A new approach to the asymptotics of Sobolev type orthogonal polynomials

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Abstract

This paper deals with Mehler-Heine type asymptotic formulas for so called discrete Sobolev orthogonal polynomials whose continuous part is given by Laguerre and generalized Hermite measures. We use a new approach which allows to solve the problem when the discrete part contains an arbitrary (finite) number of mass points.

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1 Introduction

Let $\{\mu_i\}_{i=0}^r$ be finite positive Borel measures supported on the real line. Define the Sobolev space

$$W^{2,r}(\mu_0, \mu_1, \ldots, \mu_r) := \{f : \int |f|^2 d\mu_0 + \sum_{i=1}^{r} \int |f^{(i)}|^2 d\mu_i < +\infty\}$$

with the inner product

$$(f, g) = \int f\, g d\mu_0 + \sum_{i=1}^{r} \int f^{(i)}\, g^{(i)} d\mu_i.$$

It is very well known that this inner product is nonstandard; that is, $(xf, g) \neq (f, xg)$. Consequently, some nice properties of standard orthogonal polynomials (for example, the three–term recurrence relation and the interlacing properties of zeros) are lost. More importantly, some powerful methods and techniques developed through the years to treat standard orthogonal polynomials have not found their equivalent in this setting and many questions remain unanswered.

Here, we give a general solution to one of those problems. Let μ be a finite positive Borel measure supported on the real line, $c \in \mathbb{R}$ and $M_i \geq 0$ for $i = 0, 1, ..., r$. We consider an inner product of the form

$$(f, g) = \int f(x)g(x)d\mu(x) + \sum_{i=0}^{r} M_i f^{(i)}(c) g^{(i)}(c),$$

and let $\{Q_n\}_{n\geq 0}$ be the corresponding sequence of monic orthogonal polynomials. Such products are called of discrete type. More general discrete type products, in which derivatives of different order are multiplied, have also been studied. Recently in [13] the authors prove that every symmetric bilinear form of this type can be reduced to the diagonal case; therefore, to some extent, we are considering the most general situation (as long as the reduction is plausible).

Our aim is to obtain Mehler-Heine asymptotic formulas for the sequence of Sobolev orthogonal polynomials when the measure which appears in the continuous part is Laguerre or generalized Hermite. We do this comparing the Sobolev orthogonal polynomial and its classical counterpart, and see how the addition of derivatives in the inner product affects the orthogonal system. Some applications of Sobolev discrete type orthogonality within the theory of standard orthogonal polynomials are known. For instance, some standard

classical polynomials with nonstandard parameters are not orthogonal in the usual sense, but they are orthogonal with respect to nonhermitian inner products (see e.g. [10] for the Laguerre case and [11] for the Jacobi case). Moreover, they are also orthogonal with respect to a discrete type Sobolev inner products (see e.g. [1] or [12]). This last approach has its origin in a paper by A. A. Gonchar where he studies the convergence of diagonal Padé approximation to meromorphic Markov type functions (see [7]). Its first use in the context of discrete type Sobolev orthogonal polynomials (and more general) dates to [14].

This idea allows to reinterpret Sobolev orthogonality as standard quasi–orthogonality (where some orthogonality conditions are lost). Consequently, the polynomial Q_n can be expressed as a linear combination (with a fixed number of terms) of standard orthogonal polynomials R_n corresponding to the modified measure $d\nu = (x-c)^{r+1}d\mu$; that is,

$$Q_n(x) = \sum_{j=0}^{r+1} a_n^j R_{n-j}(x)\,. \tag{1}$$

This approach has proved to be fruitful when μ has compact support and c lies in the complement of the support of the measure.

In the bounded case, a straightforward argument allows to prove that all the connection coefficients a_n^j are bounded. If the measure μ is in the Nevai class, the orthogonal polynomials R_n have ratio asymptotic which simplifies the study of (1), in order to get the relative asymptotics of Q_n/R_n. The situation is quite different in the case of measures with unbounded support. For example, consider the Laguerre probability measure, i.e. $d\mu(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}dx$ with $\alpha > -1$, and the inner product

$$(f,g)_r = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + \sum_{i=0}^{r} M_i f^{(i)}(0)g^{(i)}(0), \tag{2}$$

where $M_i > 0$, $i = 0,\dots,r$. We will see in Theorem 2 that the connection coefficients which appear in (1) are unbounded, and it is well known that the Laguerre polynomials do not have ratio asymptotics. So, new ideas must be brought in to make efficient use of (1).

We will study the difference between the Laguerre polynomials and the Sobolev polynomials $Q_{n,r}$ orthogonal with respect to (2). Because of the structure of (2), one can imagine that the main difference between two polynomials of equal degree lies around the origin, where the perturbation of the standard inner product takes place. We will see that is so.

To carry out our plan, we consider Mehler–Heine type formulas since they allow to describe precisely the Laguerre–Sobolev type polynomials close to the origin. In [4], with $r = 1$ and $M_0, M_1 > 0$ the authors find an asymptotic formula for the polynomials Q_n, and conjecture the expression this formula should take in the general case. We formulate the conjecture as stated more clearly in [15].

If $M_i > 0$ for $i = 0, \ldots, r$, in the inner product (2), then

$$\lim_{n\to\infty} \frac{(-1)^n}{n!n^\alpha} Q_{n,r}\left(\frac{x}{n}\right) = (-1)^{r+1} x^{-\alpha/2} J_{\alpha+2r+2}(2\sqrt{x}),$$

uniformly on compact subsets of the complex plane, where $\{Q_{n,r}\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to (2) *and J_α is the Bessel function of the first kind of order α.*

In Theorem 1 we prove that the conjecture is true. In [4], the authors benefit from the fact that there are explicit formulas for the connection coefficients. In fact,

$$Q_{n,1}(x) = B_0(n)L_n^\alpha(x) + B_1(n)xL_{n-1}^{\alpha+2}(x) + B_2(n)x^2L_{n-2}^{\alpha+4}(x)$$

where the coefficients $B_i(n)$ are given explicitly in [9]. For a general Laguerre–Sobolev inner product, we only know that the $B_i(n)$ are a non trivial solution of a system of $r + 1$ equations on $r + 2$ unknowns. If the system is solved, you get an intricate expression with which it is difficult to work. So we must follow general arguments. Our approach is new in the framework of Sobolev orthogonal polynomials.

Mehler–Heine type formulas are interesting twofold: they provide the scaled asymptotics for $Q_{n,r}$ on compact sets of the complex plane, and supply information on the location and asymptotic distributions of the zeros of these polynomials in terms of the zeros of known special functions. In particular, applying Hurwitz's Theorem, we prove that there exists an acceleration of the convergence of $r + 1$ zeros of these Sobolev polynomials to the origin.

In the paper, we also deal with the situation when some $M_i = 0$. We say that such Sobolev inner product have *gaps*. For example, consider the inner product

$$(f, g)_{r,s}$$
$$= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + \sum_{i=0}^{r} M_i f^{(i)}(0)g^{(i)}(0) + M_s f^{(s)}(0)g^{(s)}(0),$$

where $s \geq r + 2$ and $M_i > 0$ for $i = 0, \ldots, r$ and $i = s$.

In this situation, we also establish Mehler-Heine type formulas. We wish to remark that this case has qualitative differences with respect to the case without gaps. For example, concerning the convergence acceleration to 0 of the zeros of the polynomials, the result does not depend on the number of terms in the discrete part, but it depends on the position of the gap. So, despite the presence of the mass M_s, there only exists an acceleration of the convergence of $r+1$ zeros, such as occurs in the case of the inner product without the gap.

From the previous results, using a symmetrization process, we prove in Proposition 2 the Mehler–Heine type formulas for generalized Hermite-Sobolev type polynomials.

