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THE CONJUGATE FUNCTION IN PLANE CURVES

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ABSTRACT. We prove that the conjugate function operator is bounded in $L^p(\Gamma, wds)$, $1 < p < \infty$, if and only if $w \in A_p(\Gamma)$, where Γ is a quasiregular curve.

The weighted norm inequality problem for the conjugate function on the unit circle **T** consists in characterizing the nonnegative functions w such that

$$\int_{\mathbf{T}} |\tilde{f}(\theta)|^p w(\theta)d\theta \leqq C \int_{\mathbf{T}} |f(\theta)|^p w(\theta)d\theta$$

for a given p, $1 < p < \infty$, and constant C independent of f. When $w = 1$, the inequality turns out to be the well known M. Riesz theorem [7]. In the general case the weights are characterized as belonging to the classes A_p of Muckenhoupt, i.e., there is a constant $C_p > 0$ such that

$$\left[\frac{1}{|I|} \int_I w(\theta)d\theta\right]\left[\frac{1}{|I|} \int_I |w(\theta)|^{-1/p-1}d\theta\right]^{p-1} \leqq C_p$$

for every interval I (see [1]).

The main aim of this paper is to study the analogous problem for a special class of curves in the complex plane. Let Ω be a plane domain whose boundary Γ is a rectifiable Jordan curve, and let ϕ be the normalized conformal mapping from the unit disc D onto Ω, and ψ the inverse function of ϕ.

Let μ be a finite nonnegative measure on Γ which is absolutely continuous with respect to arc length ($d\mu = wds$). The space $L^p(\Gamma, \mu)$, $0 < p < \infty$, is the class of complex μ-measurable functions defined on Γ, such that

$$\int_\Gamma |f|^p d\mu < \infty.$$

For $1 < p < \infty$, we say that $w \in A_p(\Gamma)$ if there is a constant $C_p > 0$ such that for every interval $J \subset \Gamma$

$$\left(\frac{1}{s(J)} \int_J wds\right)\left(\frac{1}{s(J)} \int_J w^{-1/p-1}ds\right)^{p-1} \leqq C_p$$

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where $s(J)$ is the arc length of J. This is the natural definition of the A_p classes in this context.

If f is a μ-measurable complex function on Γ and $f \circ \phi \in L^1(\mathbf{T})$ we may define the conjugate function as $\tilde{f} = (f \circ \phi)^{\sim} \circ \psi$, where $(f \circ \phi)^{\sim}$ is the classical conjugate function of $f \circ \phi$. As it happens in the case of the unit circle, if $P(z)$ is a polynomial and $f(z) = \operatorname{Re} P(z)$ with $z \in \Gamma$, then $\tilde{f}(z) = \operatorname{Im} P(z)$.

In the unweighted case ($w = 1$), the conjugate function operator turns out to be bounded in $L^p(\Gamma)$, via conformal mapping, if and only if $|\phi'| \in A_p$, $1 < p < \infty$, and therefore $|\phi'| \in A_\infty$ is needed. A kind of curves verifying this last condition are the chord-arc curves, which play an important role in the study of generalized Hardy spaces and in deducing estimates for singular integrals [3, 6]. Γ is said to be a chord-arc curve if there is a constant $C > 0$ such that for all points z_1, z_2 of Γ, $s(z_1, z_2) \leqq C|z_1 - z_2|$ where $s(z_1, z_2)$ is the length of the shortest arc of Γ with endpoints z_1 and z_2. These curves are characterized by the condition $\log \phi' \in BMOA$ [5], which implies $|\phi'| \in A_\infty$.

P. Jones and M. Zinsmeister [4] proved that for every fixed p there is a chord-arc curve Γ such that $|\phi'| \notin A_p$. Thus, the conjugate function operator is not bounded in $L^p(\Gamma)$ for this curve.

Consequently, we must restrict our attention to the class of curves verifying $|\phi'| \in A_p$ for all $p > 1$.

DEFINITION 1. *Let Γ be a rectifiable Jordan curve. Γ is said quasiregular if for each $\epsilon > 0$ there is a $\eta > 0$ such that if $z_1, z_2 \in \Gamma$ verify $|z_1 - z_2| \leqq \eta$, then $s(z_1, z_2) \leqq (1 + \epsilon)|z_1 - z_2|$.*

In [5] it is shown that Γ is quasiregular if and only if $\log \phi' \in VMOA(D) = H^1(D) \cap VMO(\mathbf{T})$, where $VMO(\mathbf{T})$ is the span of trigonometric polynomials in $BMO(\mathbf{T})$. In particular, if Γ is quasiregular, then Γ is chord-arc and $|\phi'| \in A_p$ for all $p > 1$.

The following property of quasiregular curves will be needed for our main result.

