On Sobolev type orthogonal polynomials with unbounded support: asymptotic properties

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Abstract

In this expository paper we present a survey about asymptotic properties of Sobolev type orthogonal polynomials with unbounded support.

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1 Introduction

The theory of orthogonal polynomials is a very interesting field in mathematics with important applications to numerical analysis, physics, probability, and statistics among others. Orthogonal polynomials are connected with topics like moment problems, mechanical quadratures, continued fractions, spectral methods, quantum mechanics and many other concepts.

In this theory, the orthogonality is usually considered with respect to a positive linear functional defined on the linear space of polynomials or, according to the Riesz representation theorem, with respect to a positive measure. Let μ be a finite positive Borel measure supported on an interval I in the real line, we say that the sequence of polynomials $\{P_n\}_{n\geq 0}$ is a sequence of orthogonal polynomials (o.p.) with respect to either the measure μ or the inner product $(f,g)=\int_I f\,g d\mu$ if, for all $n\geq 0$, $\deg P_n = n$ and

(i) $(P_n, P_m) = 0\,,\; n \neq m\,,$

(ii) $(P_n, P_n) > 0,\; n \geq 0.$

Along the paper, such kind of inner products will be called standard inner products. They have the following remarkable property: $(xp, q) = (p, xq)$, for all polynomials p, q. As a consequence, the corresponding standard orthogonal polynomials have nice properties such as the three–term recurrence relation, the summation formula, the interlacing properties of the zeros, etc. From a numerical point of view, a useful consequence is that a Gaussian mechanical quadrature formula has exact precision when we take as nodes the zeros of appropriate standard orthogonal polynomials.

Nonstandard inner products have also been considered in the literature. In particular, the so–called Sobolev inner products of the form

$$(f,g) = \int f\, g d\mu_0 + \sum_{i=1}^{r} \int f^{(i)}\, g^{(i)} d\mu_i,$$

where $\{\mu_i\}_{i=0}^{r}$ are finite positive Borel measures supported on the real line and the functions f and g belong to the Sobolev space:

$$W^{2,r}(\mu_0, \mu_1, \ldots, \mu_r) := \{f : \int |f|^2 d\mu_0 + \sum_{i=1}^{r} \int |f^{(i)}|^2 d\mu_i < +\infty\}.$$

Studied for the first time in the 1940s, the Sobolev orthogonal polynomials have been the object of an increasing interest in the last 20 years. Obviously, Sobolev inner products are nonstandard and therefore Sobolev o.p. lose the "good" properties of the standard o.p. However, it is interesting to study these "strange" polynomials that supply us with situations different from the standard ones: no three–term recurrence relation, zeros out of the convex hull of the support of the orthogonality measure including, some times, complex zeros, and so on.

The Sobolev orthogonality can be applied in the theory of standard orthogonal polynomials. For instance, Jacobi and Laguerre polynomials with nonclassical parameters are not orthogonal in the usual sense but they are orthogonal with respect to Sobolev inner products (see among others [1] or [17]) and a similar fact occurs for some families of discrete classical polynomials. Moreover, Sobolev o.p. in two real variables are solutions of some partial differential equations (see [9], [14], [19] or [24]).

In this paper we are concerned with the so–called Sobolev type (or discrete Sobolev) orthogonal polynomials, that is, polynomials orthogonal with respect to a Sobolev inner product where $\{\mu_i\}_{i=1}^{r}$ are Dirac's deltas or, in general, discrete measures. More concretely, we consider an inner product of the form

$$(f,g) = \int f(x)g(x)d\mu(x) + \sum_{i=0}^{r} M_i f^{(i)}(c)g^{(i)}(c),$$

where μ is a finite positive Borel measure, $c \in \mathbb{R}$ and $M_i \geq 0$ for $i = 0, 1, ..., r$. In the sequel, we denote by $\{Q_{n,r}\}_{n\geq 0}$ the corresponding sequence of o.p. with the same leading coefficient as the standard o.p. with respect to μ.

More general products with terms of the form $f^{(i)}(c)g^{(j)}(c)$, $i \neq j$, in the discrete part (the so–called non–diagonal case) have also been considered. But, recently it has been proved, see [18], that every symmetric bilinear form can be reduced to a diagonal case, that is, an inner product without terms like the above ones.

In some sense, Sobolev type o.p. are not so far from the standard o.p. since there exists the possibility to transform the Sobolev type orthogonality into the standard quasi–orthogonality.