The structure of the paper is as follows. In Section 2 we introduce the notation, the basic tools, and remind some properties of classical Laguerre polynomials. In Section 3 we obtain some new results for Laguerre–Sobolev type orthogonal polynomials that we use to establish our main results in Sections 4 and 5. More precisely, Section 4 is devoted to obtain the Mehler–Heine type formulas for the orthogonal polynomials with respect to a discrete Sobolev inner product with positive masses, and in Section 5 we get the corresponding ones for orthogonal polynomials with respect to an inner product with a gap. In Section 4, we give the size of coefficients appearing in the connection formula between the Laguerre-Sobolev type and the Laguerre polynomials. The exterior strong asymptotics for the sequences of Sobolev orthogonal polynomials considered in Sections 4 and 5 are easily deduced from the relative asymptotics obtained in Section 3.

2 Notation and basic results

Throughout this work we will deal with classical Laguerre polynomials; that is, polynomials orthogonal with respect to the inner product in the space of all polynomials with real coefficients

$$(p,q) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx\,, \quad \alpha > -1.$$

We will denote by L_n^α the nth monic Laguerre polynomial.

Many of the properties of Laguerre polynomials can be seen, for example, in Szegő's book [17]. In what follows we summarize those properties which will be used in this paper.

It is known that the *monic Laguerre polynomials* are defined by

$$L_n^\alpha(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!}\,,$$

and their L_2-norm is

$$\|L_n^\alpha\|^2 = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty (L_n^\alpha(x))^2\, x^\alpha\, e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} n!\,. \tag{3}$$

The evaluation at $x = 0$ of the polynomial L_n^α and its successive derivatives are given by

$$(L_n^\alpha)^{(k)}(0) = \frac{(-1)^k\, n!}{(n-k)!} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} L_n^\alpha(0) = \frac{(-1)^{n+k}\, n!}{(n-k)!} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)}. \tag{4}$$

A useful tool for some estimates is the *Stirling formula*:

$$\Gamma(x+1) \sim x^x e^{-x} \sqrt{2\pi x} \quad (x \to +\infty),$$

where the symbol $f(x) \sim g(x) \quad (x \to a)$ stands for $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. In particular,

$$\Gamma(n+\alpha+1) \sim n!\, n^\alpha. \tag{5}$$

As a consequence, from (3), (4), and (5), we get

$$\lim_n \frac{(L_n^\alpha(0))^2}{\|L_n^\alpha\|^2\, n^\alpha} = \frac{1}{\Gamma(\alpha+1)}\,. \tag{6}$$

The following asymptotic results are known. They can be deduced from Perron's formula in Szegő's book [17],

$$\lim_n \frac{n L_{n-1}^\alpha(x)}{L_n^\alpha(x)} = -1, \tag{7}$$

$$\lim_n \frac{n^{1/2} L_n^\alpha(x)}{L_n^{\alpha+1}(x)} = \sqrt{-x}. \tag{8}$$

In both formulas, convergence is uniform on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

The nth kernel for the Laguerre polynomials $K_n(x,y) = \sum_{i=0}^{n} \frac{L_i^\alpha(x) L_i^\alpha(y)}{\|L_i^\alpha\|^2}$ satisfies the Christoffel–Darboux formula

$$K_n(x,y) = \frac{1}{\|L_n^\alpha\|^2} \frac{L_{n+1}^\alpha(x) L_n^\alpha(y) - L_{n+1}^\alpha(y) L_n^\alpha(x)}{x-y}\,.$$

As usual, we denote the derivatives of the kernels by

$$K_n^{(k,s)}(x,y) = \frac{\partial^{k+s}}{\partial x^k \partial y^s} K_n(x,y) = \sum_{i=0}^{n} \frac{(L_i^\alpha)^{(k)}(x)(L_i^\alpha)^{(s)}(y)}{\|L_i^\alpha\|^2}$$

with $k, s \in \mathbb{N} \cup \{0\}$ and the convention $K_n^{(0,0)}(x,y) = K_n(x,y)$.

In the next lemma, we show some formulas for the derivatives of the kernels, that we will need throughout the paper.

Lemma 1 *The derivatives of the kernels of the Laguerre polynomials, for $k, s \in \mathbb{N} \cup \{0\}$, satisfy*

(a)

$$K_{n-1}^{(0,s)}(x,0) = \frac{1}{\|L_{n-1}^\alpha\|^2} \frac{s!}{x^{s+1}} \left[P_s(x,0;L_{n-1}^\alpha) L_n^\alpha(x) - P_s(x,0;L_n^\alpha) L_{n-1}^\alpha(x) \right]$$

where $P_s(x,0;f)$ is the sth Taylor polynomial of f at 0.

(b)

$$K_{n-1}^{(k,s)}(0,0) = \frac{k!\,s!}{\|L_{n-1}^\alpha\|^2} \sum_{j=0}^{s} \frac{k+s+1-2j}{n-j} \frac{(L_{n-1}^\alpha)^{(j)}(0)(L_n^\alpha)^{(k+s+1-j)}(0)}{j!\,(k+s+1-j)!},$$

$$K_{n-1}^{(k,0)}(0,0) = (-1)^k \frac{\Gamma(\alpha+n+1)}{(n-(k+1))!\,\Gamma(\alpha+k+2)}.$$

Proof. (a) The result follows from the Christoffel-Darboux formula and Leibniz's rule.

(b) Observe that, according to Taylor's formula, $\frac{1}{k!} K_{n-1}^{(k,s)}(0,0)$ is precisely the coefficient of x^k in $K_{n-1}^{(0,s)}(x,0)$, therefore

$$K_{n-1}^{(k,s)}(0,0)$$
$$= \frac{k!\,s!}{\|L_{n-1}^\alpha\|^2} \sum_{j=0}^{s} \frac{(L_{n-1}^\alpha)^{(j)}(0)(L_n^\alpha)^{(k+s+1-j)}(0) - (L_n^\alpha)^{(j)}(0)(L_{n-1}^\alpha)^{(k+s+1-j)}(0)}{j!\,(k+s+1-j)!}.$$

In particular, for $s = 0$, straightforward computations lead us to conclude this lemma. □

Throughout the paper we work with sequences of monic orthogonal polynomials, and we use the acronym SMOP for them.

3 Auxiliary results

From now on $\{Q_{n,r}\}_{n\geq 0}$ denotes the sequence of monic Laguerre–Sobolev orthogonal polynomials with respect to an inner product of the form

$$(p,q)_r = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)\,x^\alpha e^{-x}\,dx + \sum_{i=0}^{r} M_i p^{(i)}(0)q^{(i)}(0), \qquad (9)$$

where $\alpha > -1$ and $M_i > 0$, $i = 0,\ldots,r$. Notice that all the masses in the discrete part of this inner product are positive.

We write $K_{n,r}$ for the corresponding nth kernel, that is $K_{n,r}(x,y) = \sum_{j=0}^{n} \frac{Q_{j,r}(x)Q_{j,r}(y)}{(Q_{j,r},Q_{j,r})_r}$, and $K_{n,r}^{(k,s)}$ for the derivatives of the kernels.

Observe that, in fact, $(.,.)_r$, $Q_{n,r}$, $K_{n,r}$ and $K_{n,r}^{(k,s)}$ also depend on the parameter α but for simplicity we have omitted it in the notations.

In the next lemma, we obtain an asymptotic estimation for $Q_{n,r}^{(k)}(0)$, $k \geq 0$, that will play an important role along this paper. To do this, we need to know the "size" of the kernels of $Q_{n,r}$ and their derivatives.

Lemma 2 *Let $Q_{n,r}$ be the monic polynomials orthogonal with respect to the inner product (9). Then the following statements hold:*

(a) For $0 \leq k \leq r$,

$$\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \sim \frac{C_{r,k}}{n^{\alpha+2k+1}},$$

where $C_{r,k}$ is a nonzero real number independent of n.

For $k \geq r+1$,

$$\lim_n \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \frac{k!}{(k-(r+1))!}\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)}\,.$$

(b)

$$\lim_n \frac{(Q_{n,r},Q_{n,r})_r}{\|L_n^\alpha\|^2} = 1\,.$$

Proof. We use mathematical induction on $r \in \mathbb{N}\cup\{0\}$.

