HILBERT, DIRICHLET AND FEJÉR FAMILIES OF OPERATORS ARISING FROM C_0-GROUPS, COSINE FUNCTIONS AND HOLOMORPHIC SEMIGROUPS

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Abstract. The main aim of this paper is to extend definitions of Hilbert transform, Dirichlet and Fejér operators (defined by convolution with suitable kernels in Lebesgue spaces) in arbitrary Banach spaces. We present a self-contained theory which includes different approaches of other authors whose starting points were usually C_0-groups or cosine functions. We present relations with holomorphic semigroups. We characterize the geometric property of UMD spaces in terms of the Dirichlet and Fejér operators. To end the paper, we give examples to illustrate our results.

Introduction

Let $(L^p(\mathbb{R}), \| \quad \|_p)$ be the usual Lebesgue space with the norm

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}}, \qquad f \in L^p(\mathbb{R}),$$

with $1 \leq p < \infty$. Take $g \in L^p(\mathbb{R})$ with $1 < p < \infty$ and $\varepsilon > 0$. Partial Hilbert transforms, $H_\varepsilon : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, are defined by

$$(H_\varepsilon g)(t) := \frac{i}{\pi} \int_{\varepsilon \leq |s|} \frac{g(t-s)}{s} ds, \qquad g \in L^p(\mathbb{R}).$$

Riesz's theorem sets up that the Hilbert transform, $Hg := \lim_{\varepsilon \to 0^+} H_\varepsilon g$, is a bounded operator $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ with $1 < p < \infty$.

With the help of the Dirichlet kernel, $(d_s)_{s \geq 0}$, defined by

$$d_s(r) := \frac{\sin(sr)}{\pi r}, \qquad r \in \mathbb{R} \backslash \{0\},$$

for $s \geq 0$, one can consider operators $\mathcal{D}(s) : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, called Dirichlet operators, $\mathcal{D}(s)(g) := d_s * g$, for $g \in L^p(\mathbb{R})$, with $1 < p < \infty$. Riesz's theorem implies that $(\mathcal{D}(s))_{s \geq 0}$ is a family of uniformly bounded operators. Another property is that $\mathcal{D}(s) g \to g$ as $s \to \infty$ for any $g \in L^p(\mathbb{R})$, $1 < p < \infty$.

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Similarly, the Fejér kernel,

$$f_s(r) := \frac{1-\cos(sr)}{\pi r^2}, \qquad s \geq 0,\ r \in \mathbb{R}\backslash\{0\},$$

gives rise to Fejér operators, $\mathcal{F}(s) : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, by $\mathcal{F}(s)g := f_s * g$ for $g \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Since $f_s \in L^1(\mathbb{R})$, $\mathcal{F}(s)$ is a bounded operator. One has $\frac{1}{s}\mathcal{F}(s)g \to g$ as $s \to \infty$ for any $g \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Note that

$$f_s(r) = \int_0^s d_t(r)dt, \qquad r \in \mathbb{R},$$

$s \geq 0$. See these known results in [19], [22] and [24].

In the first section of the paper and later in the case of bounded C_0-groups, we shall be concerned with a particular class of Banach spaces, UMD spaces. The original definition of the UMD property may be found in [9]. Every UMD space is a reflexive space and $L^p(\mathbb{R})$ is UMD if and only if $1 < p < \infty$. We shall use the following characterization of the UMD property in terms of the Hilbert transform. For $\varepsilon \in (0,1)$, $p \in (1,\infty)$ and $N > 1$, let $H_{\varepsilon,N}$ be bounded operators defined on $L^p(\mathbb{R};X)$ as

$$(H_{\varepsilon,N}g)(t) := \frac{i}{\pi}\int_{\varepsilon\leq|s|\leq N} \frac{g(t-s)}{s}ds, \qquad g \in L^p(\mathbb{R};X).$$

The Banach space X is a UMD space if and only if $H_{\varepsilon,N}$ admits a strong limit $H \in \mathcal{B}(L^p(\mathbb{R};X))$ as ε goes to 0^+ and N goes to $+\infty$ and for some, or equivalently for all $p \in (1,\infty)$, see [8, 9].

Note that if we consider the shift group $(T(s))_{s\in\mathbb{R}}$ acting by $T(s)g(t) = g(t-s)$ for $t, s \in \mathbb{R}$, then

$$Hg = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{T(s)g}{s}ds, \qquad g \in L^p(\mathbb{R};X),$$

where we follow the convention $\displaystyle\int_{-\infty}^{\infty} \frac{T(s)g}{s}ds := \lim_{\varepsilon\to 0, N\to\infty}\int_{\varepsilon\leq|s|\leq N} \frac{T(s)g}{s}ds.$

For arbitrary uniformly bounded C_0-groups in UMD spaces, $T \equiv (T(s))_{s\in\mathbb{R}} \subset \mathcal{B}(X)$, ($\mathcal{B}(X)$ is the algebra of linear and bounded operators on X), the Hilbert transform, H_0^T, where

$$H_0^T x := \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{T(s)x}{s}ds, \qquad x \in X,$$

has been studied by several authors, see for example [4, 20].

The aim of our paper is twofold. First, in the first section we characterize UMD spaces in terms of Dirichlet and Fejér operators associated with the shift group in $L^p(\mathbb{R};X)$ (Theorem 1.1 and Theorem 1.4). These characterizations show the important relationship between these operators and the geometry of the Banach space X. Second, we introduce Hilbert, Dirichlet and Fejér families of operators on an arbitrary Banach space X. Some particular cases of these general concepts have appeared through uniformly bounded C_0-groups or cosine functions of operators. The novelty of our paper is that our starting point in the three cases is an algebraic definition which allows to cover these studied cases. The main point in these definitions is a composition rule which plays a central role (in the case of Dirichlet and Fejér operators) to define algebra homomorphisms from algebras of absolutely continuous functions on $\mathbb{R}^+$ into $\mathcal{B}(X)$. We also present how to associate holomorphic semigroups with Dirichlet and Fejér families, and show that certain holomorphic semigroups give rise to Fejér families.

Algebra homomorphisms (sometimes called *functional calculi*) associated with a closed operator are also used to classify operators. The canonical example is a self-adjoint operator on a Hilbert space which has a functional calculus for any function which is Borel measurable on the spectrum of the operator. On an arbitrary Banach space such operators have been called spectral operators of scalar type and treated in [16]. From this time different functional calculi have been introduced, for example, a well-bounded operator has a functional calculus for absolutely continuous functions. See [12] for more details in this topic.

In the second section we define Hilbert families of operators in a general way. In the case of uniformly bounded C_0-groups in UMD spaces, these operators have been considered in [4, Theorems 5.12 and 5.16] and the Hilbert transform is also given in [20]. We give a definition of Hilbert transform for uniformly bounded cosine functions in UMD space (see Theorem 2.8) which coincides with the one considered in [23, Section 3].

Our definition of a Hilbert family of operators is equivalent to the concept of a spectral family, see Theorem 2.3. A *spectral family* on a Banach space X is a projection-valued function $E:\mathbb{R}\to\mathcal{B}(X)$ which satisfies the following conditions:

(i) $\sup\{\|E(\lambda)\|:\lambda\in\mathbb{R}\}<\infty$;
(ii) $E(\lambda)E(\mu)=E(\mu)E(\lambda)=E(\min\{\lambda,\mu\})$, for $\lambda,\mu\in\mathbb{R}$;
(iii) $E(\cdot)$ is right continuous on $\mathbb{R}$ in the strong operator topology;
(iv) $E(\cdot)$ has a left-hand limit in the strong operator topology at each point of $\mathbb{R}$ which we write $E(s^-)x$ for $s\in\mathbb{R}$ and $x\in X$;
(v) $E(\lambda)\to 0$ (resp., $E(\lambda)\to I$) in the strong operator topology as $\lambda\to-\infty$, (resp., $\lambda\to+\infty$).

If X is a reflexive Banach space then a projection-valued function $E:\mathbb{R}\to\mathcal{B}(X)$ satisfying (i) and (ii) of the above definition has a strong left-hand limit and a strong right-hand limit at each point of $\mathbb{R}$, see [2].

In the third section we introduce Dirichlet families of operators. Every Dirichlet family is given by some spectral family, but this might not be uniquely determined, see Theorem 3.2. To a Dirichlet family we associate an algebra homomorphism (of absolutely continuous functions on $\mathbb{R}^+$), a holomorphic C_0-semigroup and a generator (a closed and densely defined operator). We obtain that this generator is a well-bounded operator. A converse statement, that is, which operators (or which holomorphic semigroups) give rise to a Dirichlet family, is in general not known. Nevertheless, we prove that uniformly bounded C_0-groups and cosine functions in UMD spaces give rise to Dirichlet families. Some particular Dirichlet families to characterize scalar-type spectral operators in reflexive Banach spaces have been considered in [11].

In the fourth section we introduce Fejér families of operators. Dirichlet families allow to define Fejér families by integration, but it is not known whether every Fejér family is induced by some Dirichlet family. To a Fejér family we also associate an algebra homomorphism (for a subalgebra of absolutely continuous functions on $\mathbb{R}^+$), a holomorphic C_0-semigroup and an operator (generator), see Theorem 4.3. We obtain that the generator is AC^1-scalar, see the definition in [12]. Conversely, we show that certain holomorphic semigroups give rise to Fejér families. On the other hand, we show that uniformly bounded C_0-groups and cosine function determine

Fejér families of operators on an arbitrary Banach space. Note that some particular Fejér families have been introduced in [12], [13] and [17], where holomorphic C_0-semigroups with some growth properties are considered.

In the last section we give some concrete examples of uniformly bounded C_0-groups on non-UMD function spaces and we check whether they do or do not define Hilbert and Dirichlet families.

Finally, we would like to address some open problems arising in the context of this paper:

1. Which well-bounded operators come from Dirichlet families ? (The answer in reflexive spaces is given in Corollary 3.4. For the connection of well-bounded operators with Fejér families see the second remark after Theorem 4.4.)
2. Under what conditions is a Dirichlet family induced by a group ? What is the role of the UMD property of the underlying Banach space ? (For the connection between spectral families and groups see Theorem 2.4 and the paragraph below it. Every Dirichlet family is induced by a spectral family, see Theorem 3.2, but this may not be uniquely determined. Conversely, not every spectral family is induced by a Dirichlet one.)
3. When does a Lipschitz continuous Fejér family come from a Dirichlet family ? What is the role of the Radon-Nikodym property or the UMD property of the underlying Banach space for the question ? (See Theorem 4.2 and the remark following it.)
4. What is the precise connection between Fejér families and holomorphic semigroups ? (Up to a growth condition, they are equivalent, see Theorems 4.3 and 4.4.)

