Stieltjes transforms defined by C_0-semigroups

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Abstract. In this paper we use the resolvent semigroup associated to a C_0-semigroup to introduce the vector-valued Stieltjes transform defined by a C_0-semigroup. We give new results which extend known ones in the case of scalar generalized Stieltjes transform. We work with the vector-valued Weyl fractional calculus to present a deep connection between both concepts.

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Introduction

Integral transforms are useful techniques to study integral and differential equations. One of the most famous is the Laplace transform, but other ones like the Fourier, Mellin or Hankel transforms are used in different problems.

The Stieltjes transform of a function $f : [0,\infty) \to \mathbf{C}$ is defined by the integral expression

$$\mathcal{S}f(t) := \int_0^\infty \frac{f(s)}{s+t}\,ds, \qquad t > 0.$$

Its connection with the Laplace transform and some results about inversion are given, for example in [8]. The generalized Stieltjes transform for $\alpha > 0$ defined by

$$\mathcal{S}_\alpha f(t) := \int_0^\infty \frac{f(s)}{(s+t)^\alpha}\,ds, \qquad t > 0,$$

is a natural extension studied in [9]. A well-written survey about the Stieltjes transform of generalized functions with different applications to differential equations may be found in [5].

The vector-valued Stieljes transform has been introduced in [3] where some results about the inversion are shown. Note that our starting point is different.

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Given a uniformly bounded C_0-semigroup of linear and bounded operators, this short note is mainly dedicated to extend the generalized Stieltjes transform.

A C_0-semigroup of linear and bounded operators $(T(t))_{t\geq 0}$ is a natural setting to extend the Stieltjes transform. Many integro-differential operators generates C_0-semigroup and the relationship between Cauchy problems and C_0-semigroups is well-known ([1]).

In the first section we recall some basic properties about the fractional semigroup and scalar Stieltjes and Laplace transforms. Algebraic structure for the usual convolution product in the semigroup $[0,\infty)$ is considered to use it in next sections.

In the second section, we present a short introduction of C_0-semigroups and resolvent semigroups. The result Theorem 2.1 is original and let us to give solution of abstract integral equations and show new results in the forth section.

In the third section we present some facts about the vector-valued Weyl fractional calculus, and its connection with resolvent semigroups, see for example Proposition 3.1.

In the last section, we present main results of the paper. We introduce the Stieltjes transform associated to a C_0-semigroup, we give some basic results and we give a deep connection between Weyl fractional calculus and Stieltjes transform, see Theorem 4.5.

Notation. $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ are the set of natural, real and complex numbers; $\Re z$ is the real part of a complex number z; $\mathbf{C}^+ := \{z\in\mathbf{C}\ |\Re z>0\}$.

Γ is the Euler function, and χ_E is the characteristic function on the subset E. Let X be a Banach space. We denote by $\mathcal{S}_+(X)$ the Schwartz class on $[0,\infty)$ i.e., functions $f:[0,\infty)\to X$, which are infinitely differentiable and verify

$$\sup_{t\geq 0}\left\|t^m\frac{d^n}{dt^n}f(t)\right\|<\infty$$

for any $m,n\in\mathbf{N}\cup\{0\}$. We write by $\mathcal{S}_+=\mathcal{S}_+(\mathbf{C})$.

1. Fractional semigroup and integral transforms in $L^1(\mathbf{R}^+)$

We denote by $L^1(\mathbf{R}^+)$ the usual Lebesgue space of (class of) measurable functions f such that

$$\|f\|_1 := \int_0^\infty |f(t)|<\infty.$$

It is well know that $(L^1(\mathbf{R}^+),\|\ \ \|_1)$ is a Banach algebra with the convolution product $*$ given by

$$f*g(t)=\int_0^t f(t-s)g(s)ds,\qquad f,g\in L^1(\mathbf{R}^+).$$

The Laplace transform $\mathcal{L} : L^1(\mathbf{R}^+) \to H^\infty(\overline{\mathbf{C}^+})$, given by

$$\mathcal{L}f(z) := \int_0^\infty f(t)e^{-zt}dt, \qquad z \in \overline{\mathbf{C}^+},$$

is a linear and bounded algebra homomorphism, where $H^\infty(\overline{\mathbf{C}^+})$ is the algebra of analytic and bounded functions in $\overline{\mathbf{C}^+}$ with the pointwise product.

