

# Sobolev orthogonal polynomials: the discrete-continuous case

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## Abstract

In this paper, we study orthogonal polynomials with respect to the bilinear form

$$\mathcal{B}_S^{(N)}(f, g) = F(c)\mathbf{A}G(c)^T + \langle u, f^{(N)}g^{(N)} \rangle,$$

where  $u$  is a quasi-definite (or regular) linear functional on the linear space  $\mathbb{P}$  of real polynomials,  $c$  is a real number,  $N$  is a positive integer number,  $\mathbf{A}$  is a symmetric  $N \times N$  real matrix such that each of its principal submatrices are regular, and  $F(c) = (f(c), f'(c), \dots, f^{(N-1)}(c))$ ,  $G(c) = (g(c), g'(c), \dots, g^{(N-1)}(c))$ . For these non-standard orthogonal polynomials, algebraic and differential properties are obtained, as well as their representation in terms of the standard orthogonal polynomials associated with  $u$ .

*Running title:* Discrete-continuous Sobolev polynomials

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# 1 Introduction

It is well known (see [12]) that the monic generalized Laguerre polynomials  $\{L_n^{(\alpha)}\}_n$  satisfy, for any real value of  $\alpha$ , the three-term recurrence relation

$$xL_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)}(x) + \beta_n^{(\alpha)}L_n^{(\alpha)}(x) + \gamma_n^{(\alpha)}L_{n-1}^{(\alpha)}(x),$$

$$L_{-1}^{(\alpha)}(x) = 0, \quad L_0^{(\alpha)}(x) = 1,$$

where

$$\beta_n^{(\alpha)} = 2n + \alpha + 1, \quad \gamma_n^{(\alpha)} = n(n + \alpha).$$

Whenever  $\alpha$  is not a negative integer number, we have  $\gamma_n^{(\alpha)} \neq 0$  for all  $n \geq 1$  and Favard's theorem (see [2], p. 21) ensures that the sequence  $\{L_n^{(\alpha)}\}_n$  is orthogonal with respect to a quasi-definite linear functional. Besides, if  $\alpha > -1$  the functional is definite positive and the polynomials are orthogonal with respect to the weight  $x^\alpha e^{-x}$  on the interval  $(0, +\infty)$ . For  $\alpha$  a negative integer number, since  $\gamma_n^{(\alpha)}$  vanishes for some value of  $n$ , no orthogonality results can be deduced from Favard's theorem.

In the last few years, orthogonal polynomials with respect to an inner product involving derivatives (the so-called Sobolev orthogonal polynomials) have been the object of increasing interest and in this context, the case  $\{L_n^{(\alpha)}\}_n$  with  $\alpha$  a negative integer number has been solved. More precisely, Kwon and Littlejohn, in [5], established the orthogonality of the generalized Laguerre polynomials  $\{L_n^{(-k)}\}_n$ ,  $k \geq 1$ , with respect to a Sobolev inner product of the form

$$\langle f, g \rangle = F(0)\mathbf{A}G(0)^T + \int_0^{+\infty} f^{(k)}(x)g^{(k)}(x)e^{-x}dx$$

with  $\mathbf{A}$  a symmetric  $k \times k$  real matrix,  $F(0) = (f(0), f'(0), \dots, f^{(k-1)}(0))$ , and  $G(0) = (g(0), g'(0), \dots, g^{(k-1)}(0))$ . The particular case  $k = 1$  had been considered by the same authors in a previous paper, (see [6]).

In [10], Pérez and Piñar gave a unified approach to the orthogonality of the generalized Laguerre polynomials, for any real value of the parameter  $\alpha$  by proving their orthogonality with respect to a Sobolev non-diagonal inner product. So, they obtained the following result:

**Theorem ([10])** *Let  $(\cdot, \cdot)_S^{(N, \alpha)}$  be the Sobolev inner product defined by*

$$(f, g)_S^{(N, \alpha)} = \int_0^{+\infty} F(x)\mathbf{A}G(x)^T x^\alpha e^{-x}dx,$$

where the  $(i, j)$ -entry of  $\mathbf{A}$  is given by

$$m_{i,j}(N) = \sum_{p=0}^{\min\{i,j\}} (-1)^{i+j} \binom{N-p}{i-p} \binom{N-p}{j-p}, \quad 0 \leq i, j \leq N,$$

$F(x) = (f(x), f'(x), \dots, f^{(N)}(x))$ ,  $G(x) = (g(x), g'(x), \dots, g^{(N)}(x))$ . Then, for every  $\alpha \in \mathbb{R}$ , the monic generalized Laguerre polynomials  $\{L_n^{(\alpha)}\}_n$  are orthogonal with respect to  $(\cdot, \cdot)_S^{(N, \alpha+N)}$  with  $N = \max\{0, [-\alpha]\}$ , ( $[\alpha]$  denotes the greatest integer less than or equal to  $\alpha$ ).

In the case when  $\alpha$  is a negative integer, the inner product  $(\cdot, \cdot)_S^{(N, \alpha+N)}$  is the same as the one considered by Kwon and Littlejohn.

The above results justify the interest to consider such a kind of inner products. In a more general setting, our aim is to study polynomials which are orthogonal with respect to a symmetric bilinear form such as

$$\mathcal{B}_S^{(N)}(f, g) = \left( f(c), f'(c), \dots, f^{(N-1)}(c) \right) \mathbf{A} \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle, \quad (1.1)$$

where  $u$  is a quasi-definite (or regular) linear functional on the linear space  $\mathbb{P}$  of real polynomials,  $c$  is a real number,  $N$  is a positive integer number, and  $\mathbf{A}$  is a symmetric  $N \times N$  real matrix such that each of its principal submatrices is regular. By analogy with the usual terminology, we call it a discrete-continuous Sobolev bilinear form. Recently some properties of the polynomials orthogonal with respect to  $\mathcal{B}_S^{(1)}(\cdot, \cdot)$  had been considered in [4].

We will emphasize some cases in which the functional  $u$  satisfies some extra conditions, namely,  $u$  is a semiclassical or a classical linear functional (see [3], [7] and [9]). A quasi-definite linear functional  $u$  is called semiclassical if there exist polynomials  $\phi$  and  $\psi$  with  $\deg \phi \geq 0$  and  $\deg \psi \geq 1$  such that  $u$  satisfies the distributional differential equation  $\mathcal{D}(\phi u) = \psi u$ . Whenever  $\deg \phi \leq 2$  and  $\deg \psi = 1$ , the functional  $u$  is called classical. It is well known that the only classical functionals correspond to the sequences of Hermite, Laguerre, Jacobi and Bessel polynomials.

In Section 2, we give a description of the monic polynomials  $\{Q_n\}_n$  which are orthogonal with respect to  $\mathcal{B}_S^{(N)}(\cdot, \cdot)$  in terms of the monic polynomials

$\{P_n\}_n$  orthogonal with respect to the functional  $u$ . In particular, for  $n \geq N$ , we have that  $Q_n^{(N)}(x) = \frac{n!}{(n-N)!}P_{n-N}(x)$  and  $Q_n^{(k)}(c) = 0$  for  $k = 0, 1, \dots, N-1$ , while  $\{Q_n\}_{n=0}^{N-1}$  are orthogonal with respect to the discrete part of the symmetric bilinear form (1.1) and they are determined by the matrix  $\mathbf{A}$ .

By using these results, in Section 3, we give some examples of polynomials orthogonal with respect to (1.1), with an adequate choice of  $c$ , namely, Laguerre polynomials  $\{L_n^{(-N)}\}_n$  with  $c = 0$ , Jacobi polynomials  $\{P_n^{(-N, \beta)}\}_n$  with  $c = 1$ ,  $\beta + N$  not being a negative integer, and  $\{P_n^{(\alpha, -N)}\}_n$  with  $c = -1$ ,  $\alpha + N$  not being a negative integer. Note that these sequences of polynomials are not orthogonal with respect to any quasi-definite linear functional.

In Section 4, we give a new characterization of classical polynomials as the only orthogonal polynomials such that, for some positive integer number  $N$ , they have a  $N$ -th primitive satisfying a three-term recurrence relation. In particular, this result is applied to discrete-continuous Sobolev polynomials which satisfy a three-term recurrence relation and then it follows that  $u$  is classical with distributional differential equation  $\mathcal{D}(\phi u) = \psi u$ , and the point  $c$  in (1.1) is such that  $\phi(c) = 0$  and  $\psi(c) = \phi'(c)$ . Hence, the only monic discrete-continuous Sobolev polynomials which satisfy a three-term recurrence relation are the ones described in Section 3.

The link between Sobolev orthogonality and polynomials satisfying a second order differential equation is analyzed in Section 5. It is proved that if the sequence  $\{Q_n\}_n$  satisfies the equation

$$\phi(x)Q_n''(x) + \sigma(x)Q_n'(x) = \rho_n Q_n(x),$$

where  $\phi$  and  $\sigma$  are polynomials with degree less than or equal to 2 and 1, respectively, and  $\rho_n$  are real numbers, then the functional  $u$  is classical with distributional differential equation  $\mathcal{D}(\phi u) = \psi u$ ,  $\sigma(x) = \psi(x) - N\phi'(x)$  and the point  $c$  in (1.1) verifies  $\phi(c) = 0$  and  $\psi(c) = \phi'(c)$ . Hence, the only monic discrete-continuous Sobolev polynomials which satisfy a second order differential equation are again the described in Section 3.