1. **Characterizations of UMD spaces**

The main aim of this section is to characterize UMD spaces in terms of the Dirichlet and Fejér operators, Theorem 1.1 and Theorem 1.4. As we have said in the Introduction, the boundedness of the Hilbert transform on $L^p(\mathbb{R};X)$ characterizes the UMD property of X ([8, 9]).

Take X a Banach space, $g \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R};X)$ with $1 \le p \le \infty$. Then $f * g \in L^p(\mathbb{R};X)$ where

$$f * g(t) := \int_{\mathbb{R}} g(s)f(t-s)ds, \qquad t \in \mathbb{R}.$$

Then it is clear that truncated Dirichlet operators, $(\mathcal{D}_N(s))_{s,N\ge 0}$ define linear and bounded operators in $L^p(\mathbb{R};X)$, where

$$\mathcal{D}_N(s)f(t) := \int_{-N}^{N} \frac{\sin(sr)}{r} f(t-r)\frac{dr}{\pi}, \qquad t \in \mathbb{R},$$

$s, N \ge 0$, $f \in L^p(\mathbb{R};X)$ and $1 \le p \le \infty$.

Theorem 1.1. *Let X be a Banach space. The following conditions are equivalent.*

(i) *The Hilbert transform H belongs to $\mathcal{B}(L^p(\mathbb{R};X))$ with $1 < p < \infty$.*

(ii) *The Dirichlet operators $(\mathcal{D}(s))_{s\ge 0}$ belong to $\mathcal{B}(L^p(\mathbb{R};X))$ with $1 < p < \infty$ and*

$$\sup_{s>0} \|\mathcal{D}(s)\| := C < \infty,$$

where $\mathcal{D}(s)f := \lim_{N\to+\infty} \mathcal{D}_N(s)f$.

(iii) *The Dirichlet operator* $\mathcal{D}(1)$ *belongs to* $\mathcal{B}(L^p(\mathbb{R};X))$ *with* $1<p<\infty$.

Proof. To show that (i) $\Rightarrow$ (ii) we use that

$$\mathcal{D}_N(s)f(t) = \frac{1}{2}\left(\lim_{\varepsilon\to 0}(e^{-its}H_{\varepsilon,N}(e^{is(\cdot)}f)(t) - e^{its}H_{\varepsilon,N}(e^{-is(\cdot)}f)(t)\right), \quad s,t\in\mathbb{R},$$

for $N>0$. Since there exists $\lim_{\varepsilon\to 0, N\to\infty} H_{\varepsilon,N}f$ for $f\in L^p(\mathbb{R};X)$ with $1<p<\infty$, we get $\mathcal{D}(s)\subset\mathcal{B}(L^p(\mathbb{R};X))$ for $s\geq 0$, $1<p<\infty$ and

$$\sup_{s>0}\|\mathcal{D}(s)\| \leq \|H\|.$$

(ii) $\Rightarrow$ (i) It is enough to show this in the case $p=2$. Given $f\in L^2(\mathbb{R};X)$ we define the operators $L_s : L^2(\mathbb{R};X)\to L^2(\mathbb{R};X)$ by

$$(L_s f)(t) := e^{its}(d_s * f e^{-is(\cdot)})(t) - e^{-its}(d_s * f e^{is(\cdot)})(t), \qquad t\in\mathbb{R}, s>0.$$

The inequality $\|L_s f\|_2 \leq 2C\|f\|_2$ holds for every $s>0$. It is straightforward to check that

$$(L_s f)(t) = (\frac{\sin^2(sr)}{r} * f)(t), \qquad t\in\mathbb{R}, s>0.$$

Note that

$$\widehat{\frac{\sin^2(sr)}{r}}(u) = \chi_{(-s,0)}(u) - \chi_{(0,s)}(u), \qquad u\in\mathbb{R},$$

where $\hat{f}$ is the Fourier transform of f and χ_A is the characteristic function of the set A.

Let $\mathcal{S}(\mathbb{R};X)$ be the Schwartz class of functions defined on $\mathbb{R}$ with values in the Banach space X. Take $f\in\mathcal{S}(\mathbb{R};X)$ such that $\mathrm{supp}(\hat{f})\subset(-s,s)$ with $s>0$. Then

$$\widehat{L_s f} = \hat{f}(\chi_{(-s,0)} - \chi_{(0,s)}) = \widehat{Hf}.$$

Then $L_s f = Hf$ and $\|Hf\|_2 \leq 2C\|f\|_2$ for $f\in\mathcal{S}$ such that $\hat{f}$ is of compact support. By density, the last inequality holds for $f\in L^2(\mathbb{R};X)$.

(iii) $\Rightarrow$ (ii) Take $s>0$ and note that

$$\mathcal{D}(s)g(t) = (\mathcal{D}(1)g_{s^{-1}})_s(t), \qquad t\in\mathbb{R},$$

where we follow the notation $g_s(t) := g(st)$ for $s>0$. As $\|g_{s^{-1}}\|_p = s^{\frac{1}{p}}\|g\|_p$, we get

$$\|\mathcal{D}(s)g\|_p = s^{\frac{-1}{p}}\|\mathcal{D}(1)g_{s^{-1}}\|_p \leq \|\mathcal{D}(1)\|\,\|g\|_p,$$

for $g\in L^p(\mathbb{R};X)$, $1<p<\infty$ and the proof is finished. ☐

The function f_s defined in the Introduction satisfies that $\hat{f}_s(\xi) = \chi_{(-s,s)}(\xi)(s-|\xi|)$ and

$$f_s * f_t = 2\int_0^s f_u du + (t-s)f_s, \qquad 0<s\leq t. \tag{1.1}$$

In the next lemma we show that the map $s\mapsto f_s$, $\mathbb{R}^+\to L^1(\mathbb{R})$, is not a Lipschitz function. Recall that a function $g:\mathbb{R}^+\to X$ is Lipschitz if there exists $M>0$ such that

$$\|g(t)-g(s)\| \leq M|t-s|, \qquad t,s\in\mathbb{R}^+.$$

Lemma 1.2. Let $(f_s)_{s\geq 0}$ be the functions defined in the Introduction.

(i) The inequality $\|f_{s+h}-f_s\|_1 \leq \frac{2}{\pi}h|\log(h)|(e+s+\frac{\pi}{2})$ holds for $h\in(0,\frac{1}{e})$.

(ii) Given $s \geq 0$ and $\varepsilon > 0$ there does not exist a $C > 0$ such that for any $h \in (0, \varepsilon)$ one has
$$\|f_{s+h} - f_s\|_1 \leq Ch.$$

Proof. (i) Take $s, h > 0$. Then
$$\begin{aligned} \|f_{s+h} - f_s\|_1 &= 2\int_0^\infty \frac{|\cos((s+h)t) - \cos(st)|}{t^2}\frac{dt}{\pi} \\ &\leq 2\int_0^\infty \frac{|\cos(st)(1-\cos(ht))|}{t^2}\frac{dt}{\pi} + 2\int_0^\infty |\frac{\sin(st)\sin(ht)}{t^2}|\frac{dt}{\pi}. \end{aligned}$$
Note that $2\int_0^\infty \frac{|\cos(st)(1-\cos(ht))|}{t^2}\frac{dt}{\pi} \leq h$ and
$$2\int_0^\infty \frac{|\sin(st)\sin(ht)|}{t^2}\frac{dt}{\pi} \leq \frac{2sh}{\pi} + \frac{2}{\pi}h^\alpha \int_1^\infty \frac{dt}{t^{2-\alpha}} = \frac{2sh}{\pi} + \frac{2}{\pi}\frac{h^\alpha}{1-\alpha},$$
where we have used that $\sin(x) \leq x^\beta$ for $x \geq 0$ and $\beta \in [0,1]$. Fix $h \in (0, e^{-1})$, the function $\alpha \mapsto \frac{h^\alpha}{1-\alpha}$ has a minimum $eh|\log(h)|$ at the point $\alpha = 1 + \frac{1}{\log(h)} \in (0,1)$. Then
$$\|f_{s+h} - f_s\|_1 \leq h + \frac{2sh}{\pi} + \frac{2}{\pi}eh|\log(h)| \leq \frac{2}{\pi}h|\log(h)|(e + s + \frac{\pi}{2}).$$

(ii) By the part (i), it is enough to prove that given $s \geq 0$ and $\varepsilon > 0$, there does not exist $C > 0$ such that
$$\int_0^\infty \frac{|\sin(st)\sin(ht)|}{t^2}dt \leq Ch,$$
for $h \in (0, \varepsilon)$. If we suppose the existence of such C, then
$$C \geq \int_0^\infty \frac{|\sin(\frac{s}{h}t)\sin(t)|}{t^2}dt \geq \int_0^1 \frac{|\sin(\frac{s}{h}t)\sin(t)|}{t^2}dt \geq \sin(1)\int_0^1 \frac{|\sin(\frac{s}{h}t)|}{t}dt,$$
for $h \in (0, \varepsilon)$. Then
$$\frac{C}{\sin(1)} \geq \int_0^{\frac{s}{h}} \frac{|\sin(u)|}{u}du,$$
for $h \in (0, \varepsilon)$ and we obtain a contradiction. □

Remark The trick used in the proof of the first part has appeared in [23, Lemma 4].

Now we consider the Fejér operators in $L^p(\mathbb{R}; X)$ defined by
$$\mathcal{F}(s)g := f_s * g, \quad g \in L^p(\mathbb{R}; X).$$
As in the scalar case, one can show the following properties for Fejér operators.

Proposition 1.3. *Let X be a Banach space, $(\mathcal{F}(s))_{s\geq 0}$ the Fejér operators and $1 \leq p < \infty$. Then*

(i) $\|\mathcal{F}(s)\| \leq \|f_s\|_1 = s$ *for* $s \geq 0$.
(ii) *the map* $s \to \mathcal{F}(s)$ *is continuous, as function from* $[0, \infty)$ *into* $\mathcal{B}(L^p(\mathbb{R}; X))$.
(iii) $\lim_{s\to\infty} \frac{1}{s}\mathcal{F}(s)(f) = f$ *for any* $f \in L^p(\mathbb{R}; X)$.