As a consequence, several properties of the standard o.p. are partially recovered for the Sobolev type o.p., for instance, they satisfy a $2r+3$ term recurrence relation (see, [13]) and have partial interlacing properties of the zeros ([2]).

Since the polynomial $Q_{n,r}$ is quasi–orthogonal of order $r+1$ with respect to the measure μ it can be expressed as a linear combination (with a fixed number of terms: $r+2$) of standard orthogonal polynomials P_n corresponding to the modified measure $(x-c)^{r+1}d\mu$, that is,

$$Q_{n,r}(x) = \sum_{j=0}^{r+1} a_n^j P_{n-j}(x)\,. \tag{1}$$

One of the topics in the theory is to compare the Sobolev type o.p. with the standard o.p. (with respect to μ) to investigate how the addition of the discrete part in the inner product influences the orthogonal system. Many formal results are known for the polynomials $Q_{n,r}$: recurrence relation, differential formulas, location of zeros, and so on. However, little is known about the asymptotic properties and most of the general results have been obtained when the support of μ ($\operatorname{supp}\mu$) is a bounded set. For instance, in [20], the authors assume that μ is a measure of bounded support for which the asymptotic behaviour of the corresponding o.p. is known; the most relevant class of this type is the Nevai class $M(0,1)$ of o.p. with appropriately converging recurrence coefficients. There, the relative asymptotics is studied when the mass point $c \notin \operatorname{supp}\mu$. The case $c \in \operatorname{supp}\mu$ has been considered in [22].

What happens is that in the bounded case, all the coefficients a_n^j in (1) are bounded and the orthogonal polynomials P_n have an finite ratio asymptotics: these two facts allow us to study each term of (1) separately, in order to get asymptotics for $Q_{n,r}$ (see [20] and [22] where this technique is developed). However, the situation is quite different if we deal with the unbounded case because when we try to obtain some asymptotics with the same techniques as in the bounded case and we take into account the ratio asymptotics for the polynomials P_n, we come across a serious problem. Indeed, we find that each term of (1) may not have a finite limit, in fact, as we will see later, for the Laguerre–Sobolev type o.p. each term of (1) tends to infinity, all of them having the same order, but with an alternating sign.

The aim of this paper is to describe the current state of the asymptotic properties for Sobolev type polynomials when $\operatorname{supp}\mu$ is unbounded. Mainly we will analyze the case when μ is the Laguerre probability measure ($d\mu(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}dx$ with $\alpha > -1$) and $c = 0$, that is, the Laguerre–Sobolev o.p. This choice for c is due to the fact that the point $x = 0$ is a singularity of the differential equation satisfied by the classical Laguerre polynomials.

Therefore, we will deal with classical Laguerre polynomials, that is, polynomials orthogonal with respect to the inner product

$$(p,q) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\,x^\alpha e^{-x}\,dx\,, \quad \alpha > -1.$$

in the space of all polynomials with real coefficients. We will denote by L_n^α the nth Laguerre polynomial with $(-1)^n/n!$ as leading coefficient. Although many of the properties of Laguerre polynomials can be seen, for example, in the books by Chihara [10] and Szegő [23], we remind that the classical Laguerre polynomials with the normalization above quoted are defined by

$$L_n^\alpha(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-1)^k x^k}{k!},$$

and their derivatives satisfy

$$(L_n^\alpha)^{(k)}(x) = (-1)^k L_{n-k}^{\alpha+k}(x)\,. \tag{2}$$

The evaluations at $x=0$ of the polynomial L_n^α and its successive derivatives are given by

$$(L_n^\alpha)^{(k)}(0) = \frac{(-1)^k\, n!}{(n-k)!} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} L_n^\alpha(0) = \frac{(-1)^k}{(n-k)!} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)}. \tag{3}$$

From Perron's formula in Szegő's book [23], the following asymptotic results can be deduced:

$$\frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)} \rightrightarrows 1, \quad x \in \mathbb{C}\setminus[0,\infty)\,, \tag{4}$$

$$\frac{n^{1/2} L_n^\alpha(x)}{L_n^{\alpha+1}(x)} \rightrightarrows \sqrt{-x}, \quad x \in \mathbb{C}\setminus[0,\infty)\,. \tag{5}$$

where the symbol $f_n(x) \rightrightarrows f(x)$, $x \in A$, denotes that the sequence $\{f_n\}$ converges to f uniformly on compact subsets of A.

Later on we will use the symbol $f(x) \sim g(x)$ $(x \to a)$ if $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 1$.