As a consequence of the results in Sections 4 and 5, we have that if  $u$  is not a classical linear functional then the sequence  $\{Q_n\}_n$  does not satisfy neither a three-term recurrence relation nor a second order differential equation. In order to avoid this lack in our study, in Section 6, we introduce a linear differential operator  $\mathcal{F}^{(N)}$  on  $\mathbb{P}$  symmetric with respect to the bilinear form (1.1). The basic property of this operator is a relationship between the Sobolev bilinear form and the bilinear form associated with the functional

$u$ . Handling with  $\mathcal{F}^{(N)}$  we can deduce explicit relations between  $\{Q_n\}_n$  and  $\{P_n\}_n$  as well as a differential substitute of the algebraic recurrence relations. This is done in Section 7.

## 2 The Sobolev discrete-continuous bilinear form

Let  $\mathbb{P}$  be the linear space of real polynomials,  $u$  a quasi-definite linear functional on  $\mathbb{P}$  (see [2]),  $N$  a positive integer number, and  $\mathbf{A}$  a quasi-definite and symmetric real matrix of order  $N$ , that is, a symmetric and real matrix such that all the principal minors are different from zero. For a given real number  $c$ , the expression

$$\mathcal{B}_S^{(N)}(f, g) = \left( f(c), f'(c), \dots, f^{(N-1)}(c) \right) \mathbf{A} \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle, \quad (2.1)$$

defines a symmetric bilinear form on  $\mathbb{P}$ .

Since expression (2.1) involves derivatives, this bilinear form is non-standard, and by analogy with the usual terminology we will call it a *discrete-continuous Sobolev bilinear form*.

In the linear space of real polynomials, we can consider the basis given by

$$\left\{ \frac{(x-c)^m}{m!} \right\}_{m \geq 0}. \quad (2.2)$$

For  $n \leq N-1$ , the associated Gram matrix  $\mathbf{G}_n$  is given by the  $n$ -th order principal submatrix of the matrix  $\mathbf{A}$ . For  $n \geq N$ , the associated Gram matrix is given by

$$\mathbf{G}_n = \left( \begin{array}{c|c} \mathbf{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{B}_{n-N} \end{array} \right),$$

where  $\mathbf{B}_{n-N}$  is the Gram matrix associated with the quasi-definite linear functional  $u$  in the basis (2.2).

In both cases,  $\mathbf{G}_n$  is quasi-definite (that is, all the principal minors are different from zero) and therefore, we will say that the discrete-continuous Sobolev bilinear form (2.1) is quasi-definite. Thus, we can assure the existence of a sequence of monic polynomials, denoted by  $\{Q_n\}_n$ , which are orthogonal with respect to (2.1). These polynomials will be called *Sobolev orthogonal polynomials*. Our first aim is to relate this sequence with the

monic orthogonal polynomial sequence (in short MOPS)  $\{P_n\}_n$  associated with the quasi-definite linear functional  $u$ .

**Theorem 2.1** *Let  $\{Q_n\}_n$  be the sequence of monic orthogonal polynomials with respect to the bilinear form  $\mathcal{B}_S^{(N)}$ .*

*i) The polynomials  $\{Q_n\}_{n=0}^{N-1}$  are orthogonal with respect to the discrete bilinear form*

$$\mathcal{B}_D^{(N)}(f, g) = \left( f(c), f'(c), \dots, f^{(N-1)}(c) \right) \mathbf{A} \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix}. \quad (2.3)$$

*ii) If  $n \geq N$ , then*

$$Q_n^{(k)}(c) = 0, \quad k = 0, 1, \dots, N-1, \quad (2.4)$$

$$Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x). \quad (2.5)$$

**Proof.** *i)* If  $0 \leq m, n < N$ , then  $Q_n^{(N)}(x) = Q_m^{(N)}(x) = 0$ , and the value of the Sobolev bilinear form on  $(Q_n, Q_m)$  can be computed by means of the following expression

$$\begin{aligned} \mathcal{B}_S^{(N)}(Q_n, Q_m) &= \mathcal{B}_D^{(N)}(Q_n, Q_m) \\ &= \left( Q_n(c), Q_n'(c), \dots, Q_n^{(N-1)}(c) \right) \mathbf{A} \begin{pmatrix} Q_m(c) \\ Q_m'(c) \\ \vdots \\ Q_m^{(N-1)}(c) \end{pmatrix}, \end{aligned}$$

and therefore they are orthogonal with respect to the discrete bilinear form (2.3).

*ii)* Let  $n \geq N$ , then from the orthogonality of the polynomial  $Q_n$ , we deduce

$$0 = \mathcal{B}_S^{(N)}(Q_n(x), \frac{1}{k!}(x-c)^k) = \left( Q_n(c), Q_n'(c), \dots, Q_n^{(N-1)}(c) \right) \mathbf{A} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad (2.6)$$

for  $0 \leq k \leq N - 1$ . Thus, the vector

$$\left(Q_n(c), Q_n'(c), \dots, Q_n^{(N-1)}(c)\right)$$

is the only solution of a homogeneous linear system with  $N$  equations and  $N$  unknowns, whose coefficient matrix  $\mathbf{A}$  is regular. Then, we conclude  $Q_n^{(k)}(c) = 0$ ,  $k = 0, 1, \dots, N - 1$ , that is,  $Q_n$  contains the factor  $(x - c)^N$ . In this way, if  $n, m \geq N$ , the discrete part of the bilinear form  $\mathcal{B}_S^{(N)}(Q_n, Q_m)$  vanishes and we get

$$\mathcal{B}_S^{(N)}(Q_n, Q_m) = \langle u, Q_n^{(N)} Q_m^{(N)} \rangle = \tilde{k}_n \delta_{n,m}, \quad \tilde{k}_n \neq 0.$$

That is, the polynomials  $\{Q_n^{(N)}\}_{n \geq N}$  are orthogonal with respect to the linear functional  $u$ , and equality (2.5) follows from a simple inspection of the leading coefficients.  $\square$

Reciprocally, we are going to show that a system of monic polynomials  $\{Q_n\}_n$  satisfying equations (2.4) and (2.5) is orthogonal with respect to some discrete–continuous Sobolev bilinear form. This result could be considered a *Favard–type theorem*.

**Theorem 2.2** *Let  $\{P_n\}_n$  be the MOPS associated with a quasi–definite linear functional  $u$ , and  $N \geq 1$  a given integer number. Let  $\{Q_n\}_n$  be a sequence of monic polynomials satisfying*

- i)  $\deg Q_n = n$ ,  $n = 0, 1, 2, \dots$ ,*
- ii)  $Q_n^{(k)}(c) = 0$ ,  $0 \leq k \leq N - 1$ ,  $n \geq N$ ,*
- iii)  $Q_n^{(N)}(x) = \frac{n!}{(n - N)!} P_{n-N}(x)$ ,  $n \geq N$ .*

*Then, there exists a quasi–definite and symmetric real matrix  $\mathbf{A}$ , of order  $N$ , such that  $\{Q_n\}_n$  is the monic orthogonal polynomial sequence associated with the Sobolev bilinear form defined by (2.1).*

**Proof.** By using the same reasoning as above it is obvious that every polynomial  $Q_n$ , with  $n \geq N$ , is orthogonal to every polynomial with degree less than or equal to  $n - 1$  with respect to a Sobolev bilinear form like (2.1) containing an arbitrary matrix  $\mathbf{A}$  in the discrete part and the functional  $u$  in the second part.

Next, we show that we can recover the matrix  $\mathbf{A}$  from the  $N$  first polynomials  $Q_k, k = 0, 1, \dots, N - 1$ .

Let us denote

$$\mathbf{Q} = \begin{pmatrix} Q_0(c) & Q'_0(c) & \dots & Q_0^{(N-1)}(c) \\ Q_1(c) & Q'_1(c) & \dots & Q_1^{(N-1)}(c) \\ \vdots & \vdots & & \vdots \\ Q_{N-1}(c) & Q'_{N-1}(c) & \dots & Q_{N-1}^{(N-1)}(c) \end{pmatrix},$$

then  $\mathbf{Q}$  is a lower triangular and invertible matrix. Let  $\mathbf{D}$  be a diagonal matrix with non zero elements in its diagonal.

Define

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T.$$

Obviously  $\mathbf{A}$  is symmetric and quasi-definite and since

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^T = \mathbf{D},$$

the polynomials  $Q_0, \dots, Q_{N-1}$  are orthogonal with respect to the bilinear form (2.1), with the matrix  $\mathbf{A}$  in the discrete part. Besides, the elements in the diagonal of  $\mathbf{D}$  are the values  $\mathcal{B}_S^{(N)}(Q_k, Q_k)$  for  $k = 0, \dots, N-1$ .  $\square$

**Remark.** Observe that the matrix  $\mathbf{A}$  is not unique, because its construction depends on the arbitrary regular diagonal matrix  $\mathbf{D}$ .