If $r = 0$, the Fourier expansion of the polynomial $Q_{n,0}$ in the orthogonal basis $\{L_n^\alpha\}_{n\geq 0}$ leads to

$$Q_{n,0}(x) = L_n^\alpha(x) - M_0 Q_{n,0}(0)K_{n-1}(x,0)\,,$$

and therefore

$$Q_{n,0}(x) = L_n^\alpha(x) - \frac{M_0 L_n^\alpha(0)}{1 + M_0 K_{n-1}(0,0)} K_{n-1}(x,0)\,. \tag{10}$$

As a consequence of (4) and Lemma 1 (b), we obtain (a) for $r = 0$.
Using (10), we have

$$(Q_{n,0}, Q_{n,0})_0 = \|L_n^\alpha\|^2 + \frac{M_0 (L_n^\alpha(0))^2}{1 + M_0 K_{n-1}(0,0)}\,.$$

Thus, from (6) and Lemma 1 (b), it follows (b) for $r = 0$.

Suppose now that (a) and (b) hold for the SMOP $\{Q_{n,r}\}_{n\geq 0}$ with $r > 0$, then we are going to deduce that they are also true for the sequence $\{Q_{n,r+1}\}_{n\geq 0}$. To do this, we observe that

$$(p,q)_{r+1} = (p,q)_r + M_{r+1} p^{(r+1)}(0) q^{(r+1)}(0) \tag{11}$$

and therefore

$$Q_{n,r+1}(x) = Q_{n,r}(x) - M_{r+1} Q_{n,r+1}^{(r+1)}(0) K_{n-1,r}^{(0,r+1)}(x,0)\,. \tag{12}$$

Taking derivatives $r+1$ times in (12) and evaluating at $x = 0$, we obtain

$$Q_{n,r+1}^{(r+1)}(0) = \frac{Q_{n,r}^{(r+1)}(0)}{1 + M_{r+1} K_{n-1,r}^{(r+1,r+1)}(0,0)}\,. \tag{13}$$

Taking now derivatives k times in (12), evaluating at $x = 0$, and using (13) we get

$$\frac{Q_{n,r+1}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} - \frac{M_{r+1} K_{n-1,r}^{(k,r+1)}(0,0)}{1 + M_{r+1} K_{n-1,r}^{(r+1,r+1)}(0,0)} \frac{Q_{n,r}^{(r+1)}(0)}{(L_n^\alpha)^{(k)}(0)}\,. \tag{14}$$

Before taking limits in the last expression, we need to estimate $K_{n-1,r}^{(k,r+1)}(0,0)$. Applying Stolz criterion (see e.g. [8]) the induction hypothesis for $\{Q_{n,r}\}_{n\geq 0}$, (4) and (6), we obtain for $k \geq r+1$

$$\begin{aligned}
\lim_n \frac{K_{n-1,r}^{(k,r+1)}(0,0)}{n^{\alpha+k+r+2}} &= \lim_n \frac{Q_{n-1,r}^{(k)}(0) Q_{n-1,r}^{(r+1)}(0)}{\|L_{n-1}^\alpha\|^2 (\alpha+k+r+2)\; n^{\alpha+k+r+1}} \\
&= \frac{(-1)^{k+r+1}\,\Gamma(\alpha+1)}{(\alpha+k+r+2)\Gamma(\alpha+k+1)\Gamma(\alpha+r+2)} \lim_n \left[\frac{Q_{n-1,r}^{(k)}(0)}{(L_{n-1}^\alpha)^{(k)}(0)} \; \frac{Q_{n-1,r}^{(r+1)}(0)}{(L_{n-1}^\alpha)^{(r+1)}(0)} \right] \\
&= \frac{k!(r+1)!}{(k-(r+1))!} \frac{(-1)^{k+r+1}\Gamma(\alpha+1)}{\Gamma(\alpha+k+r+3)\Gamma(\alpha+2r+3)}\,,
\end{aligned} \tag{15}$$

and therefore, from (14), we get (a) for $k \geq r+1$.

Now, if $0 \leq k \leq r$, to estimate the size of $K_{n-1,r}^{(k,r+1)}(0,0)$, we use Stolz criterion again and thus, we obtain

$$\lim_n \frac{K_{n-1,r}^{(k,r+1)}(0,0)}{n^{r+1-k}} = (-1)^{r+1+k} \frac{(r+1)!}{r+1-k} \frac{C_{r,k}\,\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha+2r+3)}\,.$$

Using the induction hypothesis and substituting all these results in the right–hand side of (14) we get

$$\frac{Q_{n,r+1}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \sim \frac{C_{r+1,k}}{n^{\alpha+2k+1}}\,,$$

where $C_{r+1,k} = -\frac{\alpha+k+r+2}{r+1-k}\,C_{r,k} \neq 0$. Therefore the proof of (a) is complete.

To finish we only need to deduce (b) for $\{Q_{n,r+1}\}_{n\geq 0}$. As in the case $r=0$, from (11) and (13), we get

$$(Q_{n,r+1}, Q_{n,r+1})_{r+1} = (Q_{n,r}, Q_{n,r})_r + \frac{M_{r+1}(Q_{n,r}^{(r+1)}(0))^2}{1 + M_{r+1}K_{n-1,r}^{(r+1,r+1)}(0,0)}\,.$$

Using (a) for $k=r+1$, (3), (4) and (15) we achieve the result. $\square$

Observe that both Laguerre and Laguerre–Sobolev type polynomials have asymptotically the same global size from the point of view of the norm, while the size of the successive derivatives at the point $x=0$ is affected by the discrete part of the inner product, but only when the order of the derivatives corresponds to a positive mass.

We can also establish the following relative asymptotics.

Proposition 1 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product defined by (9). Then, for $k \geq 0$,*

$$\lim_n \frac{Q_{n,r}^{(k)}(x)}{(L_n^\alpha)^{(k)}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

Proof. From the Fourier expansion of the polynomial $Q_{n,r}$ in terms of the Laguerre polynomials and Lemma 1 (a) we get

$$\frac{Q_{n,r}(x)}{L_n^\alpha(x)} = 1 - \sum_{i=0}^{r} \frac{M_i\, i!\, Q_{n,r}^{(i)}(0) P_i(x,0;L_{n-1}^\alpha)}{\|L_{n-1}^\alpha\|^2\, x^{i+1}} \left[1 - \frac{P_i(x,0;L_n^\alpha)}{P_i(x,0;L_{n-1}^\alpha)} \frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right].$$

Since

$$\lim_n \frac{P_i(x,0;L_n^\alpha)}{(L_n^\alpha)^{(i)}(0)} = \frac{x^i}{i!}\,, \tag{16}$$

from (4) and (7), we have, for $i = 0, \ldots, r$,

$$\lim_n \left[1 - \frac{P_i(x,0;L_n^\alpha)}{P_i(x,0;L_{n-1}^\alpha)} \frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)}\right] = 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

Moreover, taking into account (4), (16), and Lemma 2 (a), there exists

$$\lim_n \frac{M_i\, i! Q_{n,r}^{(i)}(0) P_i(x,0;L_{n-1}^\alpha)}{\|L_{n-1}^\alpha\|^2\, x^{i+1}} \in \mathbb{C}\,.$$

Therefore, each one of the terms in the sum tends to 0 uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$ and the result for $k = 0$ follows.

From this result, since the functions $Q_{n,r}/L_n^\alpha$ are analytic in $\mathbb{C} \setminus [0,\infty)$, we have

$$\lim_n \left(\frac{Q_{n,r}'(x)}{(L_n^\alpha)'(x)} - \frac{Q_{n,r}(x)}{L_n^\alpha(x)}\right) \frac{(L_n^\alpha)'(x)}{L_n^\alpha(x)} = \lim_n \left(\frac{Q_{n,r}}{L_n^\alpha}\right)'(x) = 0\,,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$. Therefore, the result holds for $k = 1$, and, for $k > 1$ it suffices to use an induction procedure. $\square$

4 Main results

As we have mentioned in the introduction, if we consider a general discrete Sobolev inner product, the key used to obtain some results is the possibility to transform the Sobolev orthogonality into a standard quasi–orthogonality.