Theorem 1.4. *Let X be a Banach space and $1 < p < \infty$. The following condition is equivalent to the conditions (i), (ii) and (iii) in Theorem 1.1.*

(iv) *The map* $\mathcal{F} : s \in [0, \infty) \mapsto \mathcal{F}(s) \in \mathcal{B}(L^p(\mathbb{R}; X))$ *is a Lipschitz function.*
In this case,

$$\mathcal{F}(s)g = \int_0^s \mathcal{D}(u)g\,du, \quad g \in L^p(\mathbb{R}; X), \quad s \geq 0.$$

Proof. Since

$$\mathcal{F}(s)g(t) = f_s * g(t) = \int_0^s d_u * g(t)du = \int_0^s \mathcal{D}(u)g(t)du, \quad s \geq 0,$$

the proof of (ii)⇒ (iv) is obtained directly due to

$$\|\mathcal{F}(s)g - \mathcal{F}(t)g\|_p \leq \int_s^t \|\mathcal{D}(u)g\|_p du \leq \sup_{u>0} \|\mathcal{D}(u)\| \|g\|_p (t-s), \qquad 0 \leq s < t,$$

for $g \in L^p(\mathbb{R}; X)$ with $1 < p < \infty$. Then $\|\mathcal{F}(s) - \mathcal{F}(t)\| \leq C|s - t|$ for $0 \leq s < t$.

Now we prove the part (iv)⇒ (ii). Take $g \in L^p(\mathbb{R}; X)$ with $1 < p < \infty$ with compact support. Note that the function $\mathcal{F}(\cdot)g$ is differentiable, and

$$\frac{d}{ds}(\mathcal{F}(\cdot)g)(s) = \frac{d}{ds}(f_s * g) = d_s * g = \mathcal{D}(s)g.$$

Therefore

$$\|\mathcal{D}(s)g\|_p \leq \limsup_{h \to 0} \|\frac{\mathcal{F}(s+h)g - \mathcal{F}(s)g}{h}\|_p \leq M\|g\|_p.$$

By density, we obtain that $\|\mathcal{D}(s)\| \leq M$ for $s \geq 0$. □

2. **Hilbert families of operators**

In this section we introduce a Hilbert family of operators as a vector-valued function, $H : \mathbb{R} \to \mathcal{B}(X)$, which satisfies several conditions. The operator $H(0)$ will be called Hilbert transform, and in the case of a uniformly bounded C_0-group in a UMD space it coincides with the Hilbert transform introduced in [20]. We define a Hilbert transform for a uniformly bounded cosine function in a UMD space at the end of the section. In the main result of this section, Theorem 2.3, we prove that spectral families and Hilbert families are mutually equivalent.

Definition 2.1. A *Hilbert family* of operators on a Banach space X is a function $H : \mathbb{R} \to \mathcal{B}(X)$ which satisfies the following conditions:

(i) $\sup\{\|H(s)\| : s \in \mathbb{R}\} < \infty$;
(ii) $H(s)H(t) = H(t)H(s) = H(s) - H(t) + I$, for $s < t$;
(iii) the function $s \mapsto H(s)x$ has a left-hand limit (which we denote $H(s^-)x$) and a right-hand limit (which we denote $H(s^+)x$) at each point of $\mathbb{R}$ and $x \in X$;
(iv) $H(s)x = \dfrac{H(s^+)x + H(s^-)x}{2}$ for $s \in \mathbb{R}$ and $x \in X$;
(v) $H(s) \to -I$ (resp. $H(s) \to I$) in the strong operator topology as $s \to -\infty$ (resp. $s \to +\infty$).

The operator $H(0)$ will be called a Hilbert transform.

The identities in the next proposition will be used in the proof of Theorem 2.3. Compare the part (i) and [20, Proposition 5.3]. We omit the details of this technical proof which involves the conditions of the Definition 2.1.

Proposition 2.2. *Let* $H \equiv (H(s))_{s\in\mathbb{R}}$ *be a Hilbert family of operators on a Banach space* X*. Then the following identities hold:*

(i) $H^3(s) = H(s)$ *for* $s \in \mathbb{R}$*;*
(ii) $H(s)x - H(s^-)x = x - H^2(s)x = H(s^+)x - H(s)x$*, for* $s \in \mathbb{R}$ *and* $x \in X$*;*
(iii) $H^2(s^+)x = H^2(s^-)x = x$*, for* $s \in \mathbb{R}$ *and* $x \in X$*.*

Theorem 2.3. *Let* $E \equiv (E(\lambda))_{\lambda\in\mathbb{R}}$ *be a spectral family on a Banach space* X*. Then the family of operators* $H^E \equiv (H^E(s))_{s\in\mathbb{R}}$ *defined by*

$$H^E(s)x := E(s)x + E(s^-)x - x, \qquad x \in X, \tag{2.1}$$

is a Hilbert family of operators. In this case the following equalities hold:
(i) $(H^E)^2(s)x = E(s^-)x - E(s)x + x$*, for* $x \in X$ *and* $s \in \mathbb{R}$*;*
(ii) $E(s) = I + \frac{1}{2}\left(H^E(s) - (H^E)^2(s)\right)$ *, for* $s \in \mathbb{R}$*.*

Conversely, let $H \equiv (H(s)_{s\in\mathbb{R}}$ *be a Hilbert family of operators on* X*. Then the family of operators* $E^H \equiv (E^H(s))_{s\in\mathbb{R}}$ *defined by*

$$E^H(s) := I + \frac{1}{2}\left(H(s) - H^2(s)\right), \qquad s \in \mathbb{R}, \tag{2.2}$$

is a spectral family. In this case the following equalities hold:
(j) $H(s)x = E^H(s)x + E^H(s^-)x - x$*, for* $x \in X$ *and* $s \in \mathbb{R}$*;*
(jj) $H^2(s)x = E^H(s^-)x - E^H(s)x + x$*, for* $x \in X$ *and* $s \in \mathbb{R}$*.*

In conclusion, if E *is a spectral family and* H *a Hilbert family then* $E = E^{H^E}$ *and* $H = H^{E^H}$*.*

Proof. Let $E \equiv (E(s))_{s\in\mathbb{R}}$ be a spectral family on a Banach space X. It is direct to check that

$$H^E(s)H^E(t)x = H^E(s)(E(t)x + E(t^-)x - x) = H^E(s)x - H^E(t)x + x, \qquad x \in X,$$

for $s < t$. In a similar way it is shown that

$$H^E(t)H^E(s)x = H^E(s)x - H^E(t)x + x, \qquad x \in X,$$

and the part (i). Note that $H^E(s^-)x = 2E(s^-)x - x$ and $H^E(s^+)x = 2E(s)x - x$, therefore

$$H^E(s^+)x + H^E(s^-)x = 2(E(s)x + E(s^-)x - x) = 2H^E(s)x, \qquad s \in \mathbb{R},$$

for $x \in \mathbb{R}$. The conditions (i), (iii) and (v) of Definition 2.1 follow easily from the definition of spectral family and (2.1). We conclude that $(H^E(s))_{s\in\mathbb{R}}$ is a Hilbert family. The part (ii) is an easy consequence of (i) and (2.1).

Let $H \equiv (H(s))_{s\in\mathbb{R}}$ be a Hilbert family on a Banach space X. Take $s < t$, it is straightforward to check that

$$\begin{aligned} H^2(t)H(s) &= H(s) - H^2(t) + I, \\ H^2(s)H(t) &= H^2(s) + H(t) - I, \\ H^2(t)H^2(s) &= H^2(s) + H^2(t) - I. \end{aligned}$$

We apply these identities to show that $E^H(s)E^H(t) = E^H(s)$ for $s < t$. Using that $H^3(s) = H(s)$, we prove that $(E^H)^2(s) = E^H(s)$. The right continuity of the function $s \mapsto E(s)x$ follows from Proposition 2.2 (ii) and (iii),

$$\lim_{t\to s^+} E^H(t)x = \frac{x}{2} + \frac{1}{2}(x - H^2(s)x + H(s)x) = E^H(s)x$$

for $s \in \mathbb{R}$ and $x \in X$. We conclude that $E \equiv (E^H(s))_{s\in\mathbb{R}}$ is a spectral family.

Parts (j) and (jj) are proved from Proposition 2.2 (ii) and (iii). The identity $E = E^{H^E}$ follows from (i) and $H = H^{E^H}$ from (j). □

In the rest of this section we consider two cases when a Hilbert family may be defined. First we start with a uniformly bounded C_0-group , T, on a UMD Banach space. The Hilbert transform H_0^T and operators $(H^T(s))_{s\in\mathbb{R}}$ have been considered respectively in [20] and [4]. On the other hand, a spectral family E^T is proved to exist in [5].

In the second case we start with a uniformly bounded cosine function $C \equiv (C(t))_{t\in\mathbb{R}}$ on a UMD Banach space X. In [18] it is proved that there exists a uniformly bounded C_0-group $T \equiv (T(t))_{t\in\mathbb{R}}$ such that

$$C(t) = \frac{T(t) + T(-t)}{2}, \qquad t \in \mathbb{R},$$

and one can consider the Hilbert family $(H^T(s))_{s\in\mathbb{R}}$ and the spectral family $(E^T(t))_{t\in\mathbb{R}}$. On the other hand, we will directly define a Hilbert transform $H_0^C \in \mathcal{B}(X)$. Our aim is to find the relations between these families and the operator H_0^C, in particular to prove that $H_0^C = H_0^T$.

Let X be a Banach space. A C_0-semigroup is a family of linear and bounded operators $(T(t))_{t\geq 0}$ such that $T(t+s) = T(t)T(s)$, for any $t, s > 0$, $T(0) = I$, and $\lim_{t\to 0^+} T(t)x = x$ for any $x \in X$. If there exists $M > 0$ such that $\|T(t)\| \leq M$ for $t > 0$ then we say that $(T(t))_{t\geq 0}$ is uniformly bounded. The infinitesimal generator $(A, D(A))$ is a closed and densely defined operator such that

$$D(A) := \{x \in X \ : \text{there exists } \lim_{t\to 0^+} \frac{T(t)x - x}{t} \}, \quad Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

with $x \in D(A)$. In the case when A and $-A$ generate C_0-semigroups, $(T^+(t))_{t\geq 0}$ and $(T^-(t))_{t\geq 0}$, it is said that A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ given by

$$T(t) = \begin{cases} T^+(t), & \text{if } t \geq 0, \\ T^-(-t), & \text{if } t \leq 0. \end{cases}$$

The C_0-semigroup $(T(t))_{t\geq 0}$ is holomophic of angle $\frac{\pi}{2}$ if there exists a holomorphic extension $z \mapsto T(z)x$, $\{z \in \mathbb{C}\,; \Re z > 0\} \to X$ for $x \in X$, see for example [1].