2 Laguerre–Sobolev type polynomials

In the sequel, $\{Q_{n,r}\}_{n\geq 0}$ denotes the sequence of polynomials with leading coefficient $(-1)^n/n!$ orthogonal with respect to an inner product of the form

$$(p,q)_r = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx + \sum_{i=0}^{r} M_i p^{(i)}(0) q^{(i)}(0), \tag{6}$$

where $\alpha > -1$ and $M_i > 0$, $i = 0, \ldots, r,$. Notice that all the masses M_i are now positive.

Observe that, in fact, $(.,.)_r$ and $Q_{n,r}$ also depend on the parameter α but for simplicity we have omitted it in the notations.

These families of o.p. were considered for the first time by Koekoek and Meijer (see, among others, [15] and [16]) although no asymptotics were studied. The first asymptotic results for Laguerre–Sobolev type o.p. appear in [8]: exterior asymptotics, asymptotics on compact subsets of $(0,+\infty)$, exterior Plancherel–Rotach type asymptotics, Mehler–Heine type formulas and convergence of their zeros are obtained, but only for $r=0$ and $r=1$. Concerning the Mehler–Heine type formulas, with $r=1$ and $M_0, M_1 > 0$ the authors found a behaviour pattern and they established a conjecture. A survey including these results can be seen in [21]. Some of these properties were proved for the non–diagonal case with $r=1$ in [6] and [7], and later on in [11].

In all these papers the basic tool was the algebraic expression

$$Q_{n,1}(x) = B_0(n) L_n^\alpha(x) + B_1(n) x L_{n-1}^{\alpha+2}(x) + B_2(n) x^2 L_{n-2}^{\alpha+4}(x)$$

where the coefficients $B_i(n)$ were given explicitly in [16].

When the discrete part of the Laguerre–Sobolev inner product has an arbitrary number of terms, namely r, we have an algebraic expression like the above one with $r+1$ terms in the right hand, and the problem is that we only know that the coefficients $B_i(n)$ are a non trivial solution of a system with $r+1$ equations and $r+2$ unknowns, but we do not have their explicit expression, see [15].

Asymptotic properties of Sobolev orthogonal polynomials with respect to a general inner product such as (6), that is, with an arbitrary number of masses, have been studied in [5] where, in particular, the conjecture established in [8] is proved to be true.

In the sequel we summarize results by the authors about asymptotic behaviour of Sobolev type o.p. with unbounded support, with special emphasis in those obtained in [5].

As we have already said, the interest lies in knowing the differences in the asymptotic behaviour between the Laguerre polynomials and the Sobolev polynomials $Q_{n,r}$. Intuitively one can imagine that these differences in the complex plane should be around the perturbation of the standard inner product involved in the Sobolev inner product, that is, around the origin and therefore we cannot expect that the addition of a finite number of masses to the inner product produces a modification in the global behaviour of the polynomials. A result which supports this intuition is Lemma 2 in [5] where it has been proved:

Lemma 1 *Let $Q_{n,r}$ be the polynomials orthogonal with respect to (6) with leading coefficients $(-1)^n/n!$. Then the following statements hold:*

(a) For $0 \le k \le r$,

$$Q_{n,r}^{(k)}(0) \sim \frac{C_{r,k}}{n^{\alpha+2k+1}}(L_n^{\alpha})^{(k)}(0),$$

where $C_{r,k}$ is a nonzero real number independent of n.

(b) For $k \ge r+1$,

$$Q_{n,r}^{(k)}(0) \sim \frac{k!}{(k-(r+1))!}\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)}(L_n^{\alpha})^{(k)}(0)\,.$$

(c)

$$(Q_{n,r}, Q_{n,r})_r \sim \|L_n^{\alpha}\|^2\,.$$

Remark 1. Notice that, from the point of view of the norm, Laguerre polynomials and Laguerre–Sobolev type polynomials have the same size asymptotically (global behaviour). This is not the case concerning the successive derivatives at the point $x=0$ (local behaviour): the sizes of $Q_{n,r}^{(k)}(0)$ and $(L_n^{\alpha})^{(k)}(0)$ are different if, and only if, the order k of the derivatives corresponds to a positive mass M_k.

Now we analyze two other asymptotics of the polynomials $Q_{n,r}$: the relative asymptotics, which assures that both families $Q_{n,r}$ and L_n^{α} are identical asymptotically on compact subsets of $\mathbb{C}\setminus[0,\infty)$, and the so–called Mehler–Heine type formula which shows how the presence of the masses in the inner product changes the asymptotic behaviour around the origin.