Now, in our particular case, the sequence $\{Q_{n,r}\}_{n\geq 0}$ orthogonal with respect to the inner product defined by (9) is quasi–orthogonal of order $r+1$ with respect to the Laguerre weight $x^{\alpha+r+1}e^{-x}$, that is,

$$\int_0^{+\infty} p(x) Q_{n,r}(x) x^{\alpha+r+1} e^{-x} dx = 0,$$

for every polynomial p with $\deg p \leq n - (r+1) - 1$. Therefore, as an immediate consequence we have a *connection formula* of the form

$$Q_{n,r}(x) = \sum_{j=0}^{r+1} a_{n,r}^j L_{n-j}^{\alpha+r+1}(x)\,, \quad a_{n,r}^0 = 1. \tag{17}$$

We have just shown that both sequences of orthogonal polynomials, $\{Q_{n,r}\}_{n\geq 0}$ and $\{L_n^\alpha\}_{n\geq 0}$, are asymptotically identical on compact subsets of $\mathbb{C}\setminus[0,\infty)$. In this section, our main aim is to establish their differences through Mehler–Heine type formulas which describe the asymptotic behavior around the origin.

4.1 Mehler–Heine type formulas

First of all, we recall the corresponding formula for the monic Laguerre polynomials, (see [17, Th.8.1.3]):

$$\lim_n \frac{(-1)^n}{n!\,n^\alpha} L_n^\alpha\left(\frac{x}{n}\right) = x^{-\alpha/2} J_\alpha(2\sqrt{x}), \tag{18}$$

uniformly on compact subsets of $\mathbb{C}$, where J_α is the Bessel function of the first kind of order α ($\alpha > -1$), defined by

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}.$$

To get the Mehler–Heine type formulas for $\{Q_{n,r}\}_{n\geq 0}$, we focus on the problem in a different way than in the bounded case. As we will see later, it is worth noticing that the knowledge of the asymptotic behavior of the connection coefficients is not enough to obtain directly these asymptotic formulas.

Write the Taylor expansion of the polynomial $Q_{n,r}$

$$\frac{(-1)^n}{n!\,n^\alpha} Q_{n,r}\left(\frac{x}{n}\right) = \sum_{k=0}^{n} \frac{(-1)^n}{n!\,n^\alpha} \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \frac{(L_n^\alpha)^{(k)}(0)}{k!} \frac{x^k}{n^k}.$$

Then, to calculate the limit, we use Lebesgue's dominated convergence theorem. For this purpose, we need to find a uniform bound for the ratios $\dfrac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$. As a first step, using the connection formula (17), in the next lemma, we obtain a new algebraic expression with a nice structure which relates the derivative of order $k+1$ of the polynomials $Q_{n,r}$ to the derivative of order k.

Lemma 3 *Fixed $r \geq 1$, let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product (9). Then, we have for $0 \leq k \leq n-1$,*

$$\frac{Q_{n,r}^{(k+1)}(0)}{(L_n^\alpha)^{(k+1)}(0)} = \frac{\alpha+k+1}{\alpha+r+k+2}\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \tag{19}$$
$$+ \frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)}\sum_{i=1}^{k+1}\binom{k}{i-1}\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+i+1)}\frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)},$$

where

$$A_{n,r}^i = \sum_{j=i}^{r+1}\frac{j!}{(j-i)!}\,a_{n,r}^j\,L_{n-j}^{\alpha+r+1}(0), \quad i=1,\ldots,r+1\,,$$

and the coefficients $a_{n,r}^j$ are those of (17). By convention, $A_{n,r}^i = 0$, when $i > r+1$. Besides,

$$\lim_n \frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)} = \begin{cases} 0 & \text{if} \quad 1 \leq i \leq r\,, \\ (r+1)! & \text{if} \quad i = r+1\,. \end{cases} \tag{20}$$

Proof. Taking derivatives $k+1$ times in (17), evaluating at $x=0$, and using (4) several times we get, for $k \geq 0$,

$$Q_{n,r}^{(k+1)}(0) = \sum_{j=0}^{r+1} a_{n,r}^j (L_{n-j}^{\alpha+r+1})^{(k+1)}(0) = -\sum_{j=0}^{r+1} a_{n,r}^j \frac{n-j-k}{\alpha+r+k+2}(L_{n-j}^{\alpha+r+1})^{(k)}(0)$$
$$= -\frac{n-k}{\alpha+r+k+2}Q_{n,r}^{(k)}(0) + \frac{1}{\alpha+r+k+2}\sum_{j=1}^{r+1} j a_{n,r}^j (L_{n-j}^{\alpha+r+1})^{(k)}(0)$$
$$= -\frac{n-k}{\alpha+r+k+2}Q_{n,r}^{(k)}(0)$$
$$+ \frac{(-1)^k\,\Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)}\sum_{j=1}^{r+1} j a_{n,r}^j \frac{(n-j)!}{(n-j-k)!}L_{n-j}^{\alpha+r+1}(0)\,.$$

According to formula (5) in page 8 of [16],

$$\frac{(n-j)!}{(n-j-k)!} = k!\sum_{i=0}^{k}(-1)^i\binom{j-1}{i}\binom{n-1-i}{k-i},$$

and therefore

$$\frac{(n-j)!}{(n-j-k)!} = \frac{(j-1)!}{(n-(k+1))!}\sum_{i=1}^{k+1}(-1)^{i-1}\binom{k}{i-1}\frac{(n-i)!}{(j-i)!}\,.$$

Thus, the above expression can be written in the form

$$Q_{n,r}^{(k+1)}(0) = -\frac{n-k}{\alpha+r+k+2} Q_{n,r}^{(k)}(0)$$
$$+ \frac{(-1)^k \Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)} \sum_{i=1}^{k+1} (-1)^{i-1} \binom{k}{i-1} \frac{(n-i)!}{(n-(k+1))!} A_{n,r}^i \,,$$

which leads to

$$\frac{Q_{n,r}^{(k+1)}(0)}{(L_n^\alpha)^{(k+1)}(0)} = -\frac{n-k}{\alpha+r+k+2} \frac{(L_n^\alpha)^{(k)}(0)}{(L_n^\alpha)^{(k+1)}(0)} \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$$
$$+ \frac{(-1)^k \Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)} \sum_{i=1}^{k+1} (-1)^{i-1} \binom{k}{i-1} \frac{(n-i)!}{(n-(k+1))!} \frac{(L_n^\alpha)^{(i)}(0)}{(L_n^\alpha)^{(k+1)}(0)} \frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)}$$
$$= \frac{\alpha+k+1}{\alpha+r+k+2} \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$$
$$+ \frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)} \sum_{i=1}^{k+1} \binom{k}{i-1} \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+i+1)} \frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)} \,,$$

where we have used expression (4). So, the first part of the lemma is proved.

To deduce (20) for $i = 1$, we apply (19) for $k = 0$, that is

$$\frac{Q_{n,r}^{(1)}(0)}{(L_n^\alpha)^{(1)}(0)} = \frac{\alpha+1}{\alpha+r+2} \frac{Q_{n,r}(0)}{(L_n^\alpha)(0)} + \frac{1}{\alpha+r+2} \frac{A_{n,r}^1}{(L_n^\alpha)^{(1)}(0)}$$

and by Lemma 2 (a), we get

$$\lim_n \frac{A_{n,r}^1}{(L_n^\alpha)^{(1)}(0)} = 0$$

Now, to deduce the result for $i \geq 2$, it is enough to apply a recursive procedure. $\square$

The above lemma, although quite technical, will be very useful in what follows. In the next subsection, we will use formula (20) to estimate the size of the connection coefficients, while (19) leads us to obtain a uniform bound for the ratios $\dfrac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$.

Lemma 4 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product (9). Then, fixed $r \geq 1$ there exists a positive integer n_0 such that, for all $n \geq n_0$ and for all k with $r+1 \leq k \leq n$, the inequality*

$$\left|\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}\right| \leq 2(r+1)\frac{k!}{(k-r)!}(k-(r-1))\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)}\,,$$

holds. Furthermore, for $r \geq 0$, there exists $n_0 \in \mathbb{N}$ such that,

$$\left|\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}\right| \leq 2(r+1), \quad \forall n \geq n_0 \quad 0 \leq k \leq n\,. \tag{21}$$

Proof. We prove the Lemma using mathematical induction on k, i.e., on the order of the derivative.