Theorem 1.1. [7, Theorem 2.6] *We consider for $z \in \mathbf{C}^+$ the function I^z given by*

$$I^z(t) := \frac{t^{z-1}}{\Gamma(z)}e^{-t}, \qquad z \in \mathbf{C}^+.$$

Then it holds that

(i) $I^z * I^w = I^{z+w}$,*for* $z, w \in \mathbf{C}^+$;

(ii) *The set* $I^z * L^1(\mathbf{R}^+)$ *is dense in* $L^1(\mathbf{R}^+)$;

(iii) $\|I^z\|_1 = \dfrac{\Gamma(\Re z)}{|\Gamma(z)|}$;

(iv) $\mathcal{L}I^z(w) = (1+w)^{-z}$, *for* $z, w \in \mathbf{C}^+$.

The family $(I^z)_{\Re z>0}$ *is called the fractional semigroup in* $L^1(\mathbf{R}^+)$.

Close to the fractional semigroup, we may define functions $(e_{\varepsilon,\alpha})_{\varepsilon,\alpha>0}$ by

$$e_{\varepsilon,\alpha}(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-\varepsilon t}, \qquad t > 0.$$

Note that $(e_{\varepsilon,\alpha})_{\varepsilon,\alpha>0} \subset L^1(\mathbf{R}^+)$, $e_{\varepsilon,\alpha}(t) = \dfrac{1}{\varepsilon^{\alpha-1}}I^\alpha(\varepsilon t)$ and $\|e_{\varepsilon,\alpha}\|_1 = \dfrac{1}{\varepsilon^\alpha}$ for $\varepsilon, \alpha > 0$.

Lemma 1.2. *Given $\alpha > 0$, $\lambda \in \mathbf{C}^+$ and $\varepsilon \in \mathbf{C}$, we have that*

$$\left(\frac{t^{\alpha-1}e^{-\varepsilon t}}{\Gamma(\alpha)} * \frac{e^{-\varepsilon t}}{(t+\lambda)^\alpha}\right)(x) = \frac{e^{-\varepsilon x}x^\alpha}{\Gamma(\alpha+1)}\frac{1}{(x+\lambda)\lambda^\alpha}, \qquad x \geq 0.$$

Proof. Take $x \geq 0$ and we get that

$$\left(\frac{t^{\alpha-1}e^{-\varepsilon t}}{\Gamma(\alpha)} * \frac{e^{-\varepsilon t}}{(t+\lambda)^\alpha}\right)(x) = e^{-\varepsilon x}\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{(t+\lambda)^{\alpha-1}}dt.$$

Now, we change the variable $t = ux$ and apply the formula [2, Formula 3.198] to get

$$\left(\frac{t^{\alpha-1}e^{-\varepsilon t}}{\Gamma(\alpha)} * \frac{e^{-\varepsilon t}}{(t+\lambda)^\alpha}\right)(x) = \frac{e^{-\varepsilon x}x^\alpha}{\Gamma(\alpha+1)}\frac{1}{(x+\lambda)\lambda^\alpha},$$

and we obtain the result. □

Take $\alpha > 0$, and $f \in L^1(\mathbf{R}^+)$. The *generalized Stieltjes transform*, $\mathcal{S}_\alpha f$ of f is defined by

$$\mathcal{S}_\alpha f(y) := \int_0^\infty \frac{f(x)}{(x+y)^\alpha}dx, \qquad y > 0.$$

In the particular case $\alpha = 1$, we get the usual Stieltjes transform given by

$$\mathcal{S}_1 f(y) := \int_0^\infty \frac{f(x)}{x+y} dx, \qquad y > 0.$$

Note that $|y^\alpha \mathcal{S}_\alpha f(y)| \leq \|f\|_1$ for $y > 0$. This integral transform has been studied in detail in [9].

Proposition 1.3. *Take* $\alpha > o$ *and* $f \in L^1(\mathbf{R}^+)$. *Then*

$$\mathcal{S}_\alpha f(y) = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-yt} \mathcal{L}f(t) dt;$$

in particular $\mathcal{S}_\alpha f(1) = \int_0^\infty I^\alpha(t) \mathcal{L}f(t) dt$ *and* $\mathcal{S}_1 f = \mathcal{L}^2 f$.