As we have mentioned before, for a discrete Sobolev inner product when $\operatorname{supp}\mu$ is bounded, a tool to obtain some results is the relation between the Sobolev orthogonality and the standard quasi–orthogonality.

Now, in our particular case, the sequence $\{Q_{n,r}\}_{n\geq 0}$ is quasi–orthogonal of order $r+1$ with respect to the Laguerre weight $x^{\alpha+r+1}e^{-x}$, that is,

$$\int_0^{+\infty} p(x)Q_{n,r}(x)x^{\alpha+r+1}e^{-x}dx = 0,$$

for every polynomial p with $\deg p \leq n-(r+1)-1$. Therefore, we have a *connexion formula* of the form

$$Q_{n,r}(x) = \sum_{j=0}^{r+1} a_{n,r}^j L_{n-j}^{\alpha+r+1}(x)\,, \quad a_{n,r}^0 = 1. \tag{7}$$

In order to deduce properties of $Q_{n,r}$ it is convenient to know the size of the *connexion coefficients* $a_{n,r}^j$. In [5], it has been introduced a fruitful new technique which leads to determine their asymptotic behaviour.

Theorem 1 *Let $a_{n,r}^j$ be the connexion coefficients which appear in (7). Then, we have*

$$\lim_n a_{n,r}^j = (-1)^j \binom{r+1}{j}\,, \quad 0 \leq j \leq r+1\,. \tag{8}$$

As a token of the interest of this result we use it to deduce an asymptotics of the Laguerre–Sobolev polynomials on compact subsets of $(0,+\infty)$.

Proposition 1 *The sequence $\{n^{-(2\alpha+2r+1)/4}Q_{n,r}\}_{n\geq 1}$ is uniformly bounded on compact subsets of $(0,+\infty)$.*

Proof. The sequence $\{n^{-\alpha/2+1/4}L_n^\alpha\}_{n\geq 1}$ is uniformly bounded on compact subsets of $(0,+\infty)$ (see [23, Th. 8.22.1]), and then, for all $j=0,1,\ldots,r+1$, the sequences $\{n^{-(2\alpha+2r+1)/4}L_{n-j}^{\alpha+r+1}\}_{n\geq 1}$ are uniformly bounded on compact subsets of $(0,+\infty)$. From (8) and the connexion formula the result follows. $\square$

However, it is worth noticing that the knowledge of the asymptotic behaviour of the connexion coefficients is not enough to deduce other asymptotic properties. Indeed, concerning the relative asymptotics, from (7) we have

$$\frac{Q_{n,r}(x)}{L_n^\alpha(x)} = \sum_{j=0}^{r+1} a_{n,r}^j \frac{L_{n-j}^{\alpha+r+1}(x)}{L_n^\alpha(x)}.$$

Applying Theorem 1, and (4) and (5) each term in the above sum tends to infinity with the same order but with an alternating sign, that is,

$$a_{n,r}^j \frac{L_{n-j}^{\alpha+r+1}(x)}{L_n^\alpha(x)} \sim (-1)^j \binom{r+1}{j} \left(\frac{1}{\sqrt{-x}}\right)^{r+1} n^{\frac{r+1}{2}},$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Since the techniques used in the bounded case do not work when $\operatorname{supp}\mu$ is an unbounded set we proceed in a different way to prove:

Theorem 2 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the inner product (6) with $(-1)^n/n!$ as leading coefficient. Then, for $k \geq 0$,*

$$\lim_n \frac{Q_{n,r}^{(k)}(x)}{(L_n^\alpha)^{(k)}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Proof. From the Fourier expansion of the polynomial Q_{n0} in terms of Laguerre polynomials, using Lemma 1, (3) and (4) the result follows for $k = 0$. (For more details see [5]).

The functions $Q_{n,r}/L_n^\alpha$ are analytic in $\mathbb{C} \setminus [0, \infty)$ and according to the previous result $Q_{n,r}(x)/L_n^\alpha(x) \rightrightarrows 1$, $x \in \mathbb{C} \setminus [0, \infty)$. Then $(Q_{n,r}/L_n^\alpha)'(x) \rightrightarrows 0\,, x \in \mathbb{C} \setminus [0, \infty)$, and therefore,

$$\left(\frac{Q'_{n,r}(x)}{(L_n^\alpha)'(x)} - \frac{Q_{n,r}(x)}{L_n^\alpha(x)} \right) \frac{(L_n^\alpha)'(x)}{L_n^\alpha(x)} \rightrightarrows 0\,, x \in \mathbb{C} \setminus [0, \infty).$$

From (2), (4) and (5), we get $(L_n^\alpha)'(x)/L_n^\alpha(x) \rightrightarrows \infty\,, x \in \mathbb{C} \setminus [0, \infty)$, and then

$$\lim_n \frac{Q'_{n,r}(x)}{(L_n^\alpha)'(x)} = \lim_n \frac{Q_{n,r}(x)}{L_n^\alpha(x)} = 1$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$. So, the result holds for $k = 1$.

