Edgeworth expansions for subordinators via differential calculus for linear operators ⋆

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Abstract

We apply a differential calculus for linear operators, together with moduli of smoothness techniques, in order to obtain Edgeworth expansions for $E\phi(Z(t)) - E\phi(Z)$, where $(Z(t),\ t \geq 1)$ is a standardized subordinator, Z is a standard normal random variable and the degree of smoothness of ϕ goes from infinite differentiability to bounded variation. The main achievement of the method is to provide explicit upper bounds for the remainders, thus getting rid off the 'big or little o' terms. Other features are the relative simplicity of the proofs, the property of monotonic convergence for $E\phi(Z(t))$ under simple sufficient conditions on ϕ and, in the lattice case, the obtention of explicit lower and upper Berry-Esseen bounds of the same order of magnitude which are asymptotically sharp.

Résumé

Dans cet article, nous utilisons un calcul différentiel pour des opérateurs linéaires et des techniques basées sur des modules de continuité, afin d'obtenir des développements d'Edgeworth pour $E\phi(Z(t)) - E\phi(Z)$, où $(Z(t),\ t \geq 1)$ est un subordinateur typifié, Z est une variable aléatoire normalement distribuée, et ϕ est une fonction dont le degré de suavité va depuis la différentiabilité infinie jusqu'à la variation bornée. La caractéristique la plus remarquable de la méthode consiste à obtenir des cotes supérieures explicites pour les termes rémanents du développement. Comme traits supplémentaires, nous signalons la relative simplicité des démonstrations, la propriété de convergence monotone pour $E\phi(Z(t))$ sous des conditions appropriées sur ϕ et, au cas lattis, l'obtention de cotes inférieures et supérieures explicites du type Berry–Esseen qui sont asymptotiquement précises.

Key words: Edgeworth expansion, Berry-Esseen bounds, differential calculus, moduli of smoothness, subordinator.
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1 Introduction

Since the publication in the 1940s of the celebrated Esseen's smoothing inequality, different methods have been developed to establish rates of convergence in the central limit theorem. Apart from classical Fourier inversion techniques, we mention Stein's method (see, for instance, Stein [44], Barbour [10], and Chen and Shao [19]), the leading term approach introduced by Hall (cf. Hall [28], Bentkus [12], and Hall and Wang [29]) and Lindeberg's operator method (cf. Borisov and Skilyagina [15] and Rio [37]). Let Z be a standard normal random variable and suppose that $(S_n)_{n\geq 1}$ is a given sequence of random variables obeying the central limit theorem. Under appropriate moment assumptions, the rates of convergence for $P(S_n \leq x) - P(Z \leq x)$ take the form of either Berry-Esseen bounds or Edgeworth expansions. A big effort has been devoted to obtain uniform or non-uniform Berry-Esseen bounds with explicit upper constants (see, for instance, Alberink [8], Chen and Shao [18], Ouchti [33], and Shao [41]), or even near optimal constants in the case where S_n is a standardized sum of independent random variables (cf. Shiganov [42], Bentkus [12] and the references therein). Edgeworth expansions provide better accuracy than Berry-Esseen bounds even for moderate sample sizes, and this is confirmed by numerical computations (cf. Brown *et al.* [17] and Zhou *et al.* [50]). However, with a very few exceptions (cf. Seoh and Hallin [40]) Edgeworth expansions contain 'big or litle o' terms with unspecified upper bounds. This makes impossible to answer the question: How large must the sample size n be in order to attain some prescribed degree of accuracy in the approximation? In this respect, see the discussion in Seoh and Hallin [40] and Pfanzagl [35].

A similar picture is found in the context of Edgeworth expansions for $E\phi(S_n) - E\phi(Z)$, where ϕ is a sufficiently smooth function. Here, attention has been focused in obtaining optimal or near optimal correlations between the numer of exact terms in the Edgeworth expansion, the moment assumptions on S_n, the degree of differentiability of ϕ, and the growth requirements on the highest order derivative of ϕ (see Hipp [30], Götze and Hipp [26], Hall [27], Barbour [10], Borisov *et al.* [14], Steinsaltz [45], Rinott and Rotar [36] and Rotar [38]). Again in this context, 'big or little o' terms generally remain unspecified. We have only found a paper by Lefèvre and Utev [32] where explicit upper bounds are given.

