_{ij} \\ \Delta_x a & \Delta_x a & \Delta_x a \\ \end{array}\right) -2A\Delta t\left(\begin{array}{ccc} \Delta u & \Delta u & \Delta u \\ \Delta t & \Delta t & \Delta t \\ \end{array}\right)_{ij}^{m-1/2} -\Delta t(\gamma_{ij} \cdot \nabla u_{ij} - \Delta t(\gamma_{ij} \cdot \nabla u_{ij} - n_{ij}^x (\gamma_{ij} \cdot \nabla u_{ij}) n_{ij}^x) - 2A\Delta t(\nabla H)_{ij} + 2A\Delta t r(e_3 \cdot n)_{ij} n_{ij} - \Delta t(\begin{array}{ccc} \Delta n & \Delta n & \Delta n \\ \Delta t & \Delta t & \Delta t \\ \end{array})_{ij}^{m-1/2}$

and:

$\gamma^{**}_{ij} = \gamma_{ij} - 0.5A\Delta t(f_{ij}(\gamma_{ij} - \gamma_{i+1,j}) + g_{ij}(\gamma_{ij} - \gamma_{i+1,j}))$

$-c_0.5\Delta t\mathbf{a}^2\left(\begin{array}{ccc} \gamma_{i+1,j} - \gamma_{ij} & \gamma_{i+1,j} - \gamma_{ij} & \gamma_{i+1,j} - \gamma_{ij} \\ \Delta_x a & \Delta_x a & \Delta_x a \\ \end{array}\right) -2A\Delta t\left(\begin{array}{ccc} \Delta u & \Delta u & \Delta u \\ \Delta t & \Delta t & \Delta t \\ \end{array}\right)_{ij}^{m-1/2} -\Delta t(\gamma_{ij} \cdot \nabla u_{ij} - \Delta t(\gamma_{ij} \cdot \nabla u_{ij} - n_{ij}^x (\gamma_{ij} \cdot \nabla u_{ij}) n_{ij}^x) - 2A\Delta t(\nabla H)_{ij} + 2A\Delta t r(e_3 \cdot n)_{ij} n_{ij} - \Delta t(\begin{array}{ccc} \Delta n & \Delta n & \Delta n \\ \Delta t & \Delta t & \Delta t \\ \end{array})_{ij}^{m-1/2}$

and:

$\gamma_{ij}^{**} = (\gamma_{ij}^{**} + \gamma_{ij}^{**})/2$, where $c$ is the artificial viscosity coefficient, and:

$f = \frac{\gamma \cdot a - a \cdot b(\gamma \cdot b)}{(1 - (a \cdot b)^2)\Delta a}$

$g = \frac{\gamma \cdot b - a \cdot b(\gamma \cdot a)}{(1 - (a \cdot b)^2)\Delta b}$

the mean curvature:
\[ H_{ij}^n = \frac{1}{2(a\nabla b)} \det \left\{ \begin{align*}
0.5 \left( \frac{a_{i+1,j} - a_{ij}}{\Delta_+ a} + \frac{a_{ij} - a_{ij-1}}{\Delta_- a} \right) &+ 0.5 \left( \frac{a_{ij+1} - a_{ij}}{\Delta_+ b} + \frac{a_{ij} - a_{ij-1}}{\Delta_- b} \right) \\
+ 0.5 \left( \frac{b_{i+1,j} - b_{ij}}{\Delta_+ a} + \frac{b_{ij} - b_{ij-1}}{\Delta_- a} \right) &+ 0.5 \left( \frac{b_{ij+1} - b_{ij}}{\Delta_+ b} + \frac{b_{ij} - b_{ij-1}}{\Delta_- b} \right),
\end{align*} \right\}^m \]

and:

\[
(\nabla H)_{ij}^n = \left( \frac{a}{1 - (a \cdot b)^2} \right)^m 0.5 \left( \frac{H_{i+1,j} - H_{ij}}{\Delta_+ a} + \frac{H_{ij} - H_{ij-1}}{\Delta_- a} \right) - (a \cdot b) 0.5 \left( \frac{H_{ij+1} - H_{ij}}{\Delta_+ b} + \frac{H_{ij} - H_{ij-1}}{\Delta_- b} \right) + \left( \frac{b}{1 - (a \cdot b)^2} \right)^m 0.5 \left( \frac{H_{i+1,j} - H_{ij}}{\Delta_+ b} + \frac{H_{ij} - H_{ij-1}}{\Delta_- b} \right) - (a \cdot b) 0.5 \left( \frac{H_{ij+1} - H_{ij}}{\Delta_+ a} + \frac{H_{ij} - H_{ij-1}}{\Delta_- a} \right).
\]