The link between spectral families and C_0-semigroups have appeared in several papers, see for example [3], [5] and [7] for strongly continuous representations of compact abelian groups. The next result shows a nice decomposition of uniformly bounded C_0-groups in terms of spectral families in UMD spaces.

Theorem 2.4. [5, Théorème 4] *Let X be a UMD space and $(T(t))_{t\in\mathbb{R}}$ a uniformly bounded C_0-group on X. Then there exists a unique spectral family E on X such that*

$$T(t)x = \lim_{a\to\infty} \int_{-a}^{a} e^{itr} dE(r)x, \qquad t \in \mathbb{R}, \quad x \in X.$$

A converse result does not hold in general. For example the Laplace operator $-\Delta$ on $L^p(\mathbb{R})$, where $1 < p < \infty$, is well-bounded (i.e., there exists a "enough good" spectral family associated with it on $[0, \infty)$), see [13, Theorem 4.1 (a)] and also [3, p.457], but it is not a generator of a uniformly bounded C_0-group. It is an open problem to determine under what conditions is a spectral family induced by a uniformly bounded C_0-group.

As we have commented in the Introduction, the Hilbert transform associated with T, H_0^T, is a linear and bounded operator on a UMD space X, see [20, Proposition 5.2]. Moreover, the Hilbert transform for the perturbated group $(e^{-its}T(t))_{t\in\mathbb{R}}$ for $s\in\mathbb{R}$ has been also considered, see [4, Theorem 5.12, 5.16].

Corollary 2.5. *Let X be a UMD Banach space and $T\equiv(T(t))_{t\in\mathbb{R}}$ a uniformly bounded C_0-group on X. Then $(H^T(s))_{s\in\mathbb{R}}$ is a Hilbert family, where*

$$H^T(s)x := \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-its}T(t)x}{t}dt, \qquad x\in X,$$

and

$$H^T(s)x = E(s)x + E(s^-)x - x, \qquad x\in X, s\in\mathbb{R}, \tag{2.3}$$

where $(E(t))_{t\in\mathbb{R}}$ is the spectral family given in Theorem 2.4.

Proof. The equality (2.3) has been proved in [4, Theorem 5.12]. By Theorem 2.3, we obtain that $(H^T(s))_{s\in\mathbb{R}}$ is a Hilbert family. □

Let X be a Banach space. A family of operators $C\equiv(C(t))_{t\in\mathbb{R}}\subset\mathcal{B}(X)$ is called a cosine function on X if the map $t\mapsto C(t)x$ is strongly continuous, $C(0)=I$ and the equality $C(t+s)+C(t-s)=2C(t)C(s)$, holds for any $t,s\in\mathbb{R}$. We say that $C\equiv(C(t))_{t\in\mathbb{R}}$ is uniformly bounded if $\sup_{t\in\mathbb{R}}\|C(t)\|<\infty$. The generator of a cosine function C is defined as the unique operator A such that

$$\lambda(\lambda^2-A)^{-1}x=\int_0^{\infty}e^{-\lambda t}C(t)xdt, \qquad \lambda>0,$$

for $x\in X$. One has that $Ax = 2\lim_{t\to 0^+}\dfrac{C(t)x-x}{t^2}$ for $x\in D(A)$. We define the sine function $(S(t))_{t\geq 0}$ by

$$S(t)x := \int_0^t C(s)xds, \qquad x\in X,\ t\geq 0.$$

It is direct to show that $\int_0^t S(s)xds\in D(A)$ for $x\in X$ and $t\geq 0$ and

$$A\int_0^t S(s)xds = C(t)x-x, \qquad t\geq 0, x\in X, \tag{2.4}$$

see details in [1, Section 3.14].

It is well-known that if $(T(t))_{t\in\mathbb{R}}$ is a C_0-group generated by B then B^2 generates a cosine function given by

$$C(t)x := \frac{1}{2}\left(T(t)x+T(-t)x\right), \qquad t\in\mathbb{R},$$

for $x\in X$. The converse implication in UMD spaces was partially solved in [10] and completed in [18].

Theorem 2.6. [18, Theorem 1.1] *Let A generate a uniformly bounded cosine function $(C(t))_{t\in\mathbb{R}}$ on a UMD space. Then $i(-A)^{\frac{1}{2}}$ generates a uniformly bounded C_0-group $(T(t))_{t\in\mathbb{R}}$ such that*

$$C(t)=\frac{1}{2}\left(T(t)+T(-t)\right), \qquad t\in\mathbb{R}.$$

Due to this equivalence between C_0-groups and cosine functions in UMD spaces we obtain the following corollary.

Corollary 2.7. *Let X be a UMD space and $(C(t))_{t\in\mathbb{R}}$ a uniformly bounded cosine function on X. Then there exists a unique spectral family E on X such that*

$$C(t)x = \lim_{a\to\infty}\int_{-a}^{a}\cos(ts)dE(s)x, \qquad t\in\mathbb{R}, \quad x\in X.$$

Now we consider the associated sine function $(S(t))_{t\geq 0}$ in a UMD space X. Then $S(t)X\subset D(A^{\frac{1}{2}})$ and the map $t\mapsto A^{\frac{1}{2}}S(t)x$ is continuous for all $x\in X$, and

$$A^{\frac{1}{2}}S(t)x = \frac{2i}{\pi}\lim_{\varepsilon\to 0, T\to\infty}\int_{\varepsilon\leq|s|\leq T}\frac{C(s-t)x}{s}ds,$$

for $x\in X$, and $t\in\mathbb{R}$ see [10, Proposition 2.4, Corollary 2.6]. Moreover, the C_0-group $(T(t))_{t\in\mathbb{R}}$ in the Theorem 2.6 is given by

$$T(t)x = C(t)x + A^{\frac{1}{2}}S(t)x, \qquad x\in X,$$

where $S(-t) = -S(t)$ for $t\leq 0$ ([10]).

For any $\varepsilon>0$ and $N>1$, we define the operators

$$H^C_{\varepsilon,N}x := \frac{2i}{\pi}\int_{\varepsilon}^{N}\frac{A^{\frac{1}{2}}S(s)x}{s}ds, \qquad x\in X.$$

It is clear that $H^C_{\varepsilon,N}\in\mathcal{B}(X)$. In the next theorem we prove that $\lim_{\varepsilon\to 0,N\to\infty}H^C_{\varepsilon,N}x$ exists for $x\in X$ and defines the Hilbert transform associated with the uniformly bounded cosine function in a UMD space.

Theorem 2.8. *Let X be a UMD Banach space and $C\equiv(C(t))_{t\in\mathbb{R}}$ a uniformly bounded cosine function on X generated by A. Then $\lim_{\varepsilon\to 0,N\to\infty}H^C_{\varepsilon,N}x$ exists for $x\in X$ and defines a linear and bounded operator $H^C_0\in\mathcal{B}(X)$. Moreover*

(i) *Let $(E(t))_{t\in\mathbb{R}}$ be the spectral family given in Corollary 2.7. Then*

$$H^C_0 x = E(0)x + E(0^-)x - x, \qquad x\in X.$$

(ii) *If $x\in D(A)$, then*

$$H^C_0 A^{\frac{1}{2}}x = \frac{2i}{\pi}\int_0^{\infty}\frac{C(t)x-x}{t^2}dt.$$

Proof. As we said before, the group defined by

$$T(t) := C(t) + A^{\frac{1}{2}}S(t), \qquad t\in\mathbb{R},$$

is uniformly bounded. Note that the limit of

$$H^T_{\varepsilon,N}x := \frac{i}{\pi}\int_{\varepsilon\leq|s|\leq N}\frac{C(t)x + A^{\frac{1}{2}}S(t)x}{t}dt = \frac{2i}{\pi}\int_{\varepsilon}^{N}\frac{A^{\frac{1}{2}}S(t)x}{t}dt = H^C_{\varepsilon,N}x,$$

exists as $\varepsilon\to 0^+$ and $N\to\infty$ by [20, Proposition 5.2] and $H^C_0 = H^T_0$. The part (i) follows from the formula (2.3) for $s=0$.

Now we take $x\in D(A)$. Applying the equality (2.4) we obtain that

$$\begin{aligned} H^C_0 A^{\frac{1}{2}}x &= \lim_{\varepsilon,N}\frac{2i}{\pi}\int_{\varepsilon}^{N}\frac{AS(t)x}{t}dt = \lim_{\varepsilon,N}\frac{2i}{\pi}\left(\frac{C(N)x-x}{N} - \varepsilon\frac{C(\varepsilon)x-x}{\varepsilon^2}\right) \\ &\quad + \lim_{\varepsilon,N}\frac{2i}{\pi}\int_{\varepsilon}^{N}\frac{C(t)x-x}{t^2}dt = \frac{2i}{\pi}\int_0^{\infty}\frac{C(t)x-x}{t^2}dt. \end{aligned}$$

□

Remarks. The part (ii) of Theorem 2.8 remains true in a general Banach space X provided there exists a C_0-group $T \equiv (T(t))_{t\in\mathbb{R}}$ such that H_0^T is a linear and bounded operator and

$$C(t) = \frac{T(t) + T(-t)}{2}, \qquad t \in \mathbb{R}.$$

This was proved in [23].

Given a uniformly bounded cosine function in a UMD space X, we may also define a Hilbert family, $(H^C(s))_{s\in\mathbb{R}}$ by

$$H^C(s)x := \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{-its}(C(t)x + A^{\frac{1}{2}}S(t)x)}{t}dt, \qquad x \in X.$$

3. Dirichlet families of operators

We introduce Dirichlet families of operators as a generalization of the convolution Dirichlet operators in the space $L^p(\mathbb{R}; X)$ which were considered in Section 1. Having a Dirichlet family we define a functional calculus for absolutely continuous functions on $\mathbb{R}^+$. Uniformly bounded C_0-groups and cosine functions in UMD spaces allow us to construct examples of Dirichlet families.