The aim of this work is to propose a new method, based on the differential calculus for linear operators introduced in Adell [1], to deal with rates of con-

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vergence in the central limit theorem. We are convinced that such a method works well in other limit theorems, such as Poisson and binomial approximations (cf. Adell and Lekuona [3]). Let us sketch the main ideas. Suppose we are given a process $\mathbb{Z} := (Z(t),\ t \geq 1)$ obeying the central limit theorem, satisfying appropriate moment conditions, and such that $EZ(t) = 0$, $t \geq 1$. Assume that the linear operator L represented by $\mathbb{Z}$, i.e., $L\phi(t) := E\phi(Z(t))$, $t \geq 1$, is differentiable. The later is actually a mild assumption. For instance, every linear operator represented by a discrete time process is differentiable (see Adell [1, Section 9]). As shown in Adell [1], we can find two derived processes $\mathbb{V}_1$ and $\mathbb{V}_2$ with respect to a σ-finite integrating measure ν on $[1, \infty)$ such that, for any differentiable function ϕ satisfying suitable growth conditions, we have

$$E\phi(Z(t)) - E\phi(Z(s)) = -\int_{(t,s]} \left(E\phi^{(1)}(V_1(u)) - E\phi^{(1)}(V_2(u)) \right) d\nu(u), \quad (1.1)$$

for $1 \leq t \leq s$. Only one integrating measure ν is needed in (1.1), because $\mathbb{Z}$ has constant expectation. In usual cases, it turns out that $d\nu(u) \sim u^{-1/2}\, du$. Therefore, it is not clear from (1.1) if we can let $s \to \infty$ to obtain an integral expression for $E\phi(Z(t)) - E\phi(Z)$. In fact, we need a Taylor expansion of third order at least for $E\phi(Z(t)) - E\phi(Z(s))$ before letting $s \to \infty$. Such a Taylor expansion is possible whenever the linear operators represented by the processes $\mathbb{V}_1$ and $\mathbb{V}_2$ under the integral sign in (1.1) are repeatedly differentiable. If this is the case, repeated application of the procedure leading to (1.1) gives us the following Edgeworth expansion for a smooth enough function ϕ

$$E\phi(Z(t)) - E\phi(Z) = \sum_{k=1}^{n-2} \gamma(\phi, k) t^{-k/2} + o_{n-2}(\phi; t)\, t^{-(n-2)/2}, \quad t \geq 1, \quad (1.2)$$

where the remainder $o_{n-2}(\phi; t)$ can be written in closed form in terms of various multiple integrals. In our approach, the most natural thing is to consider a continuous parameter t and to give first Edgeworth expansions for smooth functions ϕ. By considering suitable smooth approximants, we gradually reduce the degree of smoothness of ϕ arriving at the case when ϕ has bounded variation, so that, as a final step, we tackle the problem of approximating the distribution function of $Z(t)$ by the standard normal distribution function.

Together with a relative simplicity of the proofs, the main achievement of the method is to provide Edgeworth expansions with explicit upper bounds for the remainders, thus getting rid off the 'big or little o' terms. As a counterpart, the main limitation of the method in its actual form is that the derived processes must be explicitly known, and this may be difficult in general.