Proof. Take $f \in L^1(\mathbf{R}^+)$ and $x \in X$, and by Fubini theorem, we have that

$$\int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-yt} \int_0^\infty e^{-st} f(s) ds dt = \int_0^\infty f(s) \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-(y+s)t} dt ds = \mathcal{S}_\alpha f(y),$$

for $y > 0$. Both equalities $\mathcal{S}_\alpha f(1) = \int_0^\infty I^\alpha(t) \mathcal{L}f(t) dt$ and $\mathcal{S}_1 f = \mathcal{L}^2 f$ are straightforward. ☐

There it may be proved that if $\alpha > 0$ and $f, g \in L^1(\mathbf{R}^+)$ then

$$\int_0^\infty f(x) \mathcal{S}_\alpha g(x) dx = \int_0^\infty \mathcal{S}_\alpha f(x) g(x) dx, \tag{1.1}$$

when both integrals exist, see for example [9].

2. C_0-semigroups and resolvent semigroups

Let X be a Banach space and $\mathcal{B}(X)$ the set of linear and bounded operators defined on X. A uni-parameter family of operators $(T(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called a *C_0-semigroup of operators* when the map $t \mapsto T(t)x$ is strongly continuous, $T(0) = I$, and $T(t+s) = T(t)T(s)$ for $t, s \geq 0$. A C_0-semigroup is said uniformly bounded when there exists a constant $M > 0$ such that $\|T(t)\| \leq M$ for $t \geq 0$.

Take $(T(t))_{t\geq 0}$ a uniformly bounded C_0-semigroup. Then there exists a closed densely defined operator $(A, D(A))$ such that $\mathbf{C}^+ \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A) and

$$(\lambda - A)^{-1} x = \int_0^\infty e^{-\lambda t} T(t) x dt \qquad \lambda \in \mathbf{C}^+, \quad x \in X.$$

The operator $(A, D(A))$ is called the *infinitesimal generator* of $(T(t))_{t\geq 0}$. Note that $\sup_{\lambda > 0} \|\lambda(\lambda - A)^{-1}\| < \infty$ and the fractional power $(\lambda - A)^{-\alpha}$ is defined, belongs to $\mathcal{B}(X)$,

$$(\lambda - A)^{-\alpha} x = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda t} T(t) x dt, \quad x \in X,$$

and

$$\|(\lambda - A)^{-\alpha}\| \leq \frac{C}{|\lambda|^\alpha},$$

for $\alpha > 0$ and $\lambda \in \mathbf{C}^+$ see [1, Formula (3.56)].

The family of operators $((1-A)^{-\alpha})_{\alpha>0}$ is usually called the *resolvent semigroup* associated to the C_0-semigroup $(T(t))_{t\geq 0}$. Operators $(\lambda - A)^{-\alpha}$ have dense range and then its inverse $(\lambda - A)^{\alpha}$ are densely defined for $\alpha > 0$ and $\lambda \in \mathbf{C}^+$.

It is well known that given $T \equiv (T(t))_{t\geq 0}$ a uniformly bounded C_0-semigroup, the map $\Theta_T : L^1(\mathbf{R}^+) \to \mathcal{B}(X)$ defined by

$$\Theta_T(f)x := \int_0^\infty f(t)T(t)xdt, \qquad x \in X,$$

is a linear and bounded homomorphism such that

$$\Theta_T(f * g) = \Theta_T(f)\Theta_T(g), \qquad f, g \in L^1(\mathbf{R}^+), \tag{2.1}$$

$\Theta_T(e_\lambda) = (\lambda - A)^{-1}$ where $e_\lambda(t) = e^{-\lambda t}$ and $\lambda \in \mathbf{C}^+$ and $\Theta_T(I^\alpha) = (1-A)^{-\alpha}$ for $\alpha > 0$.

Theorem 2.1. *Given $\varepsilon, \alpha > 0$ and $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup $(T(t))_{t>0}$ on a Banach space X, then*

$$\frac{t^\alpha}{\Gamma(\alpha+1)}(\varepsilon + A)^{-\alpha}(t + \varepsilon + A)^{-1}x = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\varepsilon + s + A)^{-(\alpha+1)}xds$$

for $t > 0$ and $x \in X$.