Definition 3.1. A *Dirichlet family* of operators on a Banach space X is a vector-valued function $D : [0, \infty) \to \mathcal{B}(X)$ which satisfies the following conditions:

(i) $\sup\{\|D(s)\| : s \in [0, \infty)\} < \infty$;
(ii) $D(s)D(t) = D(t)D(s) = D(s)$, for $0 \le s < t$;
(iii) $D(\cdot)x$ has a left-hand limit, $D(s^-)x$, at each point $s \in (0, \infty)$ and a right-hand limit, $D(s^+)x$ at each point $s \in [0, \infty)$ and $x \in X$;
(iv) $D(s)x = \dfrac{D(s^+)x + D(s^-)x}{2}$ for $s \in (0, \infty)$ and $x \in X$;
(v) $D(0) = 0$ and $D(s) \to I$ in the strong operator topology as $s \to \infty$.

Theorem 3.2. *Let $E \equiv (E(s))_{s\in\mathbb{R}}$ be a spectral family in a Banach space X. Then the family of operators $D^E \equiv (D^E(s))_{s\ge 0}$ defined by*

$$D^E(s)x := \frac{1}{2}\big(E(s)x + E(s^-)x - E(-s)x - E(-s^-)x\big)$$

is a Dirichlet family.

Conversely, let $D \equiv (D(s))_{s\ge 0}$ be a Dirichlet family of operators in a Banach space X. Then the family of operators $E^D \equiv (E^D(s))_{s\in\mathbb{R}}$ defined by

$$E^D(s) := D(s^+), \quad s > 0; \quad E(s) = 0, \quad s \le 0,$$

is a spectral family. Note that $D^{(E^D)} = D$.

Let now $H \equiv (H(s))_{s\in\mathbb{R}}$ be a Hilbert family of operators in a Banach space X. Then the family of operators $(D^H(s))_{s\ge 0}$ defined by

$$D^H(s) := \frac{1}{2}\left(H(s) - H(-s)\right), \qquad s \ge 0,$$

is a Dirichlet family.

Proof. It is straightforward to check that D^E satisfies the conditions of Definition 3.1, that E^D is a spectral family, and that $D^{(E^D)} = D$. For a Hilbert family H we give the details. Take $0 \le s < t$ and compute

$$D^H(s)D^H(t)x = \frac{1}{4}\left(H(s)H(t)x - H(-s)H(t)x - H(s)H(-t)x + H(-s)H(-t)x\right)$$

$$=\frac{1}{2}(H(s)x - H(-s)x) = D^H(s)x,$$

for $x \in X$. In the same way we calculate $D^H(t)D^H(s)$. To check the part (iv) of Definition 3.1, note that $D^H(s^-)x = \frac{1}{2}(H(s^-)x - H(-s^+)x$, and then

$$D^H(s)x = \frac{H(s^-)x + H(s^+)x - H(-s^+)x - H(-s^-)x}{2} = \frac{D^H(s^-)x + D^H(s^+)x}{2},$$

for $s > 0$ and $x \in X$. □

Remark. It is clear that in general $E^{(D^E)} \neq E$: for example, take a spectral family such that $E(s) \neq 0$ for $s < 0$.

An integration theory (for complex-valued functions of bounded variation on $[0, \infty)$ or equivalently for bounded Borel measures on $[0, \infty)$) is available for Dirichlet families. This theory follows the same lines as the integration theory for spectral families using Riemann-Stieltjes sums and allows us to define a functional calculus Φ_D given by

$$\Phi_D(\mu)x := \oint_{[0,\infty)} D(t)x d\mu(t), \qquad x \in X,$$

for μ bounded Borel measures on $[0, \infty)$. Note that we may recover the Dirichlet family from the above calculus. To treat this in detail would take us too far of our aims in this paper, hence we refer the interested reader to [16, Chapter XV] and [15, Chapters 16 and 17].

In the following we consider the Banach subalgebra (of bounded variation functions on $[0, \infty)$) $AC(\mathbb{R}^+)$ consisting of absolutely continuous functions on $\mathbb{R}^+$ such that $\lim_{t\to\infty} f(t) = 0$ with the norm

$$\|f\|_{AC} := \int_0^\infty |f'(s)|ds.$$

Theorem 3.3. *Let $D \equiv (D(s))_{s\geq 0}$ be a Dirichlet family on a Banach space X. Define $\Phi_D : AC(\mathbb{R}^+) \to \mathcal{B}(X)$ by*

$$\Phi_D(f)x := -\int_0^\infty f'(s)D(s)xds, \qquad f \in AC(\mathbb{R}^+),$$

for $x \in X$. Then

(i) *Φ_D is an algebra homomorphism of $AC(\mathbb{R}^+)$ into $\mathcal{B}(X)$ and*
$$\|\Phi_D(f)\| \leq \|f\|_{AC} \sup_{s\geq 0} \|D(s)\|, \qquad f \in AC(\mathbb{R}^+).$$
(ii) *There exist $M > 0$ and a holomorphic C_0-semigroup $(T_D(z))_{\Re z>0}$ such that*
$$T_D(z)x = z\int_0^\infty e^{-zs}D(s)xds, \qquad x \in X,$$
and $\|T_D(z)\| \leq M\frac{|z|}{\Re z}$ for $\Re z > 0$.
(iii) *There exists a closed and densely defined operator A on X with $\sigma(A) \subset [0, \infty)$ such that*
$$(\lambda + A)^{-1}x = \int_0^\infty \frac{D(s)x}{(\lambda + s)^2}ds, \qquad x \in X, \qquad \lambda \in \mathbb{C}\backslash(-\infty, 0].$$
We say that A is the generator of D.

Proof. It is straightforward to check that Φ_D is a linear and bounded operator and

$$\begin{aligned}\Phi_D(f)\Phi_D(g)x &= \int_0^\infty \int_0^\infty f'(t)g'(s)D(t)D(s)xdsdt \\ &= \int_0^\infty \int_0^t f'(t)g'(s)D(s)xdsdt + \int_0^\infty \int_t^\infty f'(t)g'(s)D(t)xdsdt \\ &= \int_0^\infty g'(s)D(s)x \int_s^\infty f'(t)dtds + \int_0^\infty f'(t)D(t)x \int_t^\infty g'(s)dsdt \\ &= -\int_0^\infty (f(t)g'(t) + g(t)f'(t))D(t)xdt = \Phi_D(fg)x.\end{aligned}$$

The functions $(e_z)_{\Re z>0}$ belong to $AC(\mathbb{R}^+)$, where $e_z(t) := e^{-zt}$ with $t \geq 0$ and $\Re z > 0$. Then $(\Phi_D(e_z))_{\Re z>0}$ is a holomorphic semigroup on X such that

$$\|\Phi_D(e_z)\| \leq M \int_0^\infty |ze^{-zt}|dt = M\frac{|z|}{\Re z}, \quad \Re z > 0, \tag{3.1}$$

where $M := \sup_{s\geq 0} \|D(s)\|$. Now we check that $(\Phi_D(e_t))_{t>0}$ is a C_0-semigroup. Take $\varepsilon > 0$. Since $D(s)x \to x$ as $s \to \infty$, there exists $N > 0$ such that $\|D(s)x - x\| \leq \varepsilon$ for any $s \geq N$. Take $0 < t < \frac{\varepsilon}{N}$, then

$$\begin{aligned}\|\Phi_D(e_t)x - x\| &\leq \int_0^N te^{-ts}\|D(s)x - x\|ds + \int_N^\infty te^{-ts}\|D(s)x - x\|ds \\ &\leq (M+1)\|x\| \int_0^N te^{-ts}ds + \varepsilon \int_N^\infty te^{-ts}ds \\ &\leq (M+1)\|x\|(1 - e^{-tN}) + \varepsilon e^{-tN} \leq ((M+1)\|x\| + 1)\varepsilon,\end{aligned}$$

where we use the inequality $1 - e^{-y} \leq y$ for $y \geq 0$. In conclusion, $(T_D(z))_{\Re z>0} := (\Phi_D(e_z))_{\Re z>0}$ is a holomorphic C_0-semigroup on X. We denote by $-A$ the infinitesimal generator of $(\Phi_D(e_z))_{\Re z>0}$ and $\sigma(A) \subset [0, \infty)$ by (3.1). We apply the Fubini theorem to obtain that

$$(\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda t}\Phi_D(e_t)xdt = \int_0^\infty \frac{D(s)x}{(\lambda + s)^2}ds$$

for $x \in X$ and $\Re\lambda > 0$. By holomorphy, we obtain the equality for $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. □

The algebra homomorphism Φ_D in the above theorem allows an extension $\tilde{\Phi}_D$ to functions f such that there exists $\lim_{s\to\infty} f(s)$ and $f - \lim_{s\to\infty} f(s) \in AC(\mathbb{R}^+)$ by

$$\tilde{\Phi}_D(f) := \Phi_D(f - \lim_{s\to\infty} f(s)) + \lim_{s\to\infty} f(s)I.$$

This extended functional calculus characterizes well-bounded operators, see [13, Definition 2.1]. In [13, Theorem 2.4] there are given several characterizations of well-bounded operators A with $\sigma(A) \subset [0, \infty)$. In particular, a densely defined operator A is a well-bounded operator on $[0, \infty)$ if and only if $(-\infty, 0) \subset \rho(A)$ and there exists a decomposition of the identity E for X on $[0, \infty)$ (note that $E : [0, \infty) \to \mathcal{B}(X^*)$ where X^* is the dual space of X, see [13, Definition 2.2]) such that

$$\phi((r + A)^{-1}x) = \int_0^\infty \frac{(E(s)\phi)x}{(r + s)^2}ds, \qquad x \in X, \phi \in X^*,$$

for $r > 0$. In the case when X is a reflexive space we can add the following characterization to [13, Theorem 2.4].

Corollary 3.4. *Let X be a reflexive Banach space. An operator A is well-bounded on $[0,\infty)$ if and only if A is the generator of a Dirichlet family, i.e. if and only if there exists a Dirichlet family $(D(s))_{s>0} \subset \mathcal{B}(X)$ such that*

$$(r+A)^{-1}x = \int_0^\infty \frac{D(s)x}{(r+s)^2}ds, \qquad x \in X,$$

for $r>0$.