### 3 Classical examples

#### 3.1 The Laguerre case

Let  $\alpha \in \mathbb{R}$ , the  $n$ -th monic generalized Laguerre polynomial is defined in [12], p. 102, by means of its explicit representation

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{j=0}^n \frac{(-1)^j}{j!} \binom{n+\alpha}{n-j} x^j, \quad n \geq 0, \quad (3.1)$$

where  $\binom{a}{k}$  denotes the *generalized binomial coefficient*

$$\binom{a}{k} = \frac{(a-k+1)_k}{k!}, \quad (3.2)$$

and  $(a-k+1)_k$  stands for the so-called *Pochhammer's symbol* defined by

$$(b)_0 = 1, \quad (b)_n = b(b+1)\dots(b+n-1), \quad \text{for } b \in \mathbb{R}, \quad n \geq 0. \quad (3.3)$$

In this way, we have

$$(L_n^{(\alpha)})^{(k)}(0) = (-1)^{n+k} n! \binom{n+\alpha}{n-k}, \quad n \geq k.$$

If  $\alpha$  is a negative integer number, say  $\alpha = -N$ , for  $n \geq N$ , we have

$$(L_n^{(-N)})^{(k)}(0) = 0, \quad k = 0, 1, \dots, N-1,$$

and, for  $n < N$ , we get

$$(L_n^{(-N)})^{(k)}(0) = n! \binom{N-k-1}{n-k}, \quad k = 0, 1, \dots, n.$$

On the other hand, since the derivatives of Laguerre polynomials are again Laguerre polynomials, we have

$$(L_n^{(-N)})^{(N)}(x) = \frac{n!}{(n-N)!} L_{n-N}^{(0)}(x), \quad \text{for } n \geq N.$$

Therefore, from the previous Section, we conclude that Laguerre polynomials  $L_n^{(-N)}$  are orthogonal with respect to the Sobolev bilinear form

$$\mathcal{B}_S^{(N)}(f, g) = F(0)\mathbf{A}G(0)^T + \int_0^{+\infty} f^{(N)}(x)g^{(N)}(x)e^{-x}dx,$$

with  $F(0) = (f(0), f'(0), \dots, f^{(N-1)}(0))$ ,  $G(0) = (g(0), g'(0), \dots, g^{(N-1)}(0))$ , the matrix  $\mathbf{A}$  is given by

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T,$$

$\mathbf{Q}$  is the matrix of the derivatives of Laguerre polynomials  $L_n^{(-N)}$  evaluated at zero

$$\mathbf{Q} = \begin{pmatrix} 0! \binom{N-1}{0} & 0 & \dots & 0 \\ 1! \binom{N-1}{1} & 1! \binom{N-2}{0} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (N-1)! \binom{N-1}{N-1} & (N-1)! \binom{N-2}{N-2} & \dots & (N-1)! \binom{0}{0} \end{pmatrix}$$

and  $\mathbf{D}$  is an arbitrary regular diagonal matrix. Similar results have been obtained with different techniques in [5] and [10]. We recover the results in [5] by using a diagonal matrix  $\mathbf{D}$  whose elements are  $(0!)^2, (1!)^2, \dots, ((N-1)!)^2$ .

### 3.2 The Jacobi case

For  $\alpha$  and  $\beta$  arbitrary real numbers, the *generalized Jacobi polynomials* can be defined (see [12], p. 62) by means of their explicit representation

$$\mathcal{P}_n^{(\alpha,\beta)}(x) = \sum_{m=0}^n \binom{n+\alpha}{m} \binom{n+\beta}{n-m} \left(\frac{x-1}{2}\right)^{n-m} \left(\frac{x+1}{2}\right)^m, \quad n \geq 0.$$

When  $\alpha, \beta$  and  $\alpha + \beta + 1$  are not a negative integer, Jacobi polynomials are orthogonal with respect to the quasi-definite linear functional  $u^{(\alpha,\beta)}$ . This linear functional is positive definite for  $\alpha > -1$  and  $\beta > -1$ .

For  $\alpha = -N$ , with  $N$  a positive integer, and  $\beta$  being not a negative integer, the  $n$ -th *monic generalized Jacobi polynomial* is given by

$$\begin{aligned} P_n^{(-N,\beta)}(x) \\ = \binom{2n-N+\beta}{n}^{-1} \sum_{m=0}^n \binom{n-N}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m. \end{aligned} \quad (3.4)$$

In this case, for  $n \geq N$ ,  $x = 1$  will be a zero of multiplicity  $N$  ([12], p. 65).

On the other hand, since the derivatives of Jacobi polynomials are again Jacobi polynomials, we have

$$(P_n^{(-N,\beta)})^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}^{(0,\beta+N)}(x), \quad \text{for } n \geq N.$$

Therefore, from the previous Section, we conclude that Jacobi polynomials  $P_n^{(-N,\beta)}$ , when  $\beta + N$  is not a negative integer, are orthogonal with respect to the Sobolev bilinear form

$$\mathcal{B}_S^{(N)}(f, g) = F(1)\mathbf{A}G(1)^T + \langle u^{(0,\beta+N)}, f^{(N)}g^{(N)} \rangle,$$

where the matrix  $\mathbf{A}$  is given by

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T,$$

$\mathbf{Q}$  is the matrix of the derivatives of Jacobi polynomials

$$\mathbf{Q} = \left( (P_n^{(-N,\beta)})^{(k)}(1) \right)_{n,k=0,\dots,N-1}$$

which are given by

$$(P_n^{(-N,\beta)})^{(k)}(1) = 2^{n-k} \frac{n!}{(n-k)!} \frac{(-N+k+1)_{n-k}}{(n-N+\beta+k+1)_{n-k}},$$

and  $\mathbf{D}$  is an arbitrary regular diagonal matrix.

Of course, a similar result can be stated in the case when  $\alpha + N$  is not a negative integer,  $\beta = -N$ , and  $c = -1$ .

## 4 Sobolev orthogonal polynomials and three-term recurrence relations

Laguerre and Jacobi polynomials satisfy a three-term recurrence relation even for negative integer values of their respective parameters (see [12]). In the previous Section, we have seen that Laguerre polynomials with  $\alpha$  a negative integer and Jacobi polynomials with either  $\alpha$  or  $\beta$  a negative integer are Sobolev orthogonal polynomials. In this way a natural question arises: do the Sobolev orthogonal polynomials satisfy a three-term recurrence relation? As we are going to show, the answer is very restrictive, the existence of a three-term recurrence relation for the Sobolev orthogonal polynomials implies the classical character of the linear functional  $u$  associated with the bilinear form (2.1).

**Definition 4.1** *We will say that a family of polynomials  $\{Q_n\}_{n \geq 0}$  is a monic polynomial system (MPS) if*

- i)  $\deg(Q_n) = n, \quad n \geq 0,$
- ii)  $Q_0(x) = 1, \quad Q_n(x) = x^n + \text{lower degree terms}, \quad n \geq 1.$

Obviously, every MPS is a basis of the linear space  $\mathbb{P}$  and every MOPS is a MPS.

**Definition 4.2** *A monic polynomial system  $\{Q_n\}_{n \geq 0}$  satisfies a three-term recurrence relation if there exist two sequences of real numbers  $\{b_n\}_{n=0}^{\infty}$  and  $\{g_n\}_{n=1}^{\infty}$ , such that*

$$xQ_n(x) = Q_{n+1}(x) + b_nQ_n(x) + g_nQ_{n-1}(x), \quad n \geq 0,$$

$$Q_{-1}(x) = 0, \quad Q_0(x) = 1.$$

From Favard's theorem (see [2], p. 21) we can deduce the existence of monic polynomial systems satisfying a three-term recurrence relation which are not orthogonal with respect to any linear functional. This case appears when some of the coefficients  $g_n$  are zero. For instance, Laguerre polynomials with parameter  $\alpha$  a negative integer and Jacobi polynomials with parameters either  $\alpha$  or  $\beta$  or  $\alpha + \beta + 1$  a negative integer.

**Proposition 4.3** *Let  $\{Q_n\}_{n \geq 0}$  be a monic polynomial system satisfying a three-term recurrence relation and let  $N$  be a positive integer number such that the system of monic  $N$ -th order derivatives*

$$P_n(x) := \frac{n!}{(n+N)!} Q_{n+N}^{(N)}(x), \quad n \geq 0,$$

*constitutes a monic orthogonal polynomial sequence. Then, the polynomials  $\{P_n\}_n$  are classical.*

**Proof.** Since  $\{P_n\}_{n \geq 0}$  is a MOPS, it satisfies a three-term recurrence relation

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 0, \\ P_{-1}(x) &= 0, \quad P_0(x) = 1, \end{aligned}$$

with  $\gamma_n \neq 0, n \geq 0$ .

In this way,

$$xQ_{n+N}^{(N)}(x) = \frac{n+1}{n+N+1} Q_{n+N+1}^{(N)}(x) + \beta_n Q_{n+N}^{(N)}(x) + \gamma_n \frac{n+N}{n} Q_{n+N-1}^{(N)}(x). \quad (4.1)$$

On the other hand, the monic polynomial sequence  $\{Q_n\}_{n \geq 0}$  satisfies a three-term recurrence relation

$$xQ_{n+N}(x) = Q_{n+N+1}(x) + b_{n+N} Q_{n+N}(x) + g_{n+N} Q_{n+N-1}(x).$$

Taking  $N$ -th order derivatives in this relation, we get

$$xQ_{n+N}^{(N)}(x) + NQ_{n+N}^{(N-1)}(x) = Q_{n+N+1}^{(N)}(x) + b_{n+N} Q_{n+N}^{(N)}(x) + g_{n+N} Q_{n+N-1}^{(N)}(x). \quad (4.2)$$

By eliminating the term  $xQ_{n+N}^{(N)}(x)$  between (4.1) and (4.2), we obtain

$$\begin{aligned} NQ_{n+N}^{(N-1)}(x) &= \frac{N}{n+N+1} Q_{n+N+1}^{(N)}(x) + \\ &+ (b_{n+N} - \beta_n) Q_{n+N}^{(N)}(x) + \left( g_{n+N} - \frac{n+N}{n} \gamma_n \right) Q_{n+N-1}^{(N)}(x). \end{aligned} \quad (4.3)$$

Taking again derivatives in this relation we deduce that each polynomial  $P_n$  can be expressed as a linear combination of the derivatives of three consecutive polynomials in the sequence  $\{P_n\}_n$  and, therefore, we conclude that they are classical by using the characterization of classical orthogonal polynomials obtained by Marcellán et al. in [7].  $\square$

**Remark.** This result characterizes the classical orthogonal polynomials as the only system of orthogonal polynomials having a  $N$ -th order primitive ( $N \geq 1$ ) which satisfies a three-term recurrence relation.