Proof. Take $\lambda > 0$ and we apply the Laplace transform and the Fubini theorem to obtain that

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\varepsilon + s + A)^{-(\alpha+1)}xds \\
&= \int_0^\infty (\varepsilon + s + A)^{-(\alpha+1)}x\int_s^\infty \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda t} \\
&= \int_0^\infty (\varepsilon + s + A)^{-(\alpha+1)}x\frac{1}{\lambda^\alpha}e^{-\lambda t}dt \\
&= \frac{1}{\lambda^\alpha}\int_0^\infty\int_0^\infty \frac{y^\alpha}{\Gamma(\alpha+1)}e^{-(\varepsilon+s)y}T(y)xdye^{-\lambda t}dt \\
&= \frac{1}{\lambda^\alpha}\int_0^\infty \frac{y^\alpha}{\Gamma(\alpha+1)}e^{-\varepsilon y}T(y)x\frac{dy}{y+\lambda} = \Theta_T(g_{\lambda,\varepsilon,\alpha}),
\end{aligned}$$

where $g_{\lambda,\varepsilon,\alpha}(y) := \dfrac{y^\alpha}{\lambda^\alpha\Gamma(\alpha+1)}\dfrac{e^{-\varepsilon y}}{y+\lambda}$ for $y > 0$. On the other hand,

$$\begin{aligned}
&\frac{(\varepsilon + A)^{-\alpha}}{\Gamma(\alpha+1)}\int_0^\infty e^{-\lambda t}t^\alpha(t + \varepsilon + A)^{-1}xdt \\
&= \frac{(\varepsilon + A)^{-\alpha}}{\Gamma(\alpha+1)}\int_0^\infty e^{-\lambda t}t^\alpha\int_0^\infty e^{-s(t+\varepsilon)}T(s)xdsdt
\end{aligned}$$

$$= (\varepsilon + A)^{-\alpha} \int_0^\infty \frac{e^{-\varepsilon s}}{(\lambda+s)^{\alpha+1}} T(s)xds = \Theta_T(f_{\varepsilon,\alpha})\Theta_T(h_{\lambda,\varepsilon,\alpha}),$$

where $e_{\varepsilon,\alpha}(t) := \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\varepsilon t}$ and $h_{\lambda,\varepsilon,\alpha}(t) := \dfrac{e^{-\varepsilon t}}{(\lambda+t)^{\alpha+1}}$ for $t > 0$.

By Lemma 1.2, we have that $g_{\lambda,\varepsilon,\alpha} = e_{\varepsilon,\alpha} * h_{\lambda,\varepsilon,\alpha}$ for $\varepsilon, \lambda, \alpha > 0$, and we get the equality by the identity (2.1). □

The Theorem 2.1 may be written in terms of integral equations.

Corollary 2.2. *Given $\varepsilon, \alpha > 0$ and $(-A, D(A))$ the infinitesimal generator of a C_0-semigroup on a Banach space X. The solution of the equation*

$$\frac{t^\alpha}{\Gamma(\alpha+1)} (\varepsilon + A)^{-\alpha}(t + \varepsilon + A)^{-1}x = \int_0^\infty g(t,s)(\varepsilon + s + A)^{-(\alpha+1)}xds, \tag{2.2}$$

for $t > 0$ and $x \in X$ is the function $g(t,s) := \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\chi_{(0,t)}(s)$.

Remark. Note that in the scalar case we have the following formula

$$\frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{y}\right)^\alpha \frac{1}{x+y} = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-t)^{\alpha-1}}{(y+t)^{\alpha+1}}dt.$$

for $\alpha, x, y > 0$.

3. The Weyl fractional calculus

We recall some basic properties of the Weyl fractional calculus in a Banach space X. Given $f \in \mathcal{S}_+(X)$, the *Weyl fractional integral* of f of order $\alpha > 0$ is defined by

$$W_+^{-\alpha} f(u) := \frac{1}{\Gamma(\alpha)} \int_u^\infty (t-u)^{\alpha-1} f(t)dt, \qquad u \geq 0,$$

with $\alpha > 0$. This operator $W_+^{-\alpha} : \mathcal{S}_+(X) \to \mathcal{S}_+(X)$ is one to one, and its inverse, W_+^α, is the *Weyl fractional derivative* of order α, and

$$W_+^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)ds, \qquad t \geq 0,$$

holds with $n = [\alpha] + 1$, see, for example [4], [6] in the scalar case $X = \mathbf{C}$.