Let X be a Banach space and $T \equiv (T(t))_{t\in\mathbb{R}}$ a uniformly bounded C_0-group on X. Now we consider $(D_N^T(s))_{N>1,s\geq 0} \subset \mathcal{B}(X)$ defined by

$$D_N^T(s)x := \int_{-N}^{N} \frac{\sin(sr)}{r} T(r)x\frac{dr}{\pi}, \qquad x \in X.$$

Proposition 3.5. *Let X be a UMD Banach space and $T \equiv (T(t))_{t\in\mathbb{R}}$ a uniformly bounded C_0-group on X generated by iA. Then $(D^T(s))_{s\geq 0}$ is a Dirichlet family, where*

$$D^T(s)x := \lim_{N\to\infty} D_N^T(s)x, \qquad x \in X, s\geq 0,$$

whose generator is $(A^2)^{\frac{1}{2}}$. Moreover let $(E(t))_{t\in\mathbb{R}}$ be the spectral family given in Theorem 2.4. Then

$$D^T(s)x = \frac{1}{2}(E(s)x + E(s^-)x - E(-s)x - E(-s^-)x),$$

for $x\in X$ and $s \geq 0$.

Proof. It is direct to show that

$$D^T(s) = \frac{1}{2}(H^T(s) - H^T(-s)),$$

where $(H^T(s))_{s\in\mathbb{R}}$ is the Hilbert family given in Corollary 2.5. We apply Theorem 3.2 to conclude that $(D^T(s))_{s\geq 0}$ is a Dirichlet family. Let B be the generator of $(D^T(s))_{s\geq 0}$. By Theorem 3.3 (ii), the operator $-B$ is the infinitesimal generator of the holomorphic C_0-group $(T_{-B}(z))_{\Re z>0}$, where

$$\begin{aligned} T_{-B}(z)x &= \int_0^\infty ze^{-tz}D^T(t)xdt = \lim_{N\to\infty}\int_0^\infty ze^{-tz}\int_{-N}^{N}\frac{\sin(tr)}{r}T(r)x\frac{dr}{\pi}dt \\ &= \lim_{N\to\infty}\int_{-N}^{N}\frac{zT(r)x}{r}\int_0^\infty e^{-tz}\sin(tr)dt\frac{dr}{\pi} = \int_{-\infty}^{\infty}\frac{z}{z^2+r^2}T(r)x\frac{dr}{\pi}, \end{aligned}$$

therefore $B = (A^2)^{\frac{1}{2}}$ (see similar case for the Laplacian in [1, p.174]). $\square$

Remark. If X is a UMD space and $T \equiv (T(t))_{t\in\mathbb{R}}$ is the shift group on $L^p(\mathbb{R};X)$, then $D^T(s) = \mathcal{D}(s)$ for $s\geq 0$, where $\mathcal{D}(s)$ is the Dirichlet operator considered in Section 1.

Now let X be a Banach space and $C \equiv (C(t))_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ a cosine function on X. For any $N>0$, we define $(D_N^C(s))_{N>1,s\geq 0} \subset \mathcal{B}(X)$ by

$$D_N^C(s)x := 2\int_0^N \frac{\sin(sr)}{r}C(r)x\frac{dr}{\pi}, \qquad x \in X.$$

In a UMD Banach space X,

$$D_N^C(s)x = \int_{-N}^{N}\frac{\sin(sr)}{r}T(r)x\frac{dr}{\pi} = D_N^T(s)x, \qquad x\in X,$$

for $s \geq 0$, where the C_0-group $T \equiv (T(t))_{t\in\mathbb{R}}$ is given in Theorem 2.6. The next result is an easy consequence of Proposition 3.5.

Corollary 3.6. *Let X be a UMD Banach space and $C \equiv (C(t))_{t\in\mathbb{R}}$ a uniformly bounded cosine function on X generated by A. Then $(D^C(s))_{s\geq 0}$ is a Dirichlet family, where*

$$D^C(s)x := \lim_{N\to\infty} D_N^C(s)x, \qquad x \in X, s \geq 0,$$

whose generator is $(-A)^{\frac{1}{2}}$. Moreover let $(E(t))_{t\in\mathbb{R}}$ be the spectral family given in Corollary 2.7. Then

$$D^C(s)x = \frac{1}{2}(E(s)x + E(s^-)x - E(-s)x - E(-s^-)x),$$

for $x \in X$ and $s \geq 0$.

4. **Fejér families of operators**

In this section we introduce a Fejér family of operators in an abstract setting. Every Dirichlet family defines a Fejér family by integration (Theorem 4.2). A Fejér family determines an algebra homomorphism from a subalgebra of absolutely continuous functions, see Theorem 4.3. Uniformly bounded C_0-groups, cosine functions and certain holomorphic C_0-semigroups allow us to construct examples of Fejér families.

Definition 4.1. A *Fejér family* of operators on a Banach space X is a function $F : [0,\infty) \to \mathcal{B}(X)$ which satisfies the following conditions:

(i) the map $t \mapsto F(t)x$ is continuous in any $t \geq 0$ and $x \in X$;
(ii) $F(t)F(s)x = F(s)F(t)x = 2\int_0^s F(u)xdu + (t-s)F(s)x$, for $t \geq s \geq 0$ and $x \in X$;
(iii) there exists $C > 0$ such that $\|F(t)\| \leq Ct$ for any $t \geq 0$;
(iv) $F(0) = 0$ and $\lim_{t\to\infty} \frac{1}{t}F(t)x = x$ for $x \in X$.

Theorem 4.2. *Let $D \equiv (D(s))_{s\geq 0}$ be a Dirichlet family on a Banach space X. Then $F \equiv (F(s))_{s\geq 0}$ is a Fejér family, where*

$$F(s)x := \int_0^s D(u)xdu, \qquad s \geq 0,$$

for $x \in X$. Moreover, there exists $C > 0$ such that $\|F(s) - F(t)\| \leq C|s-t|$ for $s, t \geq 0$.

Proof. We check the condition (ii) of Definition 4.1. Take $0 < s \leq t$ and $x \in X$, then

$$\begin{aligned} F(s)F(t)x &= \int_0^s D(r)\int_0^r D(u)xdudr + \int_0^s D(r)\int_r^t D(u)xdudr \\ &= \int_0^s \int_0^r D(u)xdudr + \int_0^s (t-r)D(r)xdr \\ &= \int_0^s F(r)xdr + \int_0^s F(r)xdr + (t-s)F(s)x = 2\int_0^s F(r)xdr + (t-s)F(s)x. \end{aligned}$$

Similarly one computes that $F(t)F(s)x = 2\int_0^s F(r)xdr + (t-s)F(s)x$ for $x \in X$. It is clear that $F(0) = 0$ and now we prove that $\lim_{t\to\infty} \frac{1}{t}F(t)x = x$. Take $\varepsilon > 0$.

Since $\lim_{t\to\infty} D(t)x = x$, there exists $N > 0$ such that $\|D(t)x - x\| < \varepsilon$ for $t \geq N$. Now take $t > \frac{N}{\varepsilon}$, and we obtain that

$$\begin{aligned}
&\|\frac{1}{t}F(t)x - x\| = \|\frac{1}{t}\int_0^t (D(s)x - x)ds\| \\
&\leq \|\frac{1}{t}\int_0^N (D(s)x - x)ds\| + \|\frac{1}{t}\int_N^t (D(s)x - x)ds\| \\
&\leq \frac{N}{t}(\sup_{s>0}\|D(s)\| + 1) + \varepsilon\frac{t-N}{t} \leq \varepsilon(\sup_{s>0}\|D(s)\| + 2).
\end{aligned}$$

Note that $\|F(t) - F(s)\| \leq \sup_{r>0}\|D(r)\|\,|t - s|$, for $t, s \geq 0$. □

Remark. It is clear that there exist Fejér families which do not arise by integration from Dirichlet families, for example, consider the Fejér family $(\mathcal{F}(s))_{s\geq 0} \subset \mathcal{B}(L^1(\mathbb{R}; X))$ defined in Section 1. Now we consider a Lipschitz continuous Fejér family $(F(s))_{s\geq 0} \subset \mathcal{B}(X)$ with X a Banach space with the Radon-Nikodym property, see definition of this geometric property in [1, Definition 1.2.5]. Under this condition we may conclude that there exists a projection-valued family $(P(t)) \subset \mathcal{B}(X)$ defined for a.e. $t \geq 0$ such that $F(s)x = \int_0^s P(t)xdt$, for $x \in X$ and $s \geq 0$. It remains open to conclude that, in fact, we may find a Dirichlet family $(D(t))_{t\geq 0} \subset \mathcal{B}(X)$ such that $F(s)x = \int_0^s D(t)xdt$, for $x \in X$ and $s \geq 0$.

We may define an algebra homomorphism associated with a Fejér family. To do this, we consider the Banach algebra, with the pointwise product, $AC^{(1)}(\mathbb{R}^+)$ of functions $f : [0, \infty) \to \mathbb{C}$ with absolutely continuous first derivative f' such that

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} tf'(t) = 0,$$

with the norm

$$\|f\|_{AC^{(1)}} := \int_0^\infty |f''(t)|tdt < \infty.$$

It is direct to prove that $AC^{(1)}(\mathbb{R}^+) \hookrightarrow AC(\mathbb{R}^+)$. Note that the algebra $AC^{(1)}(\mathbb{R}^+)$ belongs to the class of algebras $AC^{(\nu)}$ with $\nu = 2$ which was considered in [17].

Theorem 4.3. *Let $F \equiv (F(s))_{s\geq 0}$ be a Fejér family on a Banach space X. Define $\Phi_F : AC^{(1)}(\mathbb{R}^+) \to \mathcal{B}(X)$ by*

$$\Phi_F(f)x := \int_0^\infty f''(s)F(s)xds, \qquad f \in AC^{(1)}(\mathbb{R}^+),$$

for $x \in X$. Then

(i) *The map Φ_F is an algebra homomorphism of $AC^{(1)}(\mathbb{R}^+)$ into $\mathcal{B}(X)$ and*

$$\|\Phi_F(f)\| \leq \|f\|_{AC^{(1)}} \sup_{s>0}(\frac{1}{s}\|F(s)\|), \qquad f \in AC^{(1)}(\mathbb{R}^+).$$

(ii) *There exist $C > 0$ and a holomorphic C_0-semigroup $(T_F(z))_{\Re z>0} \subset \mathcal{B}(X)$ such that*

$$T_F(z)x = z^2\int_0^\infty e^{-zs}F(s)xds, \qquad x \in X,$$

and $\|T_F(z)\| \leq C\left(\frac{|z|}{\Re z}\right)^2$ for $\Re z > 0$.