LEMMA 2. *If Γ is quasiregular and $w \in A_p(\Gamma)$, then, $(w \circ \phi)|\phi'| \in A_p$.*

PROOF. Let J be an arc of Γ and $\psi(J) = I$ the corresponding arc of $\mathbf{T}$. As with $\mathbf{T}$ or $\mathbf{R}^n$, $w \in A_p(\Gamma)$ implies that $w \in A_{p-\epsilon}(\Gamma)$ for some $\epsilon > 0$ ([1]). Then, by using Hölder's inequality, we have

$$\left(\frac{1}{|I|}\int_I (w \circ \phi)|\phi'|\right)\left(\frac{1}{|I|}\int_I (w \circ \phi)|\phi'|)^{-1/p-1}\right)^{p-1}$$

$$\leqq \left(\frac{1}{|I|}\int_I (w \circ \phi)\cdot|\phi'|\right)\left(\frac{1}{|I|}\int_I (w \circ \phi)^{-1/p-\epsilon-1}\cdot|\phi'|\right)^{p-\epsilon-1}$$

$$\cdot\left(\frac{1}{|I|}\int_I |\phi'|^{-(p-\epsilon)/\epsilon}\right)^{\epsilon}$$

$$\leqq \left(\frac{1}{|I|}\int_I (w\circ\phi)|\phi'|\right)\left(\frac{1}{|I|}\int_I (w\circ\phi)^{-1/p-\epsilon-1}|\phi'|\right)^{p-\epsilon-1}\left(\frac{|I|}{s(J)}\right)^{p-\epsilon} C$$

$$\leqq \left(\frac{1}{s(J)}\int_J w\right)\left(\frac{1}{s(J)}\int_J w^{-1/p-\epsilon-1}\right)^{p-\epsilon-1} C \leqq C'$$

and the lemma is proved.

Before passing to the following lemma we include some well known results about A_p classes.

(A) $w \in A_\infty$ if and only if there exists $\epsilon > 0$ such that

$$\left(\frac{1}{|I|}\int_I w^{1+\epsilon}\right)^{1/1+\epsilon} \leqq K_\epsilon\left(\frac{1}{|I|}\int_I w\right),$$

which is denoted by $w \in RHI(1+\epsilon)$ (reverse Hölder inequality).

(B) Let $\phi = \log w$. Then $w \in A_p$, $1 < p < \infty$, if and only if

$$\sup_I \frac{1}{|I|}\int_I e^{\phi-\phi_I} < \infty \quad\text{and}\quad \sup_I \frac{1}{|I|}\int_I e^{-(\phi-\phi_I)/(p-1)} < \infty.$$

LEMMA 3. *Let f be a real valued function on* $\mathbf{T}$, *and* $w = \exp(f)$. *The following conditions are equivalent*:

i) $f \in \overline{L^\infty_{BMO}}(\mathbf{T})$ (*closure of* L^∞ *in* BMO).

ii) $w \in A_p \forall p > 1$ *and* $w \in RHI(q)$ *for all* $q > 1$.

iii) $w \in RHI(q)$, $w^{-1} \in RHI(q)$ *for all* $q > 1$.

iv) $w^q \in A_\infty$, $w^{-q} \in A_\infty$ *for all* $q > 1$.

v) $w \in A_p$, $w^{-1} \in A_p$ *for all* $p > 1$.

PROOF.

i) $\Rightarrow$ ii).

That $w \in A_p$ for all $p > 1$ is an immediate consequence of the Garnett-Jones theorem, see [2].

On the other hand, by applying the John-Nirenberg inequality, given $\epsilon > 0$ sufficiently small, there is a constant C such that for all $g \in BMO$ with $\|g\|_* < \epsilon$ we have

$$\frac{1}{|I|}\int_I e^{|g-g_I|} \leqq C$$

for all interval $I \subseteq T$. Hence, $\exp g \in A_2$ with A_2-constant smaller or equal than C^2 and then, there exists $\delta > 0$ so that $\exp(g) \in RHI(1+\delta)$, whenever $\|g\|_* < \epsilon$, where δ depends only on ϵ.

Since f belongs to the closure of L^∞ in BMO, for each $\epsilon > 0$ we can put $f = f_1 + f_0$, where $f_1 \in L^\infty$, $f_0 \in BMO$ and $\|f_0\|_* < \epsilon$. Thus, $w = e^{f_1}\cdot e^{f_0}$ and $w_0 = e^{f_0}$ are equivalent (i.e., there are constants c_1, $c_2 > 0$ such that

$c_1w_0 \leqq w \leqq c_2w_0$). Then, there exists $\delta > 0$ such that $w_0 \in RHI(1+\delta)$ and also $w \in RHI(1+\delta)$. By applying the same arguments to the function $qf (q > 1)$ which belongs to $\overline{L^\infty_{BMO}}$ also, we get $w^q \in RHI(1+\delta)$. Choosing $q = 1+\delta$, we obtain $w \in RHI((1+\delta)^2)$ and, by iterating this argument, we conclude that $w \in RHI(q)$, for all $q > 1$.

ii) $\Rightarrow$ iii).