Using this technique, by means of an induction procedure, the result follows for all $k \geq 0$. $\square$

Once we know that both sequences of orthogonal polynomials, $\{Q_{n,r}\}_{n\geq 0}$ and $\{L_n^\alpha\}_{n\geq 0}$, are asymptotically identical on compact subsets of $\mathbb{C} \setminus [0, \infty)$, we establish their differences.

To do this, we consider Mehler–Heine type formulas because they are nice tools to describe the polynomials around the origin. These kind of formulas are interesting twofold: they provide the scaled asymptotics for $Q_{n,r}$ on compact sets of the complex plane and they supply us with asymptotic information about the location of the zeros of these polynomials in terms of the zeros of other known special functions. More precisely, applying Hurwitz's Theorem in a straightforward way, the existence of an acceleration of the convergence of $r + 1$ zeros of these Sobolev polynomials towards the origin can be proved.

First of all, we recall the corresponding formula for the classical Laguerre polynomials, (see [23, Th. 8.1.3]):

$$n^{-\alpha} L_n^\alpha\left(\frac{x}{n}\right) \rightrightarrows x^{-\alpha/2} J_\alpha(2\sqrt{x}),\ x \in \mathbb{C}\,, \tag{9}$$

where J_α is the Bessel function of the first kind of order α ($\alpha > -1$).

As it occurs in the study of the relative asymptotics, the Mehler–Heine type formulas cannot be deduced as a consequence of the connexion formula. Indeed, from (7) we have

$$n^{-\alpha} Q_{n,r}\left(\frac{x}{n}\right) = \sum_{i=0}^{r+1} a_{n,r}^i n^{-\alpha} L_{n-i}^{\alpha+r+1}\left(\frac{x}{n}\right).$$

and, applying Theorem 1 and (9), we have that each term tends to infinity with the same order but with an alternating sign.

Thus, to get the result for $\{Q_{n,r}\}_{n\geq 0}$, the problem should be focused on in a different way. An approach consists of writing the Taylor expansion of the polynomial $Q_{n,r}$

$$n^{-\alpha}Q_{n,r}\left(\frac{x}{n}\right)=\sum_{k=0}^{n}\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}\frac{(L_n^\alpha)^{(k)}(0)}{k!}\frac{x^k}{n^{\alpha+k}}\ ,$$

and computing the limit applying the Lebesgue's dominated convergence theorem. So, we need to prove that the ratios $Q_{n,r}^{(k)}(0)/(L_n^\alpha)^{(k)}(0)$ are uniformly bounded. It is clear that taking derivatives k times in (7) the connexion coefficients do not change. Then, it could be thought about the possibility to obtain this uniform bound from this formula. But again we come across the same problem, each term of $\displaystyle\sum_{i=0}^{r+1}a_{n,r}^i\frac{(L_{n-i}^{\alpha+r+1})^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$, tends to infinity with order n^{r+1}, but with an alternating sign. To solve this problem, taking into account the expression relating $Q_{n,r}^{(k+1)}(0)/(L_n^\alpha)^{(k+1)}(0)$ and $Q_{n,r}^{(k)}(0)/(L_n^\alpha)^{(k)}(0)$, deduced in [5, Lemma 3]), a uniform bound for the ratios could be derived (see [5, Lemma 4]). Then we have

Theorem 3 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the inner product (6) with $(-1)^n/n!$ as leading coefficient. Then,*

$$\lim_n n^{-\alpha}Q_{n,r}\left(\frac{x}{n}\right)=(-1)^{r+1}x^{-\alpha/2}J_{\alpha+2r+2}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$.

This result gives a positive answer to the conjecture posed in [8]. We would like to note that the approach is totally new and the techniques used in [5] to prove the above theorem are not a simple generalization of the ones used in [8].