(iii) *There exists a closed and densely defined operator A on X with $\sigma(A) \subset [0,\infty)$ such that*

$$(\lambda + A)^{-1}x = 2\int_0^\infty \frac{F(s)x}{(\lambda+s)^3}ds, \qquad x \in X, \qquad \lambda \in \mathbb{C}\backslash(-\infty,0],$$

We say that A is the generator of F.

Proof. To check that Φ_F is an algebra homomorphism, we use the condition (ii) of Definition 4.1 and the Fubini theorem,

$$\Phi_F(f)\Phi_F(g)x = \int_0^\infty \int_0^\infty f''(t)g''(s)F(t)F(s)xdsdt$$
$$= \int_0^\infty f''(t)\int_0^t g''(s)F(t)F(s)xdsdt + \int_0^\infty f''(t)\int_t^\infty g''(s)F(t)F(s)xdsdt$$
$$= 2\int_0^\infty f''(t)\int_0^t g''(s)\int_0^s F(u)xdudsdt + \int_0^\infty f''(t)\int_0^t g''(s)(t-s)F(s)xdsdt$$
$$+ 2\int_0^\infty f''(t)\int_t^\infty g''(s)\int_0^t F(u)xdudsdt + \int_0^\infty f''(t)\int_t^\infty g''(s)(s-t)F(t)xdsdt$$
$$= 2\int_0^\infty g''(s)\int_s^\infty f''(t)\int_0^s F(u)xdudtds + \int_0^\infty g''(s)\int_s^\infty f''(t)(t-s)F(s)xdtds$$
$$- 2\int_0^\infty F(u)x\int_u^\infty f''(t)g'(t)dtdu + \int_0^\infty f''(t)g(t)F(t)xdt$$
$$= -2\int_0^\infty F(u)x\int_u^\infty (g''(s)f'(s) + f''(s)g'(s))dsdu$$
$$+ \int_0^\infty g''(t)f(t)F(t)xdt + \int_0^\infty f''(t)g(t)F(t)xdt = \int_0^\infty (fg)''(t)F(t)xdt,$$

for $x \in X$.

We take again the functions $(e_z)_{\Re z>0}$ which were used in the proof of Theorem 3.3. The functions $(e_z)_{\Re z>0}$ belong to $AC^{(1)}(\mathbb{R}^+)$ and then $(\Phi_F(e_z))_{\Re z>0}$ is a holomorphic semigroup on X such that

$$\|\Phi_F(e_z)\| \le C\int_0^\infty |z^2te^{-zt}|dt = C\left(\frac{|z|}{\Re z}\right)^2, \quad \Re z > 0, \tag{4.1}$$

where $C := \sup_{s>0}(\frac{1}{s}\|F(s)\|)$. In a similar way as in Theorem 3.3 (ii), we can check that $(\Phi_F(e_t))_{t>0}$ is a C_0-semigroup. In this case, we omit the details. In conclusion, $(T_F(z))_{\Re z>0} := (\Phi_F(e_z))_{\Re z>0}$ is a holomorphic C_0-semigroup on X and

$$\Phi_F(e_z)x = z^2\int_0^\infty e^{-zs}F(s)xds, \qquad x \in X.$$

We denote by $-A$ the infinitesimal generator of $(\Phi_F(e_z))_{\Re z>0}$ and $\sigma(A) \subset [0,\infty)$ by (4.1). Note that

$$(\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda t}\Phi_F(e_t)xdt = 2\int_0^\infty \frac{F(s)x}{(\lambda+s)^3}ds,$$

for $x \in X$, $\Re\lambda > 0$ and then for $\lambda \in \mathbb{C}\backslash(-\infty,0]$. □

Remark. In the case when D is a Dirichlet family generated by A on a Banach space X, the associated Fejér family F, given in Theorem 4.2, is also generated by A.

In the rest of this section we define Fejér families from certain families of operators on arbitrary Banach spaces. The following result is a partial converse to Theorem 4.3. Note that the assumption (4.2) below with $\alpha \in [0,1)$ implies (4.2) with $\alpha = 1$ (i.e. the property in part (ii) of Theorem 3.3), and this implies (4.2) with $\alpha = 2$ (i.e. the property in part (ii) of Theorem 4.3).

Theorem 4.4. *Let $(T(z))_{\Re z>0}$ be a holomorphic C_0-semigroup generated by $-A$ on a Banach space X such that*

$$\|T(z)\| \le C\left(\frac{|z|}{\Re z}\right)^{\alpha}, \qquad \Re z > 0, \tag{4.2}$$

with $0 \le \alpha < 1$. Then $(F^T(s))_{s\ge 0}$ is a Fejér family on X and A is its generator, where

$$F^T(s)x := \frac{1}{2\pi i}\int_{\Re z=1} \frac{T(z)x}{z^2} e^{sz}dz, \qquad x \in X,$$

and $T_{F^T} = T$ where T_{F^T} is given in Theorem 4.3 (ii).

Proof. Changing the path of integration to $\Re z = \frac{1}{s}$, we obtain that $\|F^T(s)\| \le C_1 s$ for $s \ge 0$ as in [17, Lemma 6.1]. Take $0 \le s \le t$, $u \in \mathbb{C}$ with $\Re u = 2$, the residue theorem implies that

$$\frac{1}{2\pi i}\int_{\Re z=1} \frac{e^{z(s-t)}}{z^2(z-u)^2}dz = \frac{(t-s)e^{(s-t)u}}{u^2} + 2\frac{e^{(s-t)u}}{u^3},$$

and

$$\frac{1}{2\pi i}\int_{\Re z=1} \frac{T(z)x}{z^3}dz = 0, \qquad x \in X,$$

where in the second integral we have changed the path of integration to $\Re z = R$ and let $R \to \infty$. By the Fubini theorem and the above equalities, we obtain that

$$\begin{aligned} F^T(s)F^T(t)x &= \frac{1}{2\pi i}\int_{\Re u=2} T(u)xe^{ut}\frac{1}{2\pi i}\int_{\Re z=1}\frac{e^{z(s-t)}}{z^2(z-u)^2}dzdu \\ &= \frac{(t-s)}{2\pi i}\int_{\Re u=2}\frac{T(u)xe^{us}}{u^2}du + \frac{2}{2\pi i}\int_{\Re u=2}\frac{T(u)xe^{us}}{u^3}du \\ &= (t-s)F^T(s)x + 2\int_0^s F^T(r)xdr \end{aligned}$$

for $0 \le s \le t$. In a same way we compute $F^T(s)F^T(t)$ for $0 \le s \le t$. To check that $\frac{1}{s}F^T(s)x \to x$ as $s \to \infty$, note that

$$\frac{1}{s}F^T(s)x - x = \frac{1}{2\pi i s}\int_{\Re z=\frac{1}{s}} e^{zs}\frac{T(z)x - x}{z^2}dz$$

for $s \ge 0$ and

$$\begin{aligned} \|\frac{1}{s}F^T(s)x - x\| &\le \frac{e}{2\pi s}\int_{-\infty}^{\infty}\frac{\|T(1+it)x - x\|}{|1+it|^2}dt \\ &\le \frac{C_2}{s}\int_{-\infty}^{\infty}\frac{1}{|1+it|^{2-\alpha}} + \frac{1}{|1+it|^2}dt\,\|x\|, \end{aligned}$$

where C_2 is a constant independent of s. Note that

$$\begin{aligned}\Phi_{F^T}(e_z)x &= \frac{z^2}{2\pi i}\int_{\Re u=1}\frac{T(u)x}{u^2}\int_0^\infty e^{-s(z-u)}dsdu \\ &= \frac{z^2}{2\pi i}\int_{\Re u=1}\frac{T(u)x}{u^2(z-u)}du = T(z)x,\end{aligned}$$

where we apply the vector-valued Cauchy theorem for $\Re z > 1$ and $x \in X$. By holomorphic extension, we conclude that $\Phi_{F^T}(e_z) = T(z)$ for $\Re z > 0$. Since $-A$ is the generator of $(T(z))_{\Re z>0}$, we conclude that A is the generator of $(F^T(s))_{s\geq 0}$ and $T_{F^T} = T$. □

Remark. Taking into account Theorem 4.3 and Theorem 4.4, we may conclude that certain holomorphic C_0-semigroups and Fejér families are equivalent concepts except for a growth condition. Moreover, if we take a Fejér family such that the C_0-semigroup $(T_F(z))_{\Re z>0}$ given in Theorem 4.3 (ii) satisfies (4.2) with some $0 \leq \alpha < 1$, then $F^{T_F} = F$.

Fejér families F^T defined from certain holomorphic C_0-semigroups have been considered in [17, Theorem 4.1], where they are called Bochner-Riesz kernel operators of first order, written by G^1, and used to define a functional calculus associated with the infinitesimal generator of the holomorphic semigroup.

Remark. Using [13, Theorem 3.1], well-bounded operators on $[0,\infty)$ may be characterized in terms of the existence of Lipschitz continuous Fejér families. In a precise way, suppose that A is densely defined. Then A is well-bounded on $[0,\infty)$ if and only if there exists a Lipschitz continuous Fejér family $(F(s))_{s>0}$ such that

$$(\lambda + A)^{-1}x = 2\int_0^\infty \frac{F(s)x}{(\lambda+s)^3}ds, \qquad x\in X, \qquad \lambda\in\mathbb{C}\backslash(-\infty,0],$$

for $x \in X$. Compare with Corollary 3.4 and the paragraph in front of it.

Corollary 4.5. *Let $C \equiv (C(t))_{t\in\mathbb{R}}$ be a uniformly bounded cosine function generated by A on a Banach space X. Then the family of operators $(F^C(s))_{s\geq 0}$ defined by*

$$F^C(s)x := 2\int_0^\infty \frac{1-\cos(st)}{t^2}C(t)x\frac{dt}{\pi}, \qquad x\in X,$$

is a Fejér family whose generator is $(-A)^{\frac{1}{2}}$.