It is easy to check that if $\alpha \in \mathbf{N}$ then $W_+^\alpha f = (-1)^\alpha f^{(\alpha)}$ and $W_+^{\alpha+\beta} f = W_+^\alpha(W_+^\beta f)$ with $\alpha, \beta \in \mathbf{R}$, with $W_+^0 = Id$ and $f \in \mathcal{S}_+(X)$. Moreover the following " integrating by parts" formula

$$\int_0^\infty f(t)g(t)dt = \int_0^\infty \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds W_+^\alpha g(t)dt \tag{3.1}$$

holds for $f \in \mathcal{S}_+(X)$ and $g \in \mathcal{S}_+$. Let $\lambda \in \mathbf{C}^+$ and $e_\lambda(s) := e^{-\lambda s}$ with $s \geq 0$. It is clear that $e_\lambda \in \mathcal{S}_+$ and

$$W_+^{-\alpha}(e_\lambda)(s) = \lambda^{-\alpha} e^{-\lambda s}, \qquad s \geq 0.$$

Then $W_+^{\alpha}(e_\lambda) = \lambda^{\alpha} e_\lambda$ for $\alpha \in \mathbf{R}$.

The Weyl fractional calculus is applied to several functions which do not belong to $\mathcal{S}_+(X)$, we keep the notation in these functions.

Proposition 3.1. *Let $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup $(T(t))_{t>0}$ on a Banach space X. Then the following identities hold.*

(i) $W_+^{-\alpha}(e^{-\varepsilon s}T(s)x)(t) = (\varepsilon + A)^{-\alpha} e^{-\varepsilon t} T(t)x$ *for* $x \in X$ *and* $t, \alpha > 0$.

(ii) $W_+^{\alpha}(e^{-\varepsilon s}T(s)x)(t) = (\varepsilon + A)^{\alpha} e^{-\varepsilon t} T(t)x$ *for* $x \in D(A^n)$, $n = [\alpha] + 1$ *and* $t > 0$.

(iii) $W_+^{-\alpha}((s + A)^{-\beta}x)(t) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}(t + A)^{-(\beta-\alpha)}$ *for* $x \in X$, $t > 0$ *and* $\beta > \alpha$.

(iv) $W_+^{\alpha}((s + A)^{-\beta}x)(t) = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}(t + A)^{-(\beta+\alpha)}$ *for* $x \in X$, $t > 0$ *and* $\beta, \alpha > 0$.

Proof. We check the part (i). Take $\alpha, \varepsilon > 0$ and $x \in X$. Then we have that

$$\begin{aligned} W_+^{-\alpha}(e^{-\varepsilon s}T(s)x)(t) &= \frac{1}{\Gamma(\alpha)} \int_t^\infty (s - t)^{\alpha-1} e^{-\varepsilon s} T(s)x ds \\ &= e^{-\varepsilon t} T(t) \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-\varepsilon u} T(u)x ds = (\varepsilon + A)^{-\alpha} e^{-\varepsilon t} T(t)x. \end{aligned}$$

Proofs of the other parts are left to the reader. ☐

4. The generalized Stieltjes transform defined by C_0-semigroups

In this section we introduce the generalized Stieltjes transform defined by a uniformly bounded C_0-semigroup. We give some results which extend known results in the scalar case.

Definition 4.1. Let $\alpha, \varepsilon > 0$, $f \in \mathcal{S}_+$ and $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup $T \equiv (T(t))_{t>0}$ on a Banach space X. The *generalized Stieltjes transform* defined by T, $\mathcal{S}_\alpha^T f(\varepsilon) \in \mathcal{B}(X)$ is defined by

$$\mathcal{S}_\alpha^T f(\varepsilon)x := \int_0^\infty f(t)(\varepsilon + t + A)^{-\alpha} x dt, \qquad x \in X.$$

Note that $\|\mathcal{S}_\alpha^T f(\varepsilon)x\| \leq C\|x\| \int_0^\infty \frac{|f(s)|}{(s + \varepsilon)^\alpha} ds$ for $\alpha, \varepsilon > 0$, $f \in \mathcal{S}_+$ and $x \in X$.

Next result is a natural extension of Proposition 1.3.

Proposition 4.2. *Let $\alpha > 0$, $f \in \mathcal{S}_+$ and $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup $T \equiv (T(t))_{t>0}$ on a Banach space X. Then*

$$\mathcal{S}_\alpha^T f(t) = \Theta_T(e_{t,\alpha}\mathcal{L}(f)), \qquad t > 0,$$

where $e_{t,\alpha}(s) = \dfrac{s^{\alpha-1}}{\Gamma(\alpha)} e^{-ts}$; in particular $\mathcal{S}_\alpha^T f(1) = \Theta_T(I^\alpha \mathcal{L}(f))$.