Now, we analyze the zeros of the polynomials $Q_{n,r}$ and compare them with those of L_n^α. First, remember that the zeros of the Laguerre polynomials are real, simple and they are located in $(0,\infty)$, they satisfy the interlacing property $0<x_{n+1,1}<x_{n,1}<x_{n+1,2}<\ldots$, where $(x_{n,k})_{k=1}^n$ denote the zeros of L_n^α written in an increasing order, and for each fixed k we have $x_{n,k}\underset{n}{\rightarrow}0$, (see [23]). Moreover, from (9), using Hurwitz's theorem we get a relation between the zeros of Laguerre polynomials and Bessel functions. Indeed, let $(j_{\alpha,k})_{k\geq 1}$ be the positive zeros of the Bessel function J_α in an increasing order, then $nx_{n,k}\underset{n}{\rightarrow}j_{\alpha,k}$, $k\geq 1$, and therefore $x_{n,k}\sim C_k\,n^{-1}$, $k\geq 1$, where C_k is a positive constant depending on k.

Concerning the zeros of $Q_{n,r}$, since $Q_{n,r}$ is quasi–orthogonal of order $r+1$ with respect to the Laguerre weight it has at least $n-(r+1)$ zeros with odd multiplicity in $(0,+\infty)$, or equivalently $n-(r+1)$ changes of sign (see for instance [10]). Moreover, since the mass point in the discrete part of the inner product belongs to the boundary of $(0,+\infty)$ and $M_0>0$ the number of zeros in $(0,+\infty)$ with odd multiplicity is at least $n-r$ (see [2]).

From Theorem 3, Hurwitz's theorem, and taking into account that $x=0$ is a zero of multiplicity $r+1$ of $x^{-\alpha/2}J_{\alpha+2r+2}(2\sqrt{x})$ (the limit function in Theorem 3), we achieve

Corollary 1 *Let $(\xi_{n,k}^r)_{k=1}^n$ be the zeros of $Q_{n,r}$. Then*

$$n\,\xi_{n,k}^r\underset{n}{\rightarrow}0,\quad 1\leq k\leq r+1,$$

$$n\,\xi^r_{n,k} \underset{n}{\to} j_{\alpha+2r+2,k-r-1}, \quad k \geq r+2.$$

Remark 2. Observe that this corollary shows a remarkable difference between the zeros of $Q_{n,r}$ and the ones of L^α_n concerning the convergence acceleration to 0: The presence of the positive masses M_i, $i = 0, \ldots, r$, in the inner product produces a convergence acceleration to 0 of $r+1$ zeros of the polynomials $Q_{n,r}$.

3 Laguerre–Sobolev inner products with holes

Until now, we have assumed that all the masses M_i in the discrete part of the Sobolev inner product are positive. The possibility of some $M_i = 0$ has been also studied in the literature. For instance, the case $M_0 = 0, M_1 > 0$ ([8]) and similar situations in the non–diagonal case ([7] and [11]) have been analyzed. Very recently, in [12], the authors study the particular case $M_i = 0$, $i = 0, \ldots, r-1$, for the Laguerre–Sobolev type polynomials. The results obtained in all these papers have been generalized in [5], where such kind of inner products have been called Sobolev inner products with *holes*.

More concretely, we consider the inner product

$$(f, g)_{r,s} = \tag{10}$$
$$\frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + \sum_{i=0}^{r} M_i f^{(i)}(0)g^{(i)}(0) + M_s f^{(s)}(0)g^{(s)}(0),$$

where $s \geq r+2$ and $M_i > 0$ for $i = 0, \ldots, r$ and $i = s$.

Observe that we are concerned with inner products of the form

$$(p, q)_{r,s} = (p, q)_r + M_s p^{(s)}(0)q^{(s)}(0), \quad s \geq r+2,$$

where $M_s > 0$, and in $(.,.)_r$ all the masses are positive. That is, roughly speaking, there is a "hole" in the discrete part of the inner product $(.,.)_{r,s}$. We denote by $\{T_{n,r,s}\}_{n\geq 0}$ the sequence of polynomials orthogonal with respect to the inner product $(.,.)_{r,s}$ with leading coefficients $(-1)^n/n!$.

For this situation, the relative asymptotics and the Mehler-Heine type formulas have been established in [5]. We want to remark that this case has qualitative differences with respect to the case without holes. For example, concerning the convergence acceleration to 0 of the zeros of the polynomials, as we will see below.

Arguing like in Lemma 1 it can be proved

Lemma 2 *Let $\{T_{n,r,s}\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the inner product (10) with $(-1)^n/n!$ as leading coefficient . Then the following statements hold:*

(a) For either $0 \leq k \leq r$ or $k = s$,

$$T^{(k)}_{n,r,s}(0) \sim \frac{C_{r,s,k}}{n^{\alpha+2k+1}} \,(L^\alpha_n)^{(k)}(0)\,,$$

where $C_{r,s,k}$ is a nonzero real number independent of n.