Let $T \equiv (T(t))_{t\in\mathbb{R}}$ be a uniformly bounded C_0-group generated by iA on Banach space X. Then the family of operators $(F^T(s))_{s\geq 0}$ defined by

$$F^T(s)x := \int_{-\infty}^\infty \frac{1-\cos(st)}{t^2}T(t)x\frac{dt}{\pi}, \qquad x\in X, \tag{4.3}$$

is a Fejér family whose generator is $(A^2)^{\frac{1}{2}}$.

Proof. Let $C \equiv (C(t))_{t\in\mathbb{R}}$ be a uniformly bounded cosine function generated by A. It is well-know that $-(-A)^{\frac{1}{2}}$ is the infinitesimal generator of a holomorphic C_0-semigroup $(T(z))_{\Re z>0}$ given by

$$T(z)x = 2\int_0^\infty \frac{z}{z^2+t^2}C(t)x\frac{dt}{\pi}, \qquad \Re z>0, \quad x\in X.$$

Note that

$$\|T(z)\| \le C_1 \log(2 + \frac{|z|}{\Re z}) \le C_1 \left(\frac{|z|}{\Re z}\right)^{\alpha}$$

with $C_1 > 0$ and $0 < \alpha < 1$ ([17, p. 348]). By Theorem 4.4, $(-A)^{\frac{1}{2}}$ is the generator of a Fejér family $(F^C(s))_{s\ge 0}$ such that

$$F^C(s)x := 2\int_0^{\infty} C(t)x \frac{1}{2\pi i}\int_{\Re z=1} \frac{e^{sz}}{z(z^2+t^2)}dz\frac{dt}{\pi} = 2\int_0^{\infty}\frac{1-\cos(st)}{t^2}C(t)x\frac{dt}{\pi},$$

for $x \in X$ and we obtain the result.

In the case that iA generates a uniformly bounded C_0-group $T \equiv (T(t))_{t\in\mathbb{R}}$, then $-A^2$ generates a cosine function and we apply the first part of the proof. □

Remarks. In the case when X is a UMD space, Corollary 4.5 is a consequence of Proposition 3.5, Corollary 3.6 and Theorem 4.2.

Note that if $T \equiv (T(t))_{t\in\mathbb{R}}$ is the shift group defined on $L^p(\mathbb{R}; X)$, then $F^T(s)$ is the Fejér operator $\mathcal{F}(s)$ defined in Section 1.

Fejér families defined by cosine functions have been studied in [21]. There the definition is quite different than our: they consider the limit of complex integrals which involve resolvent operators of cosine functions, see [21, p. 189]. To manage with it, they need a very skilful handling with complex variable methods. Our techniques help to simplify the proof of [21, Theorem 1].

5. **Examples**

In the next examples we consider function spaces, X, which are not UMD spaces and uniformly bounded C_0-groups of operators, T, defined on these spaces. We show that the Hilbert family, $(H^T(s))_{s\in\mathbb{R}}$, and the Dirichlet family, $(D^T(s))_{s\ge 0}$, may be or may not be defined.

Example 5.1. Let A be the multiplication operator $(Af)(t) := itf(t)$ considered on some space of functions X and let $(T(s)f)(t) := e^{ist}f(t)$,

$$\begin{aligned} H^T(s)f(t) &:= \frac{i}{\pi}\int_{\mathbb{R}} \frac{e^{-isr}}{r}(T(r)f)(t)dr = -\mathrm{sgn}(t-s)f(t), \\ D^T(s)f(t) &:= \int_{\mathbb{R}} \frac{\sin(sr)}{r}(T(r)f)(t)\frac{dr}{\pi} = (\chi_{(-s,s)}(t) + \frac{1}{2}\chi_{\{s\}}(t))f(t). \end{aligned}$$

In the case $X = (L^1(\mathbb{R}), \|\ \ \|_1)$, $T \equiv (T(t))_{t\in\mathbb{R}}$ is the uniformly bounded C_0-group generated by A and H^T resp. D^T is a Hilbert resp. Dirichlet family. But if we take $X = (C([\pi, 2\pi]), \|\ \ \|_\infty)$, one can see that $H^T(s), D^T(s)$ are not always bounded operators (e.g. $H^T(s) \in \mathcal{B}(X)$ if and only if $s \in \mathbb{R}\setminus[\pi, 2\pi]$), in particular they do not define a Hilbert resp. Dirichlet family, although T is still the C_0-group generated by A.

Example 5.2. Consider the operator $Af = -f'$ on $X = (C_{2\pi}(\mathbb{R}), \|\ \ \|_\infty)$ the Banach space of continuous, 2π-periodic complex valued functions. As a Fourier multiplier A can be expressed as $\widehat{Af}(n) = -in\widehat{f}(n)$. It is the infinitesimal generator of the shift C_0-group $T \equiv (T(t))_{t\in\mathbb{R}}$, $T(t)f(\varphi) := f(t+\varphi)$. For trigonometric polynomials $P(\varphi) = \sum_{j=-m}^{m} \hat{P}(j)e^{ij\varphi}$ we can define the (Fourier multiplier) operators

$$(H^T(s)P)(\varphi) := \frac{i}{\pi}\int_{\mathbb{R}} \frac{e^{-isr}}{r}T(r)P(\varphi)dr = \sum_{j=-m}^{m} \mathrm{sgn}(s-j)\hat{P}(j)e^{ij\varphi}, \tag{5.1}$$

$\varphi \in \mathbb{R}, s \in \mathbb{R}$.

As a consequence of [24, VII. (2.3)], one can prove that $H^T(0) \notin \mathcal{B}(X)$. Write $s = n + r$ with $n \in \mathbb{Z}$ and $r \in [0,1)$. For a trigonometric polynomial we compute

$$(H^T(n)P)(\varphi) = e^{in\varphi} H^T(0)(e^{-in\psi} P(\psi))(\varphi), \qquad \varphi \in \mathbb{R}. \tag{5.2}$$

Since $\operatorname{sgn}(s - j) = \operatorname{sgn}(n - j) + \chi_{(0,1)}(r)\chi_{\{n\}}(j)$ for $j \in \mathbb{Z}$, we obtain

$$(H^T(s)P)(\varphi) = (H^T(n)P)(\varphi) + \chi_{(0,1)}(r)\hat{P}(-n)e^{-in\varphi}, \qquad \varphi \in \mathbb{R}.$$

Therefore $H^T(s)$ is not a bounded operator on X.

Take $s > 0$, $s = n + r$ with $n \in \mathbb{Z}$, $r \in [0,1)$ and $Q(\varphi) = \sum_{j=-m}^{m} \hat{Q}(j)e^{ij\varphi}$ an arbitrary trigonometric polynomial. Then one can define

$$\begin{aligned}
(D^T(s)Q)(\varphi) &:= \int_{\mathbb{R}} \frac{\sin(ts)}{\pi t}(T(t)Q)(\varphi)dt \\
&= \sum_{j=-m}^{m} \hat{Q}(j) \int_{\mathbb{R}} \frac{\sin(ts)}{t} e^{itj} \frac{dt}{\pi} e^{ij\varphi} \\
&= (D_n(Q))(\varphi) - \frac{1}{2}\chi_{\{0\}}(r)(\hat{Q}(-n)e^{-in\varphi} + \hat{Q}(n)e^{in\varphi}),
\end{aligned}$$

where D_n is given by

$$D_n(Q)(\varphi) := \sum_{j=-\min(n,m)}^{\min(n,m)} \hat{Q}(j)e^{ij\varphi}, \qquad \varphi \in \mathbb{R}.$$

It is clear that $D^T(s) \in \mathcal{B}(X)$ if and only if $D_n \in \mathcal{B}(X)$. Since $\|D_n\| \sim \log(n)$, ([24, p. 67]), we conclude that $(D^T(s))_{s>0} \subset \mathcal{B}(X)$ although $\sup_{s>0} \|D^T(s)\| = \infty$.

Example 5.3. Now we identify the 2π-periodic functions on $\mathbb{R}$ with functions defined on the unit circle $\mathbb{T}$. The following space and C_0-group have appeared in [6, Example 5.4]. Let $C(\mathbb{T})$ be the set of continuous functions on $\mathbb{T}$. We consider X_k the linear span in $C(\mathbb{T})$ of the functions $\{z \mapsto z^j\,; -k \le j \le k\}$ and equipped with $\|\ \ \|_\infty$. We consider the Banach space $(X, \|\ \ \|_X)$ given by

$$X := \bigoplus_{k=1}^{\infty} X_k, \qquad \|f\|_X^2 := \sum_{k=1}^{\infty} \|f_k\|_\infty^2,$$

where $f = \bigoplus_{k=1}^{\infty} f_k$. The Banach space X is reflexive. We define the C_0-group $T \equiv (T(t))_{t\in\mathbb{R}}$ by $T(t) := \bigoplus_{k=1}^{\infty} T_k(t)$, where

$$T_k(t)f(z) := f(e^{it/k}z), \qquad z \in \mathbb{T},$$

for $k \ge 1$. Note that $\|T_k(t)\| \le 1$ and then $\|T(t)\| \le 1$ for $t \in \mathbb{R}$.

Take $f = \bigoplus_{k=1}^{\infty} f_k$, $f_k \in X_k$ with $f_k(z) = \sum_{j=-k}^{k} \hat{f}_k(j)z^j$ and we define

$$H^T(s)f := \bigoplus_{k=1}^{\infty} H^{T_k}(s)f_k, \qquad D^T(s)f := \bigoplus_{k=1}^{\infty} D^{T_k}(s)f_k,$$

where

$$\begin{aligned}
(H^{T_k}(s)f_k)(z) &:= \sum_{j=-k}^{k} \operatorname{sgn}(\frac{-j}{k} + s)\hat{f}_k(j)z^j, \quad z \in \mathbb{T}, \quad s \in \mathbb{R}, \\
(D^{T_k}(s)f_k)(z) &:= \sum_{|j|<ks} \hat{f}_k(j)z^j + \frac{1}{2}\sum_{|j|=sk} \hat{f}_k(j)z^j, \quad z \in \mathbb{T}, \quad s > 0,
\end{aligned}$$

$D^{T_k}(0) := 0$.

One can follow similar ideas as in the Example 5.2 to conclude that $H^T(s) \notin \mathcal{B}(X)$ for $s \in (-1, 1)$, $D^T(s) \notin \mathcal{B}(X)$ for $s \in (0, 1)$, $H^T(s) \in \mathcal{B}(X)$ for $|s| \geq 1$ and $D^T(s) \in \mathcal{B}(X)$ for $s = 0$ and $s \geq 1$.

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