Keeping in mind Lemma 2 (a) for $k = r+1$ and Lemma 3, there exists a positive integer n_0, independent of k, such that for all $n \geq n_0$, the following formulas hold,

$$\left|\frac{Q_{n,r}^{(r+1)}(0)}{(L_n^\alpha)^{(r+1)}(0)}\right| \leq 4(r+1)(r+1)!\frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+2r+3)}\,, \tag{22}$$

$$\frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+i+1)}\left|\frac{A_{n,r}^{i}}{(L_n^\alpha)^{(i)}(0)}\right| \leq 1\,, \quad i = 1,\ldots,r\,, \tag{23}$$

and

$$\left|\frac{A_{n,r}^{r+1}}{(L_n^\alpha)^{(r+1)}(0)}\right| \leq 2(r+1)!\,. \tag{24}$$

Notice that formula (22) is the required bound for $k = r+1$. Now, we assume that the result holds for a fixed k, with $k \geq r+1$, and then we will deduce that it holds for $k+1$.

Taking absolute values in (19) and using induction hypothesis, (23) and (24), we get for $n \geq n_0$ and $k+1 \leq n$

$$
\begin{aligned}
&\left|\frac{Q_{n,r}^{(k+1)}(0)}{(L_n^\alpha)^{(k+1)}(0)}\right| \leq \frac{\alpha+k+1}{\alpha+r+k+2}\left|\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}\right| \\
&+\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)}\left[\sum_{i=1}^{r}\binom{k}{i-1}+\binom{k}{r}2(r+1)!\right] \\
&\leq \left[2(r+1)\frac{k!\,(k-(r-1))}{(k-r)!}+2(r+1)\frac{k!}{(k-r)!}+\sum_{i=1}^{r}\binom{k}{i-1}\right]\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \\
&\leq \left[2(r+1)\,\frac{k!}{(k-r)!}\,(k+1-(r-1))+r\frac{k!}{(k+1-r)!}\right]\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \\
&\leq \left[2(r+1)\,\frac{(k+1)!}{(k-r)!}+\frac{(k+1)!}{(k+1-r)!}\right]\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \\
&\leq 2(r+1)\,\frac{(k+1)!}{(k+1-r)!}\,(k+1-(r-1))\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)}\,.
\end{aligned}
$$

So, the first part of Lemma is proved. For the second part, using (4), (10) and Lemma 1 (b), we deduce for every $n \geq k$ the explicit expression

$$
\frac{Q_{n,0}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = 1 - \frac{M_0 K_{n-1}(0,0)}{1+M_0 K_{n-1}(0,0)}\frac{\alpha+1}{\alpha+k+1}\frac{n-k}{n}\,.
$$

Then,

$$
0 < \frac{Q_{n,0}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} < 1\,,
$$

holds for all k with $0 \leq k \leq n$. According to this fact and the first part of Lemma 4, we have (21). $\square$

Next, we show how the presence of the masses in the inner product changes the asymptotic behavior around the origin. This result proves the conjecture posed in [4].

Theorem 1 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product (9). Then,*

$$
\lim_n \frac{(-1)^n}{n!\,n^\alpha} Q_{n,r}\left(\frac{x}{n}\right) = (-1)^{r+1} x^{-\alpha/2} J_{\alpha+2r+2}(2\sqrt{x}),
$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. Using the Taylor expansion of the polynomial $Q_{n,r}$ we can write

$$\frac{(-1)^n}{n!\,n^\alpha} Q_{n,r}\left(\frac{x}{n}\right) = \sum_{k=0}^{n} \frac{(-1)^n}{n!\,n^\alpha} \frac{Q_{n,r}^{(k)}(0)}{k!} \frac{x^k}{n^k}\,.$$

To obtain the asymptotic behavior of the above expression when $n \to \infty$, we use Lebesgue's dominated convergence theorem. Indeed, given a compact set $K \subset \mathbb{C}$, from (4), (5), and (21) in Lemma 4, there exists a positive integer n_0 such that for all $n \geq n_0$, for all $j \geq 0$ and for all $x \in K$,

$$\begin{aligned}
&\frac{1}{n!\,n^\alpha}\left|\frac{Q_{n,r}^{(k)}(0)}{k!}\frac{x^k}{n^k}\right| \leq \frac{\Gamma(\alpha+1)|L_n^\alpha(0)|}{n!\,n^\alpha}\frac{n!}{(n-k)!\,n^k}\frac{2(r+1)}{\Gamma(\alpha+k+1)}\frac{|x|^k}{k!} \\
&\leq \frac{4(r+1)}{\Gamma(\alpha+k+1)}\frac{M^k}{k!}\,,
\end{aligned}$$

for each $k \geq 0$, where M is a positive constant depending on K. As $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha+k+1)}\frac{M^k}{k!}$ converges, the assumptions of Lebesgue's dominated convergence theorem are satisfied. Then, using Lemma 2 (a), (4), and (5), we have

$$\begin{aligned}
&\lim_n \sum_{k=0}^{n} \frac{(-1)^n}{n!\,n^\alpha}\frac{Q_{n,r}^{(k)}(0)}{k!}\frac{x^k}{n^k} \\
&= \sum_{k=r+1}^{\infty} \frac{(-1)^k}{(k-(r+1))!}\frac{1}{\Gamma(\alpha+k+r+2)}\,x^k = (-1)^{r+1}x^{-\alpha/2}J_{\alpha+2r+2}(2\sqrt{x})\,,
\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$. Thus, the result follows. $\square$

We will show a remarkable difference between the zeros of the orthogonal polynomials L_n^α and the ones of $Q_{n,r}$ concerning the convergence acceleration to 0.

Before analyzing this, recall (see [17]) that the zeros of the Laguerre polynomials are real, simple and they are located in $(0,\infty)$. We denote by $(x_{n,k})_{k=1}^n$ the zeros of L_n^α in an increasing order. It is worth pointing out that they satisfy the interlacing property $0 < x_{n+1,1} < x_{n,1} < x_{n+1,2} < \dots$, and that $x_{n,k} \underset{n}{\to} 0$ for each fixed k.

Let $(j_{\alpha,k})_{k\geq 1}$ be the positive zeros of the Bessel function J_α in an increasing order. Then, formula (18) and Hurwitz's theorem lead us to

$$n x_{n,k} \underset{n}{\to} j_{\alpha,k}, \quad k \geq 1,$$

and therefore

$$x_{n,k} \sim \frac{C_k}{n}, \quad k \geq 1,$$

where C_k is a positive constant depending on k.

Concerning the zeros of $Q_{n,r}$, standard arguments (see for instance [5]) allow us to establish that $Q_{n,r}$ has at least $n-(r+1)$ zeros with odd multiplicity in $(0,+\infty)$, or equivalently $n-(r+1)$ changes of sign. Moreover, since $M_0 > 0$ and the mass point in the discrete part of the inner product belongs to the boundary of $(0,+\infty)$, then the number of zeros with odd multiplicity is at least $n-r$ (see [2]).

From Theorem 1, Hurwitz's theorem, and taking into account the multiplicity of 0 as a zero of the limit function in Theorem 1, we achieve

Corollary 1 *Let $(\xi_{n,k}^r)_{k=1}^n$ be the zeros of $Q_{n,r}$. Then*

$$n\,\xi_{n,k}^r \underset{n}{\to} 0, \quad 1 \leq k \leq r+1,$$

$$n\,\xi_{n,k}^r \underset{n}{\to} j_{\alpha+2r+2,k-r-1}, \quad k \geq r+2.$$

Remark 1. The presence of the positive masses M_i, $i = 0,\ldots,r$, in the inner product produces a convergence acceleration to 0 of $r+1$ zeros of the polynomials $Q_{n,r}$.

4.2 Connection coefficients

In this subsection, we deduce from Lemma 3 a nice result having his own interest; concretely, we obtain the size of the connection coefficients $a_{n,r}^j$.