**Theorem 4.4** *Let  $\{Q_n\}_n$  be the monic orthogonal polynomial sequence associated with the Sobolev bilinear form (2.1). If the polynomials  $\{Q_n\}_n$  satisfy a three-term recurrence relation, then the linear functional  $u$  is classical and the point  $c$  in (2.1) satisfies*

$$\phi(c) = 0, \quad (4.4)$$

$$\psi(c) - \phi'(c) = 0, \quad (4.5)$$

where  $\phi$  and  $\psi$  are the polynomials in the distributional differential equation  $\mathcal{D}(\phi u) = \psi u$  satisfied by  $u$ .

**Proof.** Let  $\{P_n\}_n$  be the monic orthogonal polynomial sequence associated with the linear functional  $u$ . From Theorem 2.1, we have

$$Q_n^{(k)}(c) = 0, \quad k = 0, 1, \dots, N-1, \quad (4.6)$$

$$Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x), \quad (4.7)$$

for all  $n \geq N$ . Therefore, using Proposition 4.3, we deduce the classical character of the polynomials  $\{P_n\}_n$  and then the classical linear functional  $u$  satisfies a distributional differential equation

$$\mathcal{D}(\phi u) = \psi u,$$

where  $\phi$  and  $\psi$  are polynomials with  $\deg \phi \leq 2$  and  $\deg \psi = 1$ . From Bochner's characterization of the classical orthogonal polynomials, (see [2]), we deduce that the polynomials  $\{P_n\}_n$  satisfy the second order differential equation

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) = \lambda_n P_n(x),$$

for all  $n \geq 0$ .

Thus the polynomials  $\{Q_n\}_n$  satisfy the differential equation

$$\phi(x)Q_{n+N}^{(N+2)}(x) + \psi(x)Q_{n+N}^{(N+1)}(x) = \lambda_n Q_{n+N}^{(N)}(x),$$

for  $n \geq 0$ . This differential equation can be written in a more convenient way

$$\left(\phi(x)Q_{n+N}^{(N+1)}(x)\right)' + \left((\psi(x) - \phi'(x))Q_{n+N}^{(N)}(x)\right)' = \kappa_n Q_{n+N}^{(N)}(x), \quad (4.8)$$

where  $\kappa_n = \lambda_n + \psi'(x) - \phi''(x)$ . Integrating (4.8) we get

$$\phi(x)Q_{n+N}^{(N+1)}(x) + (\psi(x) - \phi'(x))Q_{n+N}^{(N)}(x) = \kappa_n Q_{n+N}^{(N-1)}(x) + \mu_n,$$

where  $\mu_n$  is a constant.

For  $n \geq 2$ , let  $p$  be a polynomial with  $\deg p \leq n - 2$ , then

$$\begin{aligned} \langle u, p \left[ \phi Q_{n+N}^{(N+1)} + (\psi - \phi') Q_{n+N}^{(N)} \right] \rangle &= \langle u, p \phi Q_{n+N}^{(N+1)} \rangle - \langle u, (\phi p Q_{n+N}^{(N)})' \rangle \\ &= -\langle u, (p\phi)' Q_{n+N}^{(N)} \rangle \\ &= -\frac{(n+N)!}{n!} \langle u, (p\phi)' P_n \rangle = 0. \end{aligned}$$

Thus, the polynomial  $\phi Q_{n+N}^{(N+1)} + (\psi - \phi') Q_{n+N}^{(N)} = \kappa_n Q_{n+N}^{(N-1)} + \mu_n$  is orthogonal, with respect to  $u$ , to every polynomial of degree less than or equal to  $n-2$ , and then it can be written as a linear combination of three consecutive polynomials  $P_n$

$$\kappa_n Q_{n+N}^{(N-1)} + \mu_n = \kappa_n P_{n+1} + s_n P_n + t_n P_{n-1}.$$

From (4.3) we have that the polynomial  $Q_{n+N}^{(N-1)}$  is a linear combination of the three polynomials  $P_{n+1}$ ,  $P_n$  and  $P_{n-1}$ , and, since the sequence  $\{P_n\}_n$  constitutes a basis of the linear space of the polynomials, we conclude that  $\mu_n = 0$ , for  $n \geq 2$ .

In this way, the polynomials  $\{Q_{n+N}\}_n$  satisfy the differential equation

$$\phi(x)Q_{n+N}^{(N+1)}(x) + (\psi(x) - \phi'(x))Q_{n+N}^{(N)}(x) = \kappa_n Q_{n+N}^{(N-1)}(x), \quad (4.9)$$

for  $n \geq 2$ .

Replacing  $x = c$  in (4.9), from (4.6) we conclude

$$\phi(c)P_n'(c) + (\psi(c) - \phi'(c))P_n(c) = 0, \quad (4.10)$$

for  $n \geq 2$ .

From recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

satisfied by  $\{P_n\}_n$  and (4.10) written for  $n+1$ ,  $n$  and  $n-1$ , we obtain

$$c\phi(c)P'_n(c) + [\phi(c) + c(\psi(c) - \phi'(c))]P_n(c) = 0, \quad n \geq 3, \quad (4.11)$$

and subtracting (4.10) from (4.11), we get

$$\phi(c)P_n(c) = 0, \quad n \geq 3.$$

Therefore, we conclude  $\phi(c) = 0$  and using again (4.10),  $\psi(c) - \phi'(c) = 0$ .  
□

**Corollary 4.5** *The only sequences of monic polynomials which are orthogonal with respect to a Sobolev bilinear form (2.1) and satisfy a three-term recurrence relation are*

- a) *The generalized Laguerre polynomials  $L_n^{(-N)}$ ,*
- b) *The generalized Jacobi polynomials  $P_n^{(-N, \beta)}$ , with  $\beta + N$  not a negative integer,*
- c) *The generalized Jacobi polynomials  $P_n^{(\alpha, -N)}$ , with  $\alpha + N$  not a negative integer.*

**Proof.** Suppose that the monic polynomials  $\{Q_n\}_n$  orthogonal with respect to (2.1) satisfy a three-term recurrence relation. Theorem 4.4 assures that  $u$  is a classical linear functional. If  $\mathcal{D}(\phi u) = \psi u$  is its distributional differential equation, the polynomials  $\phi$  and  $\psi$  are given by the following table

Name	$\phi$	$\psi$	Restrictions
Hermite	1	$-2x$	
Laguerre	$x$	$(\alpha + 1) - x$	$\alpha \neq -n, n \geq 1$
Jacobi	$1 - x^2$	$(\beta - \alpha) - (\alpha + \beta + 2)x$	$\alpha \neq -n, \beta \neq -n,$ $\alpha + \beta + 1 \neq -n, n \geq 1$
Bessel	$x^2$	$(\alpha + 2)x + 2$	$\alpha \neq -n, n \geq 2$

Then, conditions (4.4) and (4.5) exclude Hermite and Bessel cases. Moreover, in Laguerre and Jacobi cases the only possibilities are the following:

- Laguerre case with  $\alpha = 0$  and  $c = 0$ .
- Jacobi case with  $\alpha = 0, \beta \neq -m, m \geq 1$  and  $c = 1$ .
- Jacobi case with  $\beta = 0, \alpha \neq -m, m \geq 1$  and  $c = -1$ .

Taking into account the results of Section 3, we conclude. □

Note that a reduction of the degree of  $P_n^{(\alpha,\beta)}$  could occur when either both  $\alpha$  and  $\beta + n$  or  $\alpha + \beta + 1$  are negative integers (see [12], p.64). An interesting open problem is to give some kind of orthogonality relations valid for these polynomials. For the particular case  $\alpha = \beta$  negative integers, this problem has been solved in [1]: the corresponding Gegenbauer polynomials are orthogonal with respect to a discrete-continuous Sobolev bilinear form, where the discrete part is concentrated in two points, namely, 1 and  $-1$ .

**Corollary 4.6** *The monic orthogonal polynomials associated to the Sobolev bilinear form (2.1) satisfy a three term recurrence relation if and only if the linear functional  $u$  is classical and the point  $c$  in (2.1) satisfies  $\phi(c) = 0$  and  $\psi(c) = \phi'(c)$ .*

**Proof.** It follows from Theorem 4.4 and Corollary 4.5 taking in mind that Laguerre and Jacobi polynomials satisfy a three-term recurrence relation for all values of their parameters.  $\square$

## 5 Sobolev orthogonal polynomials and second order differential equations

As it is well known (see [12]) Laguerre and Jacobi polynomials satisfy a second order differential equation for every value of their respective parameters.

In this Section, our aim is to characterize the sequences of monic Sobolev orthogonal polynomials satisfying a second order differential equation. We can observe that if, for every  $n$ , the polynomials  $\{Q_n\}_n$  satisfy a second order differential equation

$$\phi(x)Q_n''(x) + \sigma(x)Q_n'(x) = \rho_n Q_n(x),$$

where  $\phi$  and  $\sigma$  are fixed polynomials and  $\rho_n \in \mathbb{R}$ , then

$$\deg \phi \leq 2, \quad \deg \sigma \leq 1.$$

Moreover, if  $\rho_1 \neq 0$  then  $\deg \sigma = 1$ .