To illustrate the method, we have chosen $\mathbb{Z}$ to be a standardized subordinator, because the easy expression of its derived processes will not overshadow the main ideas. We also assume that the highest order derivative of ϕ in (1.2) has bounded variation, covering the case in which ϕ itself enjoys the same property. This setting is similar to that considered in Lefèvre and Utev [32]. Such an assumption makes appropriate the use of moduli of smoothness techniques in order to bound remainder terms. Other growth conditions on the highest order derivative of ϕ, as those considered in Götze an Hipp [26], Barbour [10], Borisov *et al.* [14] or Rinott and Rotar [36], would require other techniques. Roughly speaking, a modulus of smoothness of a certain function f is said to be of order $r = 1, 2, \ldots$, when differences of f of the same order are involved. Moduli of smoothness of first and second order are widely used in approximation theory to measure the speed of convergence in approximating a function f by a sequence $(P_n f)_{n\geq 1}$, P_n being a linear operator (see Ditzian and Totik [22], Ditzian and Ivanov [21], Adell and Sangüesa [6], and the references therein). Different kinds of moduli of smoothness of first order have been introduced in connection with rates of convergence in the central limit theorem (cf. Bhattacharya and Ranga Rao [11, Chap. 3], Barbour [10], Rinott and Rotar [36], and Rotar [38], among others). On the other hand, the usual first modulus of smoothness of a distribution function F is just the concentration function of F. It turns out that Esseen's smoothing inequality can be rewritten in terms of concentration functions (see Arak and Zaitsev [9, pp. 61-62], Stef and Tenenbaum [43], and Adell and Lekuona [4]). In turn, concentration functions can be estimated either in terms of characteristic functions (cf. Esseen [24], Petrov [34, Chap. III], and Salikhov [39]), or truncated moments (cf. Chen and Shao [18,19]). As far as we know, usual second modulus of smoothness of distribution functions have been used in Adell and Lekuona [5] for the first time to deal with Berry-Esseen bounds.

In the setting of this paper, we point out two other features of the method. On the one hand, we are able to give simple sufficient conditions on ϕ to ensure that $E\phi(Z(t))$ converges monotonically to $E\phi(Z)$ as t tends to infinity (see Theorem 1 and Corollary 14). Apparently, results in this direction seem to be rather scarce. Kane [31] has shown that $P(S_n \geq 0)$ converges monotonically to $P(Z \geq 0)$ for various choices of the law of S_n. For $a \neq 0$, the monotonicity properties of the sequence $(P(S_n \geq a))_{n\geq 1}$ may be quite surprising, as happens when S_n is Poisson distributed (see Teicher [47] and Adell and Jodrá [2]). On the other hand, and whenever $Z(t)$ has a lattice distribution F_t, we give explicit lower and upper Berry-Esseen bounds of the same order of magnitude which are asymptotically sharp. Such bounds, referring to the supremum norm on an open interval I and involving the maximum jump of F_t on I, could be considered as an alternative to the classical presentation of the lattice case (see Section 5 for more details).

The following notations will be used throughout the paper. Let $\mathbb{X} := (X(t),\ t \geq$

0) be a centred subordinator, that is, a process starting at the origin, having independent stationary increments and right-continuous nondecreasing paths, and such that $EX(t) = t$, $t \geq 0$. The characteristic function of $X(t)$ can be written as

$$Ee^{i\zeta X(t)} = \exp\left(tE\frac{e^{i\zeta T} - 1}{T}\right), \quad \zeta \in \mathbb{R}, \quad t \geq 0, \tag{1.3}$$

where T is a nonnegative random variable which determines $\mathbb{X}$ and is therefore called the characteristic random variable of $\mathbb{X}$ (see Steutel and van Harn [46, p. 107] or Adell [1, Section 11]). It follows from (1.3), or from formula (2.8) below, that $E(X(t) - t)^2 = tET$, $t \geq 0$. We assume that $0 < ET =: \sigma^2 < \infty$ and define, accordingly, the standardized subordinator $\mathbb{Z} := (Z(t),\ t \geq 1)$ by

$$Z(t) := \frac{X(t) - t}{\sigma\sqrt{t}}, \quad t \geq 1. \tag{1.4}$$

We denote by

$$\Delta(t) := \sup_{u \geq t} \|F_u - \mathfrak{N}\|, \quad t \geq 1, \tag{1.5}$$

where F_u is the distribution function of $Z(u)$, $\mathfrak{N}$ is the standard normal distribution function and $\|\cdot\|$ is the usual supremum norm. The standard normal density will be denoted by $\eta := \mathfrak{N}^{(1)}$.