Theorem 2 *Let $a_{n,r}^j$ be the connection coefficients which appear in (17). Then, we have*

$$\lim_n \frac{a_{n,r}^j}{n^j} = \binom{r+1}{j}, \quad 0 \leq j \leq r+1\,.$$

Proof. From (20) and the expression

$$A_{n,r}^{r+1} = (r+1)!\,a_{n,r}^{r+1} L_{n-r-1}^{\alpha+r+1}(0) = \frac{(r+1)!\;a_{n,r}^{r+1}}{n(n-1)\ldots(n-r)} (L_n^{\alpha})^{(r+1)}(0)\,,$$

it follows easily

$$\lim_n \frac{a_{n,r}^{r+1}}{n^{r+1}} = 1.$$

A recurrence procedure leads to the result. Indeed, we assume that the result holds for $k+1 \leq j \leq r+1$, and we will show that it is true for $j=k$. From (20), for $i=k$, we can obtain

$$\lim_n \sum_{j=k}^{r+1} \frac{j!}{k!\,(j-k)!} \frac{a_{n,r}^j}{n^{r+1-k}} \frac{L_{n-j}^{\alpha+r+1}(0)}{(L_n^\alpha)^{(k)}(0)} = 0\,.$$

From (4) and (5), we get

$$\lim_n \sum_{j=k}^{r+1} (-1)^{k-j} \binom{j}{k} \frac{a_{n,r}^j}{n^j} = 0.$$

From the assumption

$$\lim_n \frac{a_{n,r}^j}{n^j} = \binom{r+1}{j}\,, \quad k+1 \leq j \leq r+1,$$

we have

$$\begin{aligned}\lim_n \frac{a_{n,r}^k}{n^k} &= \sum_{j=k+1}^{r+1} (-1)^{k+1-j} \binom{j}{k} \binom{r+1}{j} \\ &= \binom{r+1}{k} \sum_{j=k+1}^{r+1} (-1)^{k+1-j} \binom{r+1-k}{j-k} = \binom{r+1}{k}. \quad \square\end{aligned}$$

Remark 2. As we have claimed before, the Mehler–Heine type formulas for polynomials $Q_{n,r}$ cannot be directly deduced as a consequence of the connection formula.

4.3 Generalized Hermite–Sobolev polynomials

As a consequence of the previous results, we are going to establish asymptotic properties for the orthogonal polynomials associated with the following inner product

$$(p,q) = \int_{\mathbb{R}} p(x)q(x)|x|^{2\mu}\, e^{-x^2} dx + \sum_{i=0}^{2r+1} M_i\, p^{(i)}(0)\, q^{(i)}(0), \tag{25}$$

with $\mu > -1/2$ and $M_i > 0, \quad i = 0, \ldots, 2r+1$. We denote by $S_{n,r}^\mu$ their monic orthogonal polynomials.

The polynomials H_n^μ orthogonal with respect to the weight $|x|^{2\mu}\, e^{-x^2}$ ($\mu > -1/2$) are called *generalized Hermite polynomials.*

Notice that in this case the polynomials $S_{n,r}^\mu$ are symmetric, that is, $S_{n,r}^\mu(-x) = (-1)^n\, S_{n,r}^\mu(x)$, and because of this symmetry, we can transform this inner product (25) into an inner product like (9), and so we can establish a simple relation between the polynomials $S_{n,r}^\mu$ and the polynomials $Q_{n,r}$ considered before. This technique is known as a symmetrization process. In fact, in [5] this process is considered for standard inner products associated with positive measures. The simplest case of this situation is the relation between monic Laguerre polynomials and Hermite polynomials, that is (see [5] or [17]),

$$H_{2n}(x) = L_n^{-1/2}(x^2), \quad H_{2n+1}(x) = xL_n^{1/2}(x^2), \quad n \geq 0.$$

As a consequence we have

$$S_{2n,r}^\mu(x) = Q_{n,r}^{\mu-1/2}(x^2), \quad S_{2n+1,r}^\mu(x) = xQ_{n,r}^{\mu+1/2}(x^2)$$

where $\{Q_{n,r}^{\mu-1/2}\}_{n\geq 0}$ (respectively, $\{Q_{n,r}^{\mu+1/2}\}_{n\geq 0}$) is the SMOP with respect to an inner product like (9) with $\alpha = \mu - 1/2$ (respectively, $\alpha = \mu + 1/2$).

Therefore, taking in mind Proposition 1, the relative asymptotics

$$\lim_n \frac{(S_{n,r}^\mu)^{(k)}(x)}{(H_n^\mu)^{(k)}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$, $k \geq 0$, easily follows.

Also, applying Theorem 1 in a straightforward way, we obtain

Proposition 2 *Let $\{S_{n,r}^\mu\}_{n\geq 0}$ be the SMOP with respect to the inner product (25). Then,*

$$\lim_n \frac{(-1)^n\sqrt{n}}{n!\, n^\mu} S_{2n,r}^\mu\left(\frac{x}{2\sqrt{n}}\right) = (-1)^{r+1}\left(\frac{x}{2}\right)^{-\mu+1/2} J_{\mu+2r+3/2}(x),$$
$$\lim_n \frac{(-1)^n}{n!\, n^\mu} S_{2n+1,r}^\mu\left(\frac{x}{2\sqrt{n}}\right) = (-1)^{r+1}\left(\frac{x}{2}\right)^{-\mu+1/2} J_{\mu+2r+5/2}(x),$$

uniformly on compact subsets of $\mathbb{C}$.

Remark 3. These results generalize some of the ones in [3] and solve the conjecture posed there.

5 Inner products with gaps

In this section, we are concerned with inner products such that in their discrete part at least one of the masses vanishes:

$$(p,q)_{r,s} = (p,q)_r + M_s p^{(s)}(0) q^{(s)}(0), \quad s \geq r+2, \tag{26}$$

where $M_s > 0$, and $(.,.)_r$ is defined in (9) if $r \geq 0$, and, for $r = -1$ is the classical Laguerre inner product, and therefore in this case, the polynomials $Q_{n,r}$ are the Laguerre polynomials L_n^α. So, roughly speaking, there is a "gap" in the discrete part of the inner product $(.,.)_{r,s}$. We denote by $\{T_{n,r,s}\}_{n\geq 0}$ the sequence of monic polynomials orthogonal with respect to the inner product $(.,.)_{r,s}$.

The Fourier expansion of the polynomial $T_{n,r,s}$ in the orthogonal basis $\{Q_{n,r}\}_{n\geq 0}$ gives

$$T_{n,r,s}(x) = Q_{n,r}(x) - \frac{M_s Q_{n,r}^{(s)}(0)}{1 + M_s K_{n-1,r}^{(s,s)}(0,0)} K_{n-1,r}^{(0,s)}(x,0)\,. \tag{27}$$

Using similar arguments as in Lemma 2, it can be proved

Lemma 5 *Let $\{T_{n,r,s}\}_{n\geq 0}$ be the SMOP with respect to the inner product (26). Then the following statements hold:*

(a) For either $0 \leq k \leq r$ or $k = s$,

$$\frac{T_{n,r,s}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \sim \frac{C_{r,s,k}}{n^{\alpha+2k+1}}\,,$$

where $C_{r,s,k}$ is a nonzero real number independent of n.

For $k \geq r+1$ and $k \neq s$

$$\lim_n \frac{T_{n,r,s}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \frac{k!}{(k-(r+1))!} \frac{k-s}{\alpha+s+k+1} \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)}\,.$$

(b)

$$\lim_n \frac{(T_{n,r,s} T_{n,r,s})_{r,s}}{\|L_n^\alpha\|^2} = 1\,.$$

The above lemma also allows to deduce the relative asymptotics for these orthogonal polynomials, that is, for $k \geq 0$, we have

$$\lim_n \frac{T_{n,r,s}^{(k)}(x)}{(L_n^\alpha)^{(k)}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Now, we obtain the Mehler–Heine type formula for the polynomials $\{T_{n,r,s}\}_{n\geq 0}$.