**Theorem 5.1** *Let  $\{Q_n\}_n$  be the monic orthogonal polynomial sequence associated with the Sobolev bilinear form (2.1). If, for  $n \geq N$ , every polynomial  $Q_n$  satisfies a second order differential equation*

$$\phi(x)Q_n''(x) + \sigma(x)Q_n'(x) = \rho_n Q_n(x), \quad (5.1)$$

where  $\phi$  and  $\sigma$  are fixed polynomials with degree less than or equal to 2 and 1 respectively, and  $\rho_n \in \mathbb{R}$ , then the linear functional  $u$  is classical with distributional differential equation  $\mathcal{D}(\phi u) = \psi u$ ,  $\sigma(x) = \psi(x) - N\phi'(x)$  and the point  $c$  in (2.1) satisfies

$$\phi(c) = 0, \quad (5.2)$$

$$(N-1)\phi'(c) + \sigma(c) = 0. \quad (5.3)$$

**Proof.** Taking  $k$ -th order derivatives in (5.1), from Leibniz rule, we get

$$\begin{aligned} & \phi(x)Q_n^{(k+2)}(x) + (k\phi'(x) + \sigma(x))Q_n^{(k+1)}(x) \\ &= \left( \rho_n - \frac{k(k-1)}{2}\phi''(x) - k\sigma'(x) \right) Q_n^{(k)}(x). \end{aligned} \quad (5.4)$$

Let  $k = N$  in (5.4), then we deduce that the polynomials  $P_n$  orthogonal with respect to the linear functional  $u$  satisfy the second order differential equation

$$\phi(x)P_n''(x) + \psi(x)P_n'(x) = \lambda_n P_n(x), \quad n \geq 0,$$

where

$$\begin{aligned} \psi(x) &= N\phi'(x) + \sigma(x), \\ \lambda_n &= \rho_{n+N} - \frac{N(N-1)}{2}\phi''(x) - N\sigma'(x). \end{aligned}$$

Therefore  $u$  satisfies  $\mathcal{D}(\phi u) = \psi u$  and  $\deg \psi \geq 1$ . Now, since  $\deg \phi \leq 2$  and  $\deg \sigma \leq 1$ , from Bochner's characterization of the classical orthogonal polynomials, (see [2]), we deduce the classical character of the linear functional  $u$ .

Writing equation (5.4) for  $n = N$  and  $k = N-1$  we get

$$\begin{aligned} & ((N-1)\phi'(x) + \sigma(x))Q_N^{(N)}(x) \\ &= \left( \rho_N - \frac{(N-1)(N-2)}{2}\phi''(x) - (N-1)\sigma'(x) \right) Q_N^{(N-1)}(x), \end{aligned}$$

and by substitution in  $c$ , since  $Q_N^{(N-1)}(c) = 0$ , we deduce

$$(N-1)\phi'(c) + \sigma(c) = 0.$$

Finally, if  $N = 1$  writing (5.1) for  $n = 2$ , we get  $\phi(c) = 0$  and when  $N \geq 2$ , writing equation (5.4) for  $n = N$ ,  $k = N - 2$ , we get

$$\begin{aligned} & \phi(x)Q_N^{(N)}(x) + ((N-2)\phi'(x) + \sigma(x))Q_N^{(N-1)}(x) \\ &= \left( \rho_N - \frac{(N-2)(N-3)}{2}\phi''(x) - (N-2)\sigma'(x) \right) Q_N^{(N-2)}(x) \end{aligned}$$

and by substitution in  $c$  we deduce  $\phi(c) = 0$ .  $\square$

Using the same reasoning as in Corollary (4.5), we obtain

**Corollary 5.2** *The only sequences of monic polynomials which are orthogonal with respect to a Sobolev bilinear form (2.1) and satisfy a second order differential equation (5.1) are*

- a) *The generalized Laguerre polynomials  $L_n^{(-N)}$ ,*
- b) *The generalized Jacobi polynomials  $P_n^{(-N,\beta)}$ , with  $\beta + N$  not a negative integer,*
- c) *The generalized Jacobi polynomials  $P_n^{(\alpha,-N)}$ , with  $\alpha + N$  not a negative integer.*

**Corollary 5.3** *The monic orthogonal polynomials associated to the Sobolev bilinear form (2.1) satisfy a second order differential equation like (5.1) if and only if the linear functional  $u$  is classical and the point  $c$  in (2.1) satisfies  $\phi(c) = 0$  and  $\psi(c) = \phi'(c)$ .*

**Proof.** It follows from Theorem 5.1 and Corollary 5.2 taking in mind that Laguerre and Jacobi polynomials satisfy a second order differential equation for all values of their parameters.  $\square$

## 6 A symmetric differential operator: Properties

In order to obtain explicit relations between the sequences  $\{Q_n\}_n$  and  $\{P_n\}_n$  associated with  $\mathcal{B}_S^{(N)}$  and  $u$ , respectively, we introduce a linear differential operator  $\mathcal{F}^{(N)}$  closely related to  $u$ . To do this,  $u$  must satisfy an extra condition. This is why, from now on, the functional  $u$  in (2.1) will be a semiclassical one.

**Definition 6.1** ([3], [9]) *A linear functional  $u$  on  $\mathbb{P}$  is called semiclassical, if there exist two polynomials  $\phi$  and  $\psi$ , with  $\deg \phi = p \geq 0$  and  $\deg \psi = q \geq 1$ , such that  $u$  satisfies the following distributional differential equation*

$$\mathcal{D}(\phi u) = \psi u, \quad (6.1)$$

or equivalently

$$\phi \mathcal{D}u = (\psi - \phi')u. \quad (6.2)$$

Equation (6.2) can be generalized in the following way:

**Lemma 6.2** ([8]) *Let  $u$  be a semiclassical linear functional, then for every  $n$  we have*

$$\phi^n(x) \mathcal{D}^n u = \psi(x; n)u, \quad (6.3)$$

where the polynomials  $\psi(x; n)$  are recursively defined by

$$\begin{aligned} \psi(x; 0) &= 1, \\ \psi(x; n) &= \phi(x) \psi'(x; n-1) + \psi(x; n-1) [\psi(x) - n\phi'(x)], \quad n \geq 1. \end{aligned} \quad (6.4)$$

Observe that, now, (6.2) adopts the form  $\phi(x) \mathcal{D}u = \psi(x; 1)u$ . Taking derivatives  $(n-1)$ -times in this formula, we get

$$\phi(x) \mathcal{D}^n u = \sum_{i=0}^{n-1} \left[ \binom{n-1}{i} \mathcal{D}^{n-1-i} \psi(x; 1) - \binom{n-1}{i-1} \mathcal{D}^{n-i} \phi(x) \right] \mathcal{D}^i u, \quad (6.5)$$

for  $n \geq 1$ , where  $\binom{n}{m} = 0$  whenever  $m < 0$ . Multiplying by  $\phi^{n-1}$  and using (6.3), we obtain another recursive expression for  $\psi(x; n)$ :

$$\begin{aligned} \psi(x; n) &= \sum_{i=0}^{n-1} \left[ \binom{n-1}{i} \mathcal{D}^{n-1-i} \psi(x; 1) - \binom{n-1}{i-1} \mathcal{D}^{n-i} \phi(x) \right] \phi^{n-1-i}(x) \psi(x; i) \end{aligned} \quad (6.6)$$

valid for  $n \geq 1$ .

**Lemma 6.3** *In the above conditions, we have*

$$\phi^{n-j}(x)\psi(x; j)\mathcal{D}^n u = \psi(x; n)\mathcal{D}^j u, \quad n \geq j, \quad (6.7)$$

and consequently

$$\phi^i(x)\psi(x; N-i)\mathcal{D}^{N-j} u = \phi^j(x)\psi(x; N-j)\mathcal{D}^{N-i} u, \quad 0 \leq i, j \leq N. \quad (6.8)$$

**Proof.** We will show the result by induction on  $n$ . The case  $n = 1, j = 0$  comes from (6.3), while the case  $n = j = 1$  is trivial. Suppose that

$$\phi^{n-1-j}(x)\psi(x; j)\mathcal{D}^{n-1} u = \psi(x; n-1)\mathcal{D}^j u,$$

holds for all  $j, 0 \leq j \leq n-1$ . Then,

*i)* For  $0 \leq j < n-1$ , taking derivatives, multiplying by  $\phi$ , using (6.4), and the induction hypothesis, we have

$$\begin{aligned} \phi^{n-j}(x)\psi(x; j)\mathcal{D}^n u &= \phi(x)\psi'(x, n-1)\mathcal{D}^j u + \phi(x)\psi(x; n-1)\mathcal{D}^{j+1} u - \\ &\quad - \{(n-1-j)\phi'(x)\psi(x; j) + \psi(x; j+1) - \\ &\quad - \psi(x; j)[\psi(x) - (j+1)\phi'(x)]\}\phi^{n-1-j}(x)\mathcal{D}^{n-1} u \\ &= \phi(x)\psi'(x, n-1)\mathcal{D}^j u + \phi(x)\psi(x; n-1)\mathcal{D}^{j+1} u - \\ &\quad - \{\psi(x; j+1) - \psi(x; j)[\psi(x) - n\phi'(x)]\}\phi^{n-1-j}(x)\mathcal{D}^{n-1} u \\ &= \{\phi(x)\psi'(x, n-1) + \psi(x; n-1)[\psi(x) - n\phi'(x)]\}\mathcal{D}^j u + \\ &\quad + \phi(x)\psi(x; n-1)\mathcal{D}^{j+1} u - \psi(x; j+1)\phi^{n-1-j}(x)\mathcal{D}^{n-1} u \\ &= \psi(x; n)\mathcal{D}^j u + \phi(x)\psi(x; n-1)\mathcal{D}^{j+1} u - \phi(x)\psi(x; n-1)\mathcal{D}^{j+1} u \\ &= \psi(x; n)\mathcal{D}^j u. \end{aligned}$$

*ii)* If  $j = n-1$ , multiplying (6.5) by  $\psi(x; n-1)$ , and using the induction hypothesis, we obtain the result taking into account (6.6).