(b) For $k \geq r+1$ and $k \neq s$

$$T_{n,r,s}^{(k)}(0) \sim \frac{k!}{(k-(r+1))!} \frac{k-s}{\alpha+s+k+1} \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)} (L_n^{\alpha})^{(k)}(0)\,.$$

(c)

$$(T_{n,r,s}T_{n,r,s})_{r,s} \sim \|L_n^{\alpha}\|^2\,.$$

Observe that, as in the complete case (without holes), the addition of the discrete part of the inner product modifies the size of the derivative of order k only when the corresponding mass M_k is positive.

Using this lemma the relative asymptotics for these orthogonal polynomials can be deduced:

Theorem 4 *Let $\{T_{n,r,s}\}_{n\geq 0}$ be the sequence of o.p. with respect to the inner product (10) with $(-1)^n/n!$ as leading coefficient . Then*

$$\frac{(T_{n,r,s})^{(k)}(x)}{(L_n^{\alpha})^{(k)}(x)} \rightrightarrows 1\,, \quad x \in \mathbb{C} \setminus [0,\infty)\,,\, k \geq 0.$$

The Mehler–Heine type formula adopts the form

Theorem 5 *Let $\{T_{n,r,s}\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the inner product (10) with $(-1)^n/n!$ as leading coefficient . Then,*

$$\begin{aligned} n^{-\alpha} T_{n,r,s}\left(\frac{x}{n}\right) &\rightrightarrows (-1)^{r+1} x^{-\alpha/2} \\ &\times \left[\frac{-(s-(r+1))}{\alpha+r+s+2} J_{\alpha+2r+2}(2\sqrt{x}) + \sum_{i=2}^{s-r+1} \lambda_i J_{\alpha+2r+2i}(2\sqrt{x})\right],\ x \in \mathbb{C}\,, \end{aligned} \tag{11}$$

where λ_i are nonzero real numbers.

For the particular case $s = r+2$, i.e., when there is a hole of "length one", the above result generalizes the one obtained in [8]. Theorem 5 also generalizes the corresponding result in [12].

Now, we comment the acceleration of the convergence towards the origin of the zeros of the polynomials $T_{n,r,s}$. The quasi–orthogonality of order $s+1$ of the sequence $\{T_{n,r,s}\}_{n\geq 0}$ with respect to the positive weight $x^{\alpha+s+1}e^{-x}$ assures that $T_{n,r,s}$ has at least $n-(s+1)$ changes of sign in $(0,+\infty)$. However, in [2] the authors proved that the number of zeros in $(0,+\infty)$ does not depend on the order of the derivatives but on the number of terms in the discrete part of the inner product. So, $T_{n,r,s}$ has at least $n-(r+1)$ zeros with odd multiplicity in $(0,+\infty)$. Proceeding like in Corollary 1, we get:

Corollary 2 *Let $(\zeta_{n,k}^{r,s})_{k=1}^n$ be the zeros of $T_{n,r,s}$. Then*

$$n\,\zeta_{n,k}^{r,s} \underset{n}{\to} 0, \quad 1 \leq k \leq r+1,$$

$$n\,\zeta_{n,k}^{r,s} \underset{n}{\to} j_{\alpha+2r+2,k-r-1}, \quad k \geq r+2.$$

Remark 3. We want to highlight that this result is in a way surprising since it does not depend on the number of terms in the discrete part, but on the position of the hole. So, despite the presence of the mass M_s, there only exists an acceleration of the convergence of $r+1$ zeros such as it occurs in the case of the inner products without holes. That is, the convergence acceleration to 0 of the zeros of the polynomials $Q_{n,r}$ and $T_{n,r,s}$ is the same and the addition of a mass M_s *after a hole* in the inner product does not affect the convergence acceleration to 0.

4 Generalized Hermite–Sobolev type polynomials

As a consequence of the previous results, asymptotic properties for the orthogonal polynomials $S^{\mu}_{n,r}$ associated with the inner product

$$(p,q) = \int_{\mathbb{R}} p(x)q(x)|x|^{2\mu}\, e^{-x^2} dx + \sum_{i=0}^{2r+1} M_i\, p^{(i)}(0)\, q^{(i)}(0), \tag{12}$$

with $\mu > -1/2$ and $M_i > 0$, $i = 0, \ldots, 2r+1$, can be established. We assume that the leading coefficient of $S^{\mu}_{n,r}$ is 2^n.