Theorem 3 *Let $\{T_{n,r,s}\}_{n\geq 0}$ be the SMOP with respect to the inner product (26). Then,*

$$\lim_n \frac{(-1)^n}{n!\, n^\alpha} T_{n,r,s}\left(\frac{x}{n}\right) = (-1)^{r+1} x^{-\frac{\alpha}{2}} \left[\frac{-(s-(r+1))}{\alpha+r+s+2} J_{\alpha+2r+2}(2\sqrt{x}) + \sum_{i=2}^{s-r+1} \lambda_i J_{\alpha+2r+2i}(2\sqrt{x}) \right], \tag{28}$$

where λ_i are nonzero real numbers. The limit holds uniformly on compact subsets of $\mathbb{C}$.

Proof. From (27),

$$\begin{aligned} &\frac{(-1)^n}{n!\, n^\alpha} T_{n,r,s}\left(\frac{x}{n}\right) \\ &= \frac{(-1)^n}{n!\, n^\alpha} Q_{n,r}\left(\frac{x}{n}\right) - \frac{M_s Q_{n,r}^{(s)}(0)}{1 + M_s K_{n-1,r}^{(s,s)}(0,0)} \frac{(-1)^n}{n!\, n^\alpha} \sum_{k=0}^{n-1} \frac{K_{n-1,r}^{(k,s)}(0,0)}{k!} \left(\frac{x}{n}\right)^k. \end{aligned} \tag{29}$$

To estimate the kernels $K_{n-1,r}^{(k,s)}(0,0)$, we apply Stolz criterion, Lemma 2, (4) and (6), obtaining

$$\lim_n \frac{K_{n-1,r}^{(k,s)}(0,0)}{n^{\alpha+k+s+1}} = \begin{cases} 0 & \text{if } 0 \leq k \leq r\,, \\ \frac{k!}{(k-(r+1))!} \frac{s!}{(s-(r+1))!} \frac{(-1)^{k+s}\Gamma(\alpha+1)}{(\alpha+k+s+1)\Gamma(\alpha+k+r+2)\Gamma(\alpha+s+r+2)} & \text{if } k \geq r+1\,. \end{cases}$$

Moreover, it is not difficult to check that

$$\lim_n \frac{(-1)^n\, n^{s+1}}{n!} \frac{Q_{n,r}^{(s)}(0)}{K_{n-1,r}^{(s,s)}(0,0)} = (-1)^s \, \frac{(s-(r+1))!}{s!} \frac{(\alpha+2s+1)\Gamma(\alpha+s+r+2)}{\Gamma(\alpha+1)}.$$

According to the two above results, we get the asymptotic behavior of the coefficients in the sum appearing in (29),

$$\begin{aligned}
&\lim_n \frac{(-1)^n}{n!\, n^{\alpha+k}} \frac{M_s Q_{n,r}^{(s)}(0) K_{n-1,r}^{(k,s)}(0,0)}{1+M_s K_{n-1,r}^{(s,s)}(0,0)} \\
&= \begin{cases} 0 & \text{if } 0 \le k \le r\,, \\ \frac{(-1)^k k!}{(k-(r+1))!} \frac{(\alpha+2s+1)}{(\alpha+k+s+1)\Gamma(\alpha+k+r+2)} & \text{if } k \ge r+1\,. \end{cases}
\end{aligned}$$

On the other hand, from (21), there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, and for every $k, s \ge 0$, we have

$$\left| K_{n-1,r}^{(k,s)}(0,0) \right| \le 4(r+1)^2 \left| K_{n-1}^{(k,s)}(0,0) \right| .$$

Now, to obtain a bound for the kernels $K_{n-1}^{(k,s)}(0,0)$, we consider the expression which appears in Lemma 1 (b). Then, when $k \ge s-1$, it is easy to check that $j!\,(k+s+1-j)! \ge s!\,(k+1)!$, and $\Gamma(\alpha+j+1)\,\Gamma(\alpha+k+s+2-j) \ge \Gamma(\alpha+s+1)\,\Gamma(\alpha+k+2)$ for $0 \le j \le s$. Therefore,

$$\left| K_{n-1}^{(k,s)}(0,0) \right| \le C_s \, \frac{n^{\alpha+k+s+1}}{\Gamma(\alpha+k+2)} \, .$$

Indeed, given a compact set $K \subset \mathbb{C}$, from (4), (5), and (21), there exists a positive integer n_0, such that for all $n \ge n_0$, for all $k \ge s-1$, for all $j \ge 0$ and for all $x \in K$,

$$\frac{1}{n^{\alpha+s+1}} \left| \frac{K_{n-1,r}^{(k,s)}(0,0)}{k!} \frac{x^k}{n^k} \right| \le C_s \frac{4(r+1)^2}{\Gamma(\alpha+k+2)} \frac{M^k}{k!},$$

where M is a positive constant depending on K. As $\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha+k+2)} \frac{M^k}{k!}$ converges, we can apply Lebesgue's dominated convergence in the last term of (29). Then, using Theorem 1, we obtain

$$\begin{aligned}
&\lim_n \frac{(-1)^n}{n!\, n^{\alpha}} T_{n,r,s}\left(\frac{x}{n}\right) = (-1)^{r+1} x^{-\alpha/2} J_{\alpha+2r+2}(2\sqrt{x}) \\
&- \sum_{k=r+1}^{\infty} \frac{(-1)^k(\alpha+2s+1)}{(\alpha+k+s+1)\Gamma(\alpha+k+r+2)} \frac{x^k}{(k-(r+1))!}.
\end{aligned} \tag{30}$$

If we write $s = r + 1 + h$ with $h \geq 1$, then the above series is read as

$$(-1)^{r+1}x^{r+1}(\alpha+2r+2h+3)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k+2r+h+3)}{\Gamma(\alpha+k+2r+3)\Gamma(\alpha+k+2r+h+4)}\frac{(-1)^k x^k}{k!}.$$

Observe that $\Gamma(\alpha+k+2r+h+3)/\Gamma(\alpha+k+2r+3)$ is a polynomial in k of degree h (the number of gaps) and so we can write

$$\frac{\Gamma(\alpha+k+2r+h+3)}{\Gamma(\alpha+k+2r+3)} = \frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)} + \sum_{l=1}^{h}\beta_l k^l$$

where β_l, $l = 1, \ldots, h$ are positive coefficients. Thus, the above series can be expressed as

$$\frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha+k+2r+h+4)}\frac{(-1)^k x^k}{k!}$$
$$+\sum_{l=1}^{h}\beta_l\sum_{k=0}^{\infty}\frac{k^l}{\Gamma(\alpha+k+2r+h+4)}\frac{(-1)^k x^k}{k!}.$$

For the first one, using the recurrence relation repeatedly (see [17]),

$$J_{\alpha-1}(2\sqrt{x}) + J_{\alpha+1}(2\sqrt{x}) = \alpha\, x^{-\frac{1}{2}}\, J_\alpha(2\sqrt{x}),$$

we get

$$\frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha+k+2r+h+4)}\frac{(-1)^k x^k}{k!}$$
$$= \frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)}\, x^{-\frac{\alpha+2r+h+3}{2}}\, J_{\alpha+2r+h+3}(2\sqrt{x})$$
$$= x^{-\frac{\alpha+2r+2}{2}}\left[\frac{1}{\alpha+2r+h+3}J_{\alpha+2r+2}(2\sqrt{x}) + \sum_{i=2}^{h+2}\mu_i J_{\alpha+2r+2i}(2\sqrt{x})\right],$$

where μ_i are real numbers which can be computed explicitly.

Moreover, for the remaining series, using the same arguments, it can be seen that each one of the terms can be written as a combination of Bessel functions of order bigger than $\alpha + 2r + 2$. More precisely, for $l = 1, \ldots, h$

$$\sum_{k=0}^{\infty}\frac{k^l}{\Gamma(\alpha+k+2r+h+4)}\frac{(-1)^k x^k}{k!} = -x\sum_{k=0}^{\infty}\frac{(k+1)^{l-1}}{\Gamma(\alpha+k+2r+h+5)}\frac{(-1)^k x^k}{k!}$$
$$= x^{-\frac{\alpha+2r+2}{2}}\left[-\frac{\Gamma(\alpha+2r+5)}{\Gamma(\alpha+2r+h+5)}J_{\alpha+2r+4}(2\sqrt{x}) + \sum_{i=3}^{h+2}\mu_i^* J_{\alpha+2r+2i}(2\sqrt{x})\right],$$

where μ_i^* are again real numbers which can be computed explicitly.