Now, from *i)* and *ii)*, we conclude, since the case  $j = n$  is trivial.  $\square$

We define a linear differential operator  $\mathcal{F}^{(N)}$  on the linear space of real polynomials  $\mathbb{P}$  in the following way

$$\mathcal{F}^{(N)} = (-1)^N (x-c)^N \sum_{i=0}^N \binom{N}{i} \phi^i(x)\psi(x; N-i)\mathcal{D}^{N+i}, \quad (6.9)$$

where  $\mathcal{D}$  denotes the derivative operator and the polynomials  $\psi(x; n)$  are defined as in Lemma 6.2.

**Remark.** In the particular case of a semiclassical linear functional  $u$  defined from a weight function, expression (6.9) can be written in a very compact form

$$\mathcal{F}^{(N)} = (-1)^N (x - c)^N \frac{\phi^N(x)}{\rho(x)} \left( \rho(x) \mathcal{D}^N \right)$$

where  $\rho$  denotes the weight function associated with the semiclassical linear functional  $u$ .

In the next Lemma, we recall a very useful formula involving derivatives.

**Lemma 6.4 ([8])** *Let  $f$  and  $g$  be  $n$ -times and  $2n$ -times differentiable functions, respectively. Then,*

$$f^{(n)} g^{(n)} = \sum_{i=0}^n (-1)^i \binom{n}{i} (fg^{(n+i)})^{(n-i)}.$$

In the following Proposition, we show how the linear operator  $\mathcal{F}^{(N)}$  allows us to obtain a representation for the Sobolev bilinear form (2.1), in terms of the consecutive derivatives of the semiclassical linear functional  $u$ .

**Proposition 6.5** *Let  $\mathcal{B}_S^{(N)}$  be a Sobolev bilinear form with  $u$  semiclassical and  $f, g$  arbitrary polynomials. Then, for  $0 \leq i \leq N$ , we have*

$$\mathcal{B}_S^{(N)} \left( (x - c)^N \phi^i(x) \psi(x; N - i) f, g \right) = \langle \mathcal{D}^{N-i} u, f \mathcal{F}^{(N)} g \rangle.$$

**Proof.** From Lemmas 6.2, 6.3 and 6.4, we get

$$\begin{aligned} & \mathcal{B}_S^{(N)} \left( (x - c)^N \phi^i(x) \psi(x; N - i) f, g \right) \\ &= \langle u, \left( (x - c)^N \phi^i(x) \psi(x; N - i) f \right)^{(N)} g^{(N)} \rangle \\ &= \sum_{j=0}^N (-1)^j \binom{N}{j} \langle u, \left( (x - c)^N \phi^i(x) \psi(x; N - i) f g^{(N+j)} \right)^{(N-j)} \rangle \\ &= \sum_{j=0}^N (-1)^N \binom{N}{j} \langle \mathcal{D}^{N-j} u, (x - c)^N \phi^i(x) \psi(x; N - i) f g^{(N+j)} \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^N (-1)^N \binom{N}{j} \langle \phi^i(x) \psi(x; N-i) \mathcal{D}^{N-j} u, (x-c)^N f g^{(N+j)} \rangle \\
&= \sum_{j=0}^N (-1)^N \binom{N}{j} \langle \phi^j(x) \psi(x; N-j) \mathcal{D}^{N-i} u, (x-c)^N f g^{(N+j)} \rangle \\
&= \langle \mathcal{D}^{N-i} u, f [(-1)^N (x-c)^N \sum_{j=0}^N \binom{N}{j} \phi^j(x) \psi(x; N-j) g^{(N+j)}] \rangle \\
&= \langle \mathcal{D}^{N-i} u, f \mathcal{F}^{(N)} g \rangle. \quad \square
\end{aligned}$$

**Theorem 6.6** *The linear operator  $\mathcal{F}^{(N)}$  is symmetric with respect to the Sobolev bilinear form (2.1), that is*

$$\mathcal{B}_S^{(N)}(\mathcal{F}^{(N)} f, g) = \mathcal{B}_S^{(N)}(f, \mathcal{F}^{(N)} g).$$

**Proof.** From Proposition 6.5 and Lemma 6.4, we can deduce

$$\begin{aligned}
\mathcal{B}_S^{(N)}(\mathcal{F}^{(N)} f, g) &= \sum_{i=0}^N (-1)^N \binom{N}{i} \mathcal{B}_S^{(N)}\left((x-c)^N \phi^i(x) \psi(x; N-i) f^{(N+i)}, g\right) \\
&= \sum_{i=0}^N (-1)^N \binom{N}{i} \langle \mathcal{D}^{N-i} u, f^{(N+i)} \mathcal{F}^{(N)} g \rangle \\
&= \sum_{i=0}^N (-1)^i \binom{N}{i} \langle u, (f^{(N+i)} \mathcal{F}^{(N)} g)^{(N-i)} \rangle \\
&= \langle u, f^{(N)} (\mathcal{F}^{(N)} g)^{(N)} \rangle = \mathcal{B}_S^{(N)}(f, \mathcal{F}^{(N)} g). \quad \square
\end{aligned}$$

Now, we study the degree of the polynomial  $\mathcal{F}^{(N)} x^n$  for every  $n$ . Observe that  $\mathcal{F}^{(N)}$  vanishes on every polynomial with degree less than  $N$ .

**Proposition 6.7** *For every  $n \geq 0$ , we have*

$$\deg(\mathcal{F}^{(N)} x^n) \leq n + N \max\{p-1, q\},$$

where  $p = \deg \phi$  and  $q = \deg \psi$ .

**Proof.** By using the induction method it is very easy to see that  $\deg \psi(x; n) \leq n + \max\{p-1, q\}$  for all  $n \geq 0$ . Taking into account the definition of the linear operator  $\mathcal{F}^{(N)}$ , the conclusion follows.  $\square$

On the other hand, the linear operator  $\mathcal{F}^{(N)}$  never reduces the degree for all polynomials.

**Proposition 6.8** *There exists  $n_0 \geq N$  such that*

$$\deg \mathcal{F}^{(N)} x^{n_0} \geq n_0.$$

**Proof.** Suppose that  $\deg \mathcal{F}^{(N)} x^n < n$ , for all  $n \geq N$ . Then, we can expand

$$\mathcal{F}^{(N)} Q_n = \sum_{i=0}^{n-1} a_{n,i} Q_i.$$

Thus, since  $\mathcal{F}^{(N)}$  vanishes on every polynomial of degree less than  $N$ , using its symmetry property, we have

$$a_{n,i} = \frac{\mathcal{B}_S^{(N)}(\mathcal{F}^{(N)} Q_n, Q_i)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = \frac{\mathcal{B}_S^{(N)}(Q_n, \mathcal{F}^{(N)} Q_i)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = 0, \quad i = 0, 1, \dots, n-1,$$

and the result follows.  $\square$

To study the degree of  $\mathcal{F}^{(N)} x^n$ , we need to know the degree of the polynomials  $\psi(x; N-i)$ ,  $i = 0, \dots, N$  in formula (6.9). The following Lemma provides us some combinatorial identities, that will be useful for our purpose.

**Lemma 6.9** *Let  $a$  and  $b$  arbitrary real numbers. Then for every non negative integer  $n$ , we have*

$$i) \quad (a)_n = (-1)^n (-a - n + 1)_n,$$

$$ii) \quad \sum_{i=0}^n \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n},$$

$$iii) \quad \sum_{i=0}^n (-1)^i \binom{a}{i} \binom{b-i}{n-i} = \binom{b-a}{n},$$

where  $\binom{a}{k}$  and  $(a)_k$  are given by (3.2) and (3.3).

**Proof.** *i)* It is a direct consequence of the definition of the Pochhammer's symbol.

*ii)* It can be derived from the power series expansion of the identity

$$(1+x)^a(1+x)^b = (1+x)^{a+b},$$

comparing the coefficients.

*iii)* This formula can be deduced from *i)* and *ii)*.  $\square$

**Remark.** For  $a$  and  $b$  positive integer numbers, formulas *ii)* and *iii)* can be found on page 8 in [11].

Let assume that the explicit representation for the polynomials  $\phi$  and  $\psi$  is given by

$$\phi(x) = \sum_{i=0}^p a_i x^i, \quad a_p \neq 0, \quad p \geq 0, \quad \psi(x) = \sum_{i=0}^q b_i x^i, \quad b_q \neq 0, \quad q \geq 1,$$

and without loss of generality, we can suppose that  $a_p = 1$ .

Next Lemma gives us the degree of the polynomial  $\psi(x; n)$  and its leading coefficient in terms of  $p, q$  and  $b_q$ .

**Lemma 6.10** *The following assertions are true:*

*i)* If  $p - 1 < q$ , then

$$\psi(x; n) = b_q^n x^{nq} + \text{lower degree terms},$$

and  $\deg \psi(x; n) = nq, \quad n \geq 0$ .