Remember that the polynomials H^{μ}_n orthogonal with respect to the weight $|x|^{2\mu}\, e^{-x^2}$ ($\mu > -1/2$) are called generalized Hermite polynomials. So, we are concerned with generalized Hermite–Sobolev type orthogonal polynomials.

Notice that in this case the polynomials $S^{\mu}_{n,r}$ are symmetric, that is, $S^{\mu}_{n,r}(-x) = (-1)^n\, S^{\mu}_{n,r}(x)$, and because of this symmetry, we can transform the inner product (12) into an inner product like (6) and so we can establish a simple relation between the polynomials $S^{\mu}_{n,r}$ and the polynomials $Q_{n,r}$ considered before. This technique is known as a symmetrization process. In fact, in [10] this process is considered for standard inner products associated with positive measures. The simplest case of this situation is the relation between Laguerre polynomials and Hermite polynomials, that is (see [10] or [23]), for $n \geq 0$,

$$H_{2n}(x) = (-1)^n\, 2^{2n}\, n!\, L_n^{-1/2}(x^2),\ H_{2n+1}(x) = (-1)^n\, 2^{2n+1}\, n!\, x L_n^{1/2}(x^2)\,.$$

Later in [3] the authors generalize the symmetrization process in the framework of Sobolev type orthogonal polynomials, (see Theorem 2 in [3]). Thus,

$$S^{\mu}_{2n,r}(x) = (-1)^n\, 2^{2n}\, n!\, Q^{\mu-1/2}_{n,r}(x^2),\ S^{\mu}_{2n+1,r}(x) = (-1)^n\, 2^{2n+1}\, n!\, x Q^{\mu+1/2}_{n,r}(x^2)$$

where $\{Q^{\mu-1/2}_{n,r}\}_{n\geq 0}$ (respectively, $\{Q^{\mu+1/2}_{n,r}\}_{n\geq 0}$) is the sequence of polynomials orthogonal with respect to an inner product like (6) with $\alpha = \mu - 1/2$ (respectively, $\alpha = \mu + 1/2$) and leading coefficient $(-1)^n/n!$.

Asymptotics for monic Hermite–Sobolev type polynomials (i.e., $\mu = 0$ in (12)) has been analyzed in [4] where a non–diagonal case with $r = 0$ is considered. Then, the symmetrization process does not work and the starting–point is an algebraic formula which relates Hermite–Sobolev type and Hermite polynomials (see [4, Proposition 1]). A first conclusion of this analysis is that, in this case, Hermite–Sobolev type polynomials have the same outer strong asymptotic and Plancherel–Rotach type behaviour as Hermite polynomials. Moreover, from the Mehler–Heine type formulas there obtained (see [4, Theorem 1]) it can be deduced that the non–diagonal case does not add further additional information to the one obtained in the diagonal case concerning the asymptotic behaviour of the scaled polynomials. For this reason, in the rest of the paper [4] a diagonal inner product like (12), with $\mu = 0, r = 1$, and $M_i \geq 0, i = 0, 1, 2, 3$, is studied.

Coming back to the general inner product (12), using the symmetrization process, we can prove the relative asymptotics and the Mehler–Heine type formulas for generalized Hermite-Sobolev type polynomials.

Proposition 2 *Let $\{S^{\mu}_{n,r}\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the inner product (12) with 2^n as leading coefficient . Then,*

(a)

$$\frac{(S^{\mu}_{n,r})^{(k)}(x)}{(H^{\mu}_{n})^{(k)}(x)} \rightrightarrows 1,\; x \in \mathbb{C} \setminus \mathbb{R}\,,\; k \geq 0.$$

(b)

$$n^{-\mu+1/2}\, S^{\mu}_{2n,r}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows (-1)^{r+1}\left(\frac{x}{2}\right)^{-\mu+1/2} J_{\mu+2r+3/2}(x),\; x \in \mathbb{C}$$
$$n^{-\mu+1/2}\, S^{\mu}_{2n+1,r}\left(\frac{x}{2\sqrt{n}}\right) \rightrightarrows (-1)^{r+1}\left(\frac{x}{2}\right)^{-\mu+1/2} J_{\mu+2r+5/2}(x),\; x \in \mathbb{C}.$$

Remark 4. These results generalize some of the results in [4] and solve the conjecture posed there.

Using a symmetrization process, we can deduce the relative asymptotics and the Mehler–Heine type formulas for generalized Hermite–Sobolev polynomials with holes in the discrete part of the inner product.

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