Finally, taking these results into account in (30), we achieve

$$\lim_n \frac{(-1)^n}{n!\,n^\alpha} T_{n,r,s}\left(\frac{x}{n}\right) = (-1)^{r+1} x^{-\alpha/2} \left[1 - \frac{\alpha+2r+2h+3}{\alpha+2r+h+3}\right] J_{\alpha+2r+2}(2\sqrt{x})$$
$$+ (-1)^{r+1} x^{-\alpha/2} \sum_{i=2}^{h+2} \lambda_i J_{\alpha+2r+2i}(2\sqrt{x}),$$

and the proof is concluded. $\square$

For the particular case $s = r+2$ in the inner product (26), i.e., when there is a gap of "length one", the result established in the above theorem generalizes the one obtained in [4]. In fact, handling the right–hand side of the expression (30) and using the recurrence relation of the Bessel functions, we obtain:

$$\lim_n \frac{(-1)^n}{n!\,n^\alpha} T_{n,r,r+2}\left(\frac{x}{n}\right) = (-1)^{r+1} x^{-\alpha/2} \Bigg[J_{\alpha+2r+2}(2\sqrt{x})$$
$$- \frac{(\alpha+2r+5)(\alpha+2r+3)}{x} J_{\alpha+2r+4}(2\sqrt{x}) + \frac{\alpha+2r+5}{\sqrt{x}} J_{\alpha+2r+5}(2\sqrt{x}) \Bigg]$$
$$= (-1)^{r+1} x^{-\alpha/2} \Bigg[J_{\alpha+2r+2}(2\sqrt{x}) -$$
$$\frac{(\alpha+2r+5)(\alpha+2r+3)}{(\alpha+2r+4)\sqrt{x}} J_{\alpha+2r+3}(2\sqrt{x}) + \frac{\alpha+2r+5}{(\alpha+2r+4)\sqrt{x}} J_{\alpha+2r+5}(2\sqrt{x}) \Bigg]$$
$$= (-1)^{r+1} x^{-\alpha/2}$$
$$\times \left[\frac{-1}{\alpha+2r+4} J_{\alpha+2r+2}(2\sqrt{x}) - J_{\alpha+2r+4}(2\sqrt{x}) + \frac{1}{\alpha+2r+4} J_{\alpha+2r+6}(2\sqrt{x}) \right].$$

Concretely, we recover the corresponding result in [4] taking $r = -1$ in the above formula.

We also generalize other asymptotic results. For example, very recently

in [6], the authors consider the inner product

$$\begin{aligned}(p,q)_{-1,s} &= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)\,x^\alpha e^{-x}\,dx + M_s p^{(s)}(0)q^{(s)}(0)\\ &= (p,q) + M_s p^{(s)}(0)q^{(s)}(0), \qquad (31)\end{aligned}$$

where $s \geq 1$. Notice that the inner product (31) is a particular case of the inner product (26) taking $r = -1$. Thus, we recover some asymptotic results obtained in [6].

On the other hand, as a consequence of the above theorem, we present the situation about the acceleration of the convergence towards the origin of the zeros of the polynomials $T_{n,r,s}$. The quasi–orthogonality of order $s+1$ of the sequence $\{T_{n,r,s}\}_{n\geq 0}$ with respect to the positive measure $x^{\alpha+s+1}e^{-x}$ assures that $T_{n,r,s}$ has at least $n-(s+1)$ changes of sign in $(0,+\infty)$. However, in [2], the authors proved that the number of zeros in $(0,+\infty)$ does not depend on the order of the derivatives, but on the number of terms in the discrete part of the inner product. So, $T_{n,r,s}$ has at least $n-(r+1)$ zeros with odd multiplicity in $(0,+\infty)$.

The situation is quite different if we are concerned with the study of the acceleration convergence to 0 of the zeros of the polynomials. So, from Theorem 3 and Hurwitz's theorem and taking into account that $x = 0$ is a zero of multiplicity $r+1$ of the limit function in (28), we achieve the following result:

Corollary 2 *Let $(\zeta_{n,k}^{r,s})_{k=1}^n$ be the zeros of $T_{n,r,s}$. Then*

$$n\,\zeta_{n,k}^{r,s} \underset{n}{\to} 0, \quad 1 \leq k \leq r+1,$$

$$n\,\zeta_{n,k}^{r,s} \underset{n}{\to} h_{\alpha,r,k-r-1}, \quad k \geq r+2,$$

where $h_{\alpha,r,k-r-1}$ is the (k-r-1)-th zero of the limit function in (28).

Again, using a symmetrization process, we can also obtain the Mehler–Heine type formulas, as well as the relative asymptotics for generalized Hermite–Sobolev polynomials with gaps in the discrete part of the inner product.

Remark 4. We wish to highlight that the convergence acceleration to 0 of the zeros of the polynomials $Q_{n,r}$ and $T_{n,r,s}$ is the same. That is, the addition of a mass M_s *after a gap* in the inner product does not affect the

convergence acceleration to 0. So the breakpoint is given by the position of the *gap*.

References

[1] M. Alfaro, M. Álvarez de Morales and M.L. Rezola, *Orthogonality of the Jacobi polynomials with negative parameters*, J. Comput. Appl. Math. **145** (2002), 379–386.

[2] M. Alfaro, G. López and M.L. Rezola, *Some properties of zeros of Sobolev-type orthogonal polynomials*, J. Comput. Appl. Math. **69** (1996), 171–179.

[3] M. Alfaro, J.J. Moreno–Balcázar, A. Peña and M.L. Rezola, *Asymptotics for a generalization of Hermite polynomials*, Asymptotic Analysis, **66** (2010), 103–117.

[4] R. Álvarez–Nodarse and J.J. Moreno–Balcázar, *Asymptotic properties of generalized Laguerre orthogonal polynomials*, Indag. Mathem., N.S. **15** (2004), 151–165.

[5] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York, 1978.

[6] H. Dueñas and F. Marcellán, *The Laguerre–Sobolev–Type Orthogonal Polynomials*, J. Approx. Theory **162** (2010), 421–440.

[7] A. A. Gonchar, *On the convergence of Padé approximants for certain classe of meromorphic functions*, Math. USSR. Sb. **26** (1975), 555–575.

[8] G. Klambauer, *Aspects of Calculus*, Springer–Verlag, New York, 1986.

[9] R. Koekoek and H.G. Meijer, *A generalization of Laguerre polynomials*, SIAM J. Math. Anal. **24** (1993), 768–782.

[10] A.B.J. Kuijlaars and K. T-R McLaughlin, *Riemann–Hilbert analysis for Laguerre polynomials with large negative parameter*, Comput. Methods Funct. Theory **1** (2001), 205–233.

[11] A.B.J. Kuijlaars, A. Martínez–Finkelshtein and R. Orive, *Orthogonality of Jacobi polynomials with general parameters*, Electron. Trans. Numer. Anal. **19** (2005), 1–17.

[12] K.H. Kwon and L.L. Littlejohn, *The orthogonality of the Laguerre polynomials* $\{L_n^{(-k)}(x)\}$ *for a positive integer* k, Ann. Numer. Math. **2** (1995), 289–304.

[13] K.H. Kwon, L.L. Littlejohn and G.J. Yoon, *Ghost matrices and a characterization of symmetric Sobolev bilinears forms*, Linear Algebra Appl. **431** (2009), 104–119.

[14] G. López Lagomasino, F. Marcellán, and W. Van Assche, *Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product*, Constr. Approx., **11** (1995), 107-137.

[15] F. Marcellán and J.J. Moreno–Balcázar, *Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports*, Acta Appl. Math. **94** (2006), 163–192.

[16] J. Riordan, *Combinatorial identities*, Wiley, New York, 1968.

[17] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. vol. **23**, Amer. Math. Soc., Providence R.I., 1975. Fourth Edition.