*ii)* If  $p - 1 > q$ , then

$$\psi(x; n) = (-1)^n (p)_n x^{n(p-1)} + \text{lower degree terms},$$

and  $\deg \psi(x; n) = n(p-1), \quad n \geq 0$ .

*iii)* If  $p - 1 = q$ , then

$$\psi(x; n) = (-1)^n (p - b_q)_n x^{n(p-1)} + \text{lower degree terms}.$$

Therefore,

*iii.1)* If  $p - b_q \neq 0, -1, \dots, -(n-1)$  then  $\deg \psi(x; n) = n(p-1), \quad n \geq 0$ ,

*iii.2)* If  $p - b_q = -k, \quad k \geq 0$ , then  $\deg \psi(x; n) = n(p-1), \quad 0 \leq n \leq k$ , and  $\deg \psi(x; n) < n(p-1), \quad n > k$ .

**Proof.** These results can be obtained by induction on  $n$ .  $\square$

As we are going to see, the equality in Proposition 6.7 is true for almost all  $n$ , that is, the action of  $\mathcal{F}^{(N)}$  on a polynomial of degree bigger than or equal to  $N$  increases its degree exactly in  $Nt$  being  $t = \max\{p-1, q\}$ . Therefore, we can write

$$\mathcal{F}^{(N)}x^n = F(n; N, t)x^{n+Nt} + \dots,$$

where  $F(n; N, t)$  denotes the leading coefficient of the polynomial  $\mathcal{F}^{(N)}x^n$ . We want to notice that this coefficient can be zero for some specific values of  $n$  and in particular for every  $n < N$ .

To prove this, we decompose the operator  $\mathcal{F}^{(N)}$  in  $N+1$  differential operators defined by

$$\mathcal{F}_i^{(N)} = (-1)^N (x-c)^N \binom{N}{i} \phi^i(x) \psi(x; N-i) \mathcal{D}^{N+i}, \quad i = 0, 1, \dots, N. \quad (6.10)$$

Thus,

$$\mathcal{F}^{(N)} = \sum_{i=0}^N \mathcal{F}_i^{(N)}, \quad \text{and} \quad \mathcal{F}^{(N)}x^n = \sum_{i=0}^{\min\{N, n-N\}} \mathcal{F}_i^{(N)}x^n, \quad n \geq N,$$

where, for  $i = 0, \dots, \min\{N, n-N\}$ ,

$$\mathcal{F}_i^{(N)}x^n = (-1)^N (x-c)^N \binom{N}{i} \phi^i(x) \psi(x; N-i) \frac{n!}{(n-N-i)!} x^{n-N-i}.$$

Let us denote by  $F_i(n)$  the leading coefficient of the polynomial  $\mathcal{F}_i^{(N)}x^n$  and, for the sake of simplicity, we will put  $F(n; N, t) = F(n)$ .

**Theorem 6.11** *Let  $t = \max\{p-1, q\}$ . Except for finitely many values of  $n \geq N$ , we have*

$$\deg \mathcal{F}^{(N)}x^n = n + Nt,$$

that is,

$$\mathcal{F}^{(N)}x^n = F(n)x^{n+Nt} + \text{lower terms degree},$$

with  $F(n) \neq 0$ . More precisely

i) If  $p-1 < q$ , then

$$F(n) = (-b_q)^N \frac{n!}{(n-N)!},$$

and  $\deg \mathcal{F}^{(N)}x^n = n + Nt$ ,  $n \geq N$ .

ii) If  $p - 1 > q$ , then

$$F(n) = (p - n + N)_N \frac{n!}{(n - N)!},$$

and

$$\begin{aligned} \deg \mathcal{F}^{(N)}x^n &< n + Nt, & N + p \leq n \leq 2N - 1 + p, \\ \deg \mathcal{F}^{(N)}x^n &= n + Nt, & N \leq n < N + p, \quad n \geq 2N + p. \end{aligned}$$

iii) If  $p - 1 = q$ , then

$$F(n) = (p - b_q - n + N)_N \frac{n!}{(n - N)!},$$

and

iii.1) if  $p - b_q = -k$ ,  $k = 0, 1, \dots, N - 1$ , then

$$\begin{aligned} \deg \mathcal{F}^{(N)}x^n &< n + Nt, & N \leq n \leq 2N - 1 - k, \\ \deg \mathcal{F}^{(N)}x^n &= n + Nt, & n \geq 2N - k, \end{aligned}$$

iii.2) if  $p - b_q$  is a positive integer, then

$$\begin{aligned} \deg \mathcal{F}^{(N)}x^n &< n + Nt, & N + p - b_q \leq n \leq 2N - 1 + p - b_q, \\ \deg \mathcal{F}^{(N)}x^n &= n + Nt, & N \leq n < N + p - b_q, \quad n \geq 2N + p - b_q. \end{aligned}$$

iii.3) in another case,

$$\deg \mathcal{F}^{(N)}x^n = n + Nt, \quad n \geq N.$$

**Proof.** To prove the theorem a basic tool will be Lemma 6.10. For this reason, we distinguish three different cases.

i) Case  $p - 1 < q$ . In this situation, we have

$$\deg \mathcal{F}_i^{(N)}x^n = n + Nq - i(q - (p - 1)), \quad i = 0, 1, \dots, \min\{N, n - N\},$$

and then,  $\deg \mathcal{F}^{(N)}x^n = \deg \mathcal{F}_0^{(N)}x^n = n + Nq$ , for all  $n \geq N$ .

The explicit expression for  $\mathcal{F}_0^{(N)}x^n$  is

$$\begin{aligned}\mathcal{F}_0^{(N)}x^n &= (-1)^N(x-c)^N\psi(x;N)\mathcal{D}^Nx^n \\ &= (-b_q)^N\frac{n!}{(n-N)!}x^{n+Nq} + \text{lower terms degree,}\end{aligned}$$

and the leading coefficient for  $\mathcal{F}^{(N)}x^n$  is

$$F(n) = F_0(n) = (-b_q)^N\frac{n!}{(n-N)!}, \quad n \geq N.$$

ii) Case  $p-1 > q$ . In this case, as  $p > 2$ , we get

$$\deg \mathcal{F}_i^{(N)}x^n = n + N(p-1), \quad i = 0, \dots, \min\{N, n-N\},$$

and then  $\deg \mathcal{F}^{(N)}x^n \leq n + N(p-1)$ , for all  $n \geq N$ . The leading coefficient of  $\mathcal{F}_i^{(N)}x^n$  is

$$F_i(n) = (-1)^i \binom{N}{i} (p)_{N-i} \frac{n!}{(n-N-i)!}, \quad i = 0, 1, \dots, \min\{N, n-N\}.$$

Taking into account that

$$F(n) = \sum_{i=0}^{\min\{N, n-N\}} F_i(n), \quad (6.11)$$

and using Lemma 6.9 *iii*), we can show that:

If  $N \leq n < 2N$ , since  $\min\{N, n-N\} = n-N$ , we obtain

$$\begin{aligned}F(n) &= \sum_{i=0}^{n-N} F_i(n) = n! (p)_{2N-n} \sum_{i=0}^{n-N} (-1)^i \binom{N}{i} \binom{p-1+N-i}{n-N-i} \\ &= n! (p)_{2N-n} \binom{p-1}{n-N} = (p-n+N)_N \frac{n!}{(n-N)!}.\end{aligned}$$

On the other hand, if  $n \geq 2N$ ,

$$\begin{aligned}F(n) &= \sum_{i=0}^N F_i(n) = \frac{n! N!}{(n-N)!} \sum_{i=0}^N (-1)^i \binom{n-N}{i} \binom{p-1+N-i}{N-i} \\ &= \frac{n! N!}{(n-N)!} \binom{p-1-n+2N}{N} = (p-n+N)_N \frac{n!}{(n-N)!}.\end{aligned}$$

Observe that, since  $p$  is an integer bigger than 2, from the definition of the Pochhammer's symbol  $(p - n + N)_N$ , we have  $F(n) = 0$  if and only if  $N + p \leq n \leq 2N - 1 + p$ . Then, there exist exactly  $N$  values of  $n$  such that  $\deg \mathcal{F}^{(N)} x^n < n + N(p - 1)$ . In another case, we have  $\deg \mathcal{F}^{(N)} x^n = n + N(p - 1)$ .

iii) Case  $p - 1 = q$ .

First, we assume that  $p - b_q = -k$ ,  $k = 0, 1, \dots, N - 1$ . In this case, by Lemma 6.10,  $\deg \psi(x; N - i) = (N - i)(p - 1)$ , if  $0 \leq N - i \leq k$  and  $\deg \psi(x; N - i) < (N - i)(p - 1)$  for  $k + 1 \leq N - i \leq N$ .

Therefore, for  $n \geq N$ , we have  $\deg \mathcal{F}_i^{(N)} x^n < n + N(p - 1)$  when  $i = 0, 1, \dots, N - k - 1$ , and  $\deg \mathcal{F}_i^{(N)} x^n = n + N(p - 1)$  if  $N - k \leq i \leq N$ . In this way,  $\deg \mathcal{F}^{(N)} x^n < n + N(p - 1)$  when  $N \leq n \leq 2N - k - 1$ .