Proof. We apply the Fubini theorem to get that

$$\begin{aligned} \mathcal{S}_\alpha^T f(t)x &= \int_0^\infty f(r)(t+r+A)^{-\alpha} x dr \\ &= \int_0^\infty f(r) \int_0^\infty \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{-(r+t)s} T(s) x ds dr \\ &= \int_0^\infty \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{-ts} T(s) x \int_0^\infty f(r) e^{-rs} dr ds = \Theta_T(e_{t,\alpha}\mathcal{L}(f))x, \end{aligned}$$

and we conclude the result. □

Proposition 4.3. *Let $0 \le \alpha$, $f, g \in \mathcal{S}_+$ such that*

$$\int_0^\infty \int_0^\infty \frac{|f(t)g(s)| dt ds}{(t+s)^\alpha} < \infty$$

and $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup $T \equiv (T(t))_{t>0}$ on a Banach space X.

$$\int_0^\infty g(t) \mathcal{S}_\alpha^T f(t) x dt = \int_0^\infty \mathcal{S}_\alpha^T g(s) x f(s) ds, \qquad x \in X. \tag{4.1}$$

Proof. Take $f, g \in \mathcal{S}_+$ and by the Fubini theorem, we have that

$$\begin{aligned} \int_0^\infty g(t) \mathcal{S}_\alpha^T f(t) x dt &= \int_0^\infty g(t) \int_0^\infty f(s)(s+t+A)^{-\alpha} x ds dt \\ &= \int_0^\infty f(s) \int_0^\infty g(t)(s+t+A)^{-\alpha} x dt ds \\ &= \int_0^\infty \mathcal{S}_\alpha^T g(s) x f(s) ds, \end{aligned}$$

and we conclude the proof. □

Remark. Note that the equality (4.1) contains the scalar case given in (1.1).

Proposition 4.4. *Take $\mu, \alpha > 0$, $f \in \mathcal{S}_+$ and $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup $T \equiv (T(t))_{t>0}$ on a Banach space X. The equality*

$$W_+^\alpha \left(\mathcal{S}_\mu^T f(\,\cdot\,) x \right)(t) = \frac{\Gamma(\alpha+\mu)}{\Gamma(\alpha)} \mathcal{S}_{\mu+\alpha}^T f(t) x,$$

holds for $x \in X$ and $t > 0$.

Proof. We use Proposition 3.1 (iv) to get that

$$\begin{aligned} W_+^\alpha\left(\mathcal{S}_\mu^T f(\,\cdot\,)x\right)(t) &= \frac{\Gamma(\alpha+\mu)}{\Gamma(\alpha)}\int_0^\infty f(s)(s+t+A)^{-(\alpha+\mu)}xds \\ &= \frac{\Gamma(\alpha+\mu)}{\Gamma(\alpha)}\mathcal{S}_{\mu+\alpha}^T f(t)x, \end{aligned}$$

with $x \in X$ and $t > 0$. □

Theorem 4.5. *Let $\alpha > 0$, $f \in \mathcal{S}_+$ and $(-A, D(A))$ the infinitesimal generator of a uniformly bounded C_0-semigroup on a Banach space X. Then we have that*

$$\mathcal{S}_{\alpha+1}^T f(t)x = \frac{(t+A)^{-\alpha}}{\Gamma(\alpha+1)}\mathcal{S}_1^T(y^\alpha W_+^\alpha f)(t)x \qquad x \in X, t > 0.$$

Proof. We apply the formula (3.1) and the Theorem 2.1 to get

$$\begin{aligned} \mathcal{S}_{\alpha+1}^T f(t)x &= \int_0^\infty W_+^\alpha f(y)\frac{1}{\Gamma(\alpha)}\int_0^y (y-r)^{\alpha-1}(r+t+A)^{\alpha+1}xdrdy \\ &= \frac{(t+A)^{-\alpha}}{\Gamma(\alpha+1)}\int_0^\infty W_+^\alpha f(y)y^\alpha(y+t+A)^{-1}xdy \\ &= \frac{(t+A)^{-\alpha}}{\Gamma(\alpha+1)}\mathcal{S}_1^T(y^\alpha W_+^\alpha f)(t)x, \end{aligned}$$

for $t > 0$ and $x \in X$. □

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