**u equation**

\[
u_{ij}^{n+1} = \frac{1}{4\pi} \sum_k \sum_l \left( \frac{(x_{ij} - x_{kl}) \cdot n_{ij} \gamma_{kl} - (x_{ij} - x_{kl}) \cdot \gamma_{kl} n_{ij}}{(x_{ij} - x_{kl}) \cdot (x_{ij} - x_{kl}) + \eta^2)^{3/2}} \right)^m.
\]

**x equation**

\[
x_{ij}^{n+1} = x_{ij}^n + u_{ij}^n \Delta t.
\]

**Corrector γ equation**

\[
\gamma_j^m = \gamma_j^m - 0.5 A \Delta t (f_{ij}^m (\gamma_j^{m+1} - \gamma_j^m) + g_{ij}^m (\gamma_j^{m+1} + \gamma_j^m))
\]

\[
- c_0.5 \Delta t a^2 \left( \frac{\gamma_j^{m+1} - \gamma_j^m}{\Delta_+ a} + \frac{\gamma_j^{m+1} - \gamma_j^{m-1}}{\Delta_- a} \right)
\]

\[
- c_0.5 \Delta t b^2 \left( \frac{\gamma_j^{m+1} - \gamma_j^m}{\Delta_+ b} + \frac{\gamma_j^{m+1} - \gamma_j^{m-1}}{\Delta_- b} \right) - 2A \Delta t \left( \frac{\Delta u}{\Delta t} \right)^{m+1/2} \cdot n \left( \gamma \right)^{m+1/2}
\]

\[
- \Delta t (\gamma_j^m \cdot \nabla u_{ij}^m - n_{ij}^m (\gamma_j^m \cdot \nabla u_{ij}^m) n_{ij}^m - 2B \Delta t (\nabla H)_{ij}^m + 2A \Delta t g (j-n) n_{ij}^m - \Delta t n \left( \gamma \right)^{m+1/2},
\]

and:

\[
\gamma_{ij}^{m+1} = \gamma_{ij}^m - 0.5 A \Delta t (f_{ij}^m (\gamma_j^m - \gamma_j^{m-1}) + g_{ij}^m (\gamma_j^m - \gamma_j^{m-1}))
\]

\[
- c_0.5 \Delta t a^2 \left( \frac{\gamma_j^{m+1} - \gamma_j^m}{\Delta_+ a} + \frac{\gamma_j^{m+1} - \gamma_j^{m-1}}{\Delta_- a} \right)
\]

\[
- c_0.5 \Delta t b^2 \left( \frac{\gamma_j^{m+1} - \gamma_j^m}{\Delta_+ b} + \frac{\gamma_j^{m+1} - \gamma_j^{m-1}}{\Delta_- b} \right) - 2A \Delta t \left( \frac{\Delta u}{\Delta t} \right)^{m+1/2} \cdot n \left( \gamma \right)^{m+1/2}
\]

\[
- \Delta t (\gamma_j^m \cdot \nabla u_{ij}^m - n_{ij}^m (\gamma_j^m \cdot \nabla u_{ij}^m) n_{ij}^m - 2B \Delta t (\nabla H)_{ij}^m + 2A \Delta t g (j-n) n_{ij}^m - \Delta t n \left( \gamma \right)^{m+1/2},
\]

and:

\[
\gamma_{ij}^{m+1} = (\gamma_{ij}^m + \gamma_{ij}^{m+1})/2.
\]

**u equation**

\[
u_{ij}^{n+1} = \frac{1}{4\pi} \sum_k \sum_l \left( \frac{(x_{ij} - x_{kl}) \cdot n_{ij} \gamma_{kl} - (x_{ij} - x_{kl}) \cdot \gamma_{kl} n_{ij}}{(x_{ij} - x_{kl}) \cdot (x_{ij} - x_{kl}) + \eta^2)^{3/2}} \right)^{m+1}.
\]

**x equation**

\[
x_{ij}^{n+1} = x_{ij}^m + u_{ij}^{n+1/2} \Delta t.
\]

And finally γ calculated in predictor and corrector are averaged:

\[
\gamma_j^{m+1} = 0.5 \{ \gamma_j^{m+1} \text{(predictor)} + \gamma_j^{m+1} \text{(corrector)} \}.
\]