For  $n \geq 2N - k$ , we can observe that

$$F(n) = \sum_{i=N-k}^{\min\{N, n-N\}} F_i(n),$$

where, by Lemma 6.9 i),

$$\begin{aligned} F_i(n) &= (-1)^N \binom{N}{i} (-1)^{N-i} (-k)_{N-i} \frac{n!}{(n - N - i)!} \\ &= (-1)^N \binom{N}{i} (i + 1 - (N - k))_{N-i} \frac{n!}{(n - N - i)!}. \end{aligned}$$

Now, we give an explicit expression of  $F(n)$ . Suppose  $2N - k \leq n < 2N$ , then  $\min\{N, n - N\} = n - N$ , and using Lemma 6.9 ii)

$$\begin{aligned} F(n) &= (-1)^N \sum_{i=N-k}^{n-N} \binom{N}{i} (i + 1 - (N - k))_{N-i} \frac{n!}{(n - N - i)!} \\ &= (-1)^N \frac{n! N!}{(n - N)!} \sum_{i=0}^{n-(2N-k)} \binom{k}{i} \binom{n - N}{n - (2N - k) - i} \\ &= (-1)^N \frac{n! N!}{(n - N)!} \binom{n + k - N}{n - (2N - k)} \\ &= (-1)^N (n + 1 - (2N - k))_N \frac{n!}{(n - N)!}. \end{aligned}$$

If  $n \geq 2N$ , again by Lemma 6.9 *ii*), we have

$$\begin{aligned}
F(n) &= (-1)^N \sum_{i=N-k}^N \binom{N}{i} (i+1 - (N-k))_{N-i} \frac{n!}{(n-N-i)!} \\
&= (-1)^N \frac{n! k!}{(n - (2N-k))!} \sum_{i=0}^k \binom{n - (2N-k)}{i} \binom{N}{k-i} \\
&= (-1)^N \frac{n! k!}{(n - (2N-k))!} \binom{n+k-N}{k} \\
&= (-1)^N (n+1 - (2N-k))_N \frac{n!}{(n-N)!}.
\end{aligned}$$

Hence, if  $n \geq 2N - k$ ,

$$F(n) = (-1)^N (n+1 - (2N-k))_N \frac{n!}{(n-N)!} = (-k-n+N)_N \frac{n!}{(n-N)!} \neq 0.$$

Now, we assume that  $p - b_q \neq 0, -1, \dots, -(N-1)$  and thus, from Lemma 6.10,  $\deg \psi(x; N-i) = (N-i)(p-1)$ ,  $i = 0, 1, \dots, N$ . As in the case  $p-1 > q$ , we have  $\deg \mathcal{F}_i^{(N)} x^n = n + N(p-1)$ ,  $i = 0, 1, \dots, \min\{N, n-N\}$ ,  $\deg \mathcal{F}^{(N)} x^n \leq n + N(p-1)$ , and also (6.11), where

$$F_i(n) = (-1)^i \binom{N}{i} (p-b_q)_{N-i} \frac{n!}{(n-N-i)!}, \quad i = 0, 1, \dots, \min\{N, n-N\}.$$

Using the same technique, we obtain

$$F(n) = (p-b_q-n+N)_N \frac{n!}{(n-N)!}, \quad n \geq N.$$

If  $p - b_q$  is a positive integer, then  $F(n) = 0$  if and only if  $N + p - b_q \leq n \leq 2N - 1 + p - b_q$ , that is, there exist precisely  $N$  values of  $n$  such that  $\deg \mathcal{F}^{(N)} x^n < n + N(p-1)$ , and for the other values of  $n \geq N$ ,  $F(n) \neq 0$  and hence  $\deg \mathcal{F}^{(N)} x^n = n + N(p-1)$ .

In another case,  $\deg \mathcal{F}^{(N)} x^n = n + N(p-1)$ , for all  $n \geq N$ .  $\square$

## 7 Recurrence relations and differential operators

As a direct consequence of Proposition 6.5, which relates the discrete-continuous Sobolev bilinear form  $\mathcal{B}_S^{(N)}$  and the one defined from a semi-classical linear functional  $u$ , we can establish some relations between the

monic Sobolev orthogonal polynomials  $\{Q_n\}_n$  and the monic orthogonal polynomials  $\{P_n\}_n$ , associated with the semiclassical linear functional  $u$ . In the sequel, for the sake of simplicity, we will denote

$$k_n = \langle u, P_n^2 \rangle \neq 0, \quad \tilde{k}_n = \mathcal{B}_S^{(N)}(Q_n, Q_n) \neq 0, \quad \forall n \geq 0.$$

**Proposition 7.1** *The following formulas hold:*

$$i) \quad (x-c)^N \phi^N(x) P_n(x) = \sum_{i=r}^{n+N(p+1)} \alpha_i^{(n)} Q_i(x), \quad n \geq 0, \quad (7.1)$$

$$\text{where } r = \max\{N, n-Nt\}, \quad \alpha_{n+N(p+1)}^{(n)} = 1 \quad \text{and} \quad \alpha_r^{(n)} = \frac{\langle u, P_n \mathcal{F}^{(N)} Q_r \rangle}{\tilde{k}_r}.$$

$$ii) \quad \mathcal{F}^{(N)} Q_n(x) = \sum_{i=n-N(p+1)}^{n+Nt} \beta_i^{(n)} P_i(x), \quad n \geq N(p+1), \quad (7.2)$$

$$\text{where } \beta_{n+Nt}^{(n)} = F(n), \quad \beta_{n-N(p+1)}^{(n)} = \frac{\tilde{k}_n}{k_{n-N(p+1)}}.$$

**Proof.**

*i)* Expanding the polynomial  $(x-c)^N \phi^N P_n$  in terms of the Sobolev polynomials  $Q_n$ , we have

$$(x-c)^N \phi^N(x) P_n(x) = \sum_{i=0}^{n+N(p+1)} \alpha_i^{(n)} Q_i(x),$$

where, taking into account Proposition 6.5,

$$\alpha_i^{(n)} = \frac{\mathcal{B}_S^{(N)}((x-c)^N \phi^N P_n, Q_i)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = \frac{\langle u, P_n \mathcal{F}^{(N)} Q_i \rangle}{\tilde{k}_i}.$$

From the orthogonality of  $\{P_n\}_n$  and since  $\mathcal{F}^{(N)} Q_i = 0$  for  $i < N$ , we deduce that  $\alpha_i^{(n)} = 0$  when  $0 \leq i < r = \max\{N, n-Nt\}$ .

*ii)* Because of Proposition 6.7, the expansion of the polynomial  $\mathcal{F}^{(N)} Q_n$  in terms of  $P_n$  is

$$\mathcal{F}^{(N)} Q_n(x) = \sum_{i=0}^{n+Nt} \beta_i^{(n)} P_i(x).$$

The coefficients  $\beta_i^{(n)}$  can be computed using again Proposition 6.5, and therefore

$$\beta_i^{(n)} = \frac{\langle u, P_i \mathcal{F}^{(N)} Q_n \rangle}{\langle u, P_i^2 \rangle} = \frac{\mathcal{B}_S^{(N)} \left( (x-c)^N \phi^N P_i, Q_n \right)}{k_i}.$$

Finally, from the orthogonality of  $\{Q_n\}_n$  it follows  $\beta_i^{(n)} = 0$  for  $0 \leq i < n - N(p+1)$ .  $\square$

From the symmetry of the linear operator  $\mathcal{F}^{(N)}$ , we can obtain a difference–differential relation satisfied by the Sobolev orthogonal polynomials with respect to the Sobolev bilinear form (2.1), where  $u$  is a semiclassical linear functional.

**Proposition 7.2 (Difference–Differential Relation)** *For every  $n \geq N$ , the following relation holds*

$$\mathcal{F}^{(N)} Q_n(x) = \sum_{i=r}^{n+Nt} \gamma_i^{(n)} Q_i(x), \quad (7.3)$$

where  $r = \max\{N, n - Nt\}$ ,  $\gamma_{n+Nt}^{(n)} = F(n)$  and  $\gamma_r^{(n)} = \frac{\mathcal{B}_S^{(N)}(Q_n, \mathcal{F}^{(N)} Q_r)}{\tilde{k}_r}$ .

**Proof.** Consider the Fourier expansion of the polynomial  $\mathcal{F}^{(N)} Q_n$  in terms of  $Q_n$  which, by Proposition 6.7, is

$$\mathcal{F}^{(N)} Q_n(x) = \sum_{i=0}^{n+Nt} \gamma_i^{(n)} Q_i(x).$$

Then

$$\gamma_i^{(n)} = \frac{\mathcal{B}_S^{(N)}(\mathcal{F}^{(N)} Q_n, Q_i)}{\mathcal{B}_S^{(N)}(Q_i, Q_i)} = \frac{\mathcal{B}_S^{(N)}(Q_n, \mathcal{F}^{(N)} Q_i)}{\tilde{k}_i},$$

where we have used Theorem 6.6. Notice that  $\gamma_i^{(n)} = 0$  for  $0 \leq i < N$  and that the orthogonality of the polynomials  $\{Q_n\}_n$  leads to  $\gamma_i^{(n)} = 0$  for  $0 \leq i < n - Nt$ . So the result follows.  $\square$

**Remark.** In formulas (7.1) and (7.3), when  $r = n - Nt$ , the coefficients  $\alpha_r^{(n)}$  and  $\gamma_r^{(n)}$  can explicitly be given by

$$\alpha_r^{(n)} = F(r) \frac{k_n}{\tilde{k}_r}, \quad \gamma_r^{(n)} = F(r) \frac{\tilde{k}_n}{\tilde{k}_r}.$$

Recall that the values of  $F(n)$  had been calculated in Theorem 6.11.

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