If $w \in A_p$ for all $p > 1$ then $w^{-1} \in A_\infty$ and, by using (A), $w^{-1} \in RHI(1+\epsilon)$ for some $\epsilon > 0$. Both, this last condition and Hölder's inequality, lead us to

$$1 \leqq K_\epsilon\left(\frac{1}{|I|}\int_I w^{-1}\right)\left(\frac{1}{|I|}\int_I w^r\right)^{1/r}$$

with $r = 1 + 1/\epsilon$. Now, by applying $w \in RHI(q)$ for all $q > 1$, it follows that

$$\left(\frac{1}{|I|}\int_I w^{-q}\right)^{1/q} \leqq C_q\left(\frac{1}{|I|}\int_I w\right)^{-1}$$

$$\leqq C_qK_r\left(\frac{1}{|I|}\int_I w^r\right)^{-1/r} \leqq C_qK_rK_\epsilon\left(\frac{1}{|I|}\int_I w^{-1}\right)$$

iii) $\Rightarrow$ iv).

(A) and Hölder's inequality lead us to

$$\left(\frac{1}{|I|}\int_I (w^q)^{1+\epsilon}\right)^{1/1+\epsilon} \leqq K_\epsilon\left(\frac{1}{|I|}\int_I w\right)^q \leqq \frac{K_\epsilon}{|I|}\int_I w^q.$$

The verification for w^{-1} is similar.

iv) $\Rightarrow$ v).

It follows from $w^q \in A_\infty$ and $w^{-q} \in A_\infty$ for all $q > 1$ that

$$\sup_I \frac{1}{|I|}\int_I e^{q(\phi-\phi_I)} < +\infty \quad \text{and}$$

$$\sup_I \frac{1}{|I|}\int_I e^{-q(\phi-\phi_I)} < \infty. \quad \text{Then}$$

$$\sup_I \frac{1}{|I|}\int_I e^{\phi-\phi_I} < \infty \quad \text{and}$$

$$\sup_I \frac{1}{|I|}\int_I e^{-(\phi-\phi_I)/p-1} < +\infty \quad \text{for all } p > 1.$$

Therefore, $w \in A_p$ for all $p > 1$. The same argument works for w^{-1}.

v) $\Rightarrow$ i). It is obvious from (B) and the Garnett-Jones theorem.

THEOREM 4. *Let* Γ *be a quasiregular curve. Then the conjugation operator is bounded on* $L^p(\Gamma, wds)$ $(1 < p < \infty)$ *if and only if* $w \in A_p(\Gamma)$.

PROOF. Since the conjugate function operator is bounded on $L^p(T, w \circ \phi \cdot |\phi'| d\theta)$ if and only if $(w \circ \phi)|\phi'| \in A_p$, the "if part" of the theorem is an immediate consequence of Lemma 2.

For the converse, we suppose that $(w \circ \phi)|\phi'| \in A_p$ and then $(w \circ \phi)|\phi'| \in A_{p-\epsilon}$ for some $\epsilon > 0$. Since Γ is quasiregular, $\log|\phi'| \in VMO \subset \overline{L^\infty_{BMO}}$, and therefore, by Lemma 3, $|\phi'|$ and $|\phi'|^{-1}$ verify $RHI(q)$ for all $q > 1$. Thus

$$\left(\frac{1}{s(J)}\int_J w\right)\left(\frac{1}{s(J)}\int_J w^{-1/p-1}\right)^{p-1} \leqq \left(\frac{1}{|I|}\int_I w\circ\phi\cdot|\phi'|\right)$$

$$\cdot\left(\frac{1}{|I|}\int_I (w\circ\phi|\phi'|)^{-1/p-\epsilon-1}\right)^{p-\epsilon-1}\left(\frac{1}{|I|}\int_I |\phi'|^{p/\epsilon}\right)^{\epsilon}\left(\frac{|I|}{s(J)}\right)^{p} \leqq C.$$

REMARK. In the proof of the preceding theorem we only use the fact that $\log|\phi'| \in \overline{L^\infty_{BMO}}$. Quasiregular curves satisfy this condition and also every curve which is transformed of a quasiregular curve by a conformal mapping with bounded derivate (such curves are not necessarily quasiregular). The class of curves (boundaries of Jordan domains) for which Theorem 4 is verified, strictly contains the quasiregular curves and it is contained in the class of chord-arc

On the other hand, $w = 1$, satisfies $w \in A_p(\Gamma)$ which implies that the conjugation operator is bounded on $L^p(\Gamma, ds)$ and so $|\phi'| \in A_p$ for all $p > 1$. Now, the theorem is proved by using Lemma 3.

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