By $\mathbb{R}$ and $\mathbb{N}$ we denote the sets of real and positive integer numbers, respectively. The indicator function of a set A is denoted by 1_A. Every function ϕ is a real measurable function defined on $\mathbb{R}$. For any $n \in \mathbb{N}$, $\mathcal{A}^{(n)}$ is the set of $n-1$ times differentiable functions ϕ such that $\phi^{(n-1)}$ is absolutely continuous with respect to the Lebesgue measure having Radon-Nikodym derivative $\phi^{(n)}$, and $\mathcal{A}^{\infty} := \cap_{n \geq 1} \mathcal{A}^{(n)}$. By $\mathcal{AV}^{(0)}$ we denote the set of right-continuous functions ϕ whose total variation $V(\phi)$ on $\mathbb{R}$ is finite, and by $\mathcal{AV}^{(n)}$ the subset of $\mathcal{A}^{(n)}$ consisting of those functions ϕ such that $\phi^{(n)} \in \mathcal{AV}^{(0)}$, $n \in \mathbb{N}$. For any $p \geq 0$, $\mathcal{B}_p$ and $\mathcal{E}_p$ stand, respectively, for the sets of functions ϕ such that $|\phi(x)| \leq C(1 + |x|^p)$ and $|\phi(x)| \leq Ce^{p|x|}$, $x \in \mathbb{R}$, for some constant $C \geq 0$.

On the other hand, we shall use the following random variables: Z is a standard normal random variable, T is the characteristic random variable of the subordinator $\mathbb{X}$, S_1 is a Bernoulli random variable with success parameter $1/2$, S_2 is uniformly distributed on $[0, 1]$ and, for any $n \in \mathbb{N}$, β_n has the beta density $\rho_n(\theta) := n(1 - \theta)^{n-1}$, $\theta \in [0, 1]$ ($\beta_0 := 1$). Finally, unless otherwise specified, all of the random variables appearing under the same expectation sign are supposed to be mutually independent.

The paper is organized as follows. The main tools are presented in Section 2, while the remaining sections are devoted to obtain Edgeworth expansions for $E\phi(Z(t)) - E\phi(Z)$, starting with smooth functions (Section 3), continuing with moderately smooth functions (Section 4), and concluding with functions of bounded variation (Section 5). All of the sections have a common structure, namely, the results are stated and discussed at the beginning, and their proofs are differed to the end of each section.

2 Differential calculus

In this section, we introduce the main tools of the paper. On the one hand, we apply the differential calculus developed in Adell [1] to the linear operator L represented by the standardized subordinator $\mathbb{Z} := (Z(t),\ t \geq 1)$ defined in (1.4) in order to obtain the basic differentiation formulae which will be used in the following sections. Such differentiation formulae are established under different integrability assumptions on $\mathbb{Z}$ and therefore for different growth conditions on the functions ϕ under consideration. In essence, we give exact integral expressions for $E\phi(Z(s)) - E\phi(Z(t))$, $t \leq s$, whenever ϕ is smooth enough. From these formulae, it is clear that $(E\phi(Z(t)),\ t \geq 1)$ satisfies the Cauchy criterion and therefore converges to some limit as $t \to \infty$. We have not been able to produce a simple argument to identify this limit as $E\phi(Z)$. Instead, we appeal to uniform integrability arguments or to the statement of monotonic convergence in (2.2) below, which is interesting by itself, depending on the integrability conditions imposed on $\mathbb{Z}$ (see the proof of Theorem 2).

On the other hand, we apply moduli of smoothness techniques to give precise upper bounds for the remainders (Corollary 3). Only the first modulus of smoothness of the standard normal distribution function $\mathfrak{N}$ at a random length is needed. In spite of this, we consider some properties of general mth modulus of smoothness to be used in Section 5.

We list here the main results of this section.

Theorem 1 *Let $\phi \in \mathcal{A}^{(1)}$, $p \geq 1$, $\lambda > 0$ and $0 < \alpha < \lambda\sigma$. Assume either that (i) $ET^p < \infty$ and $\phi^{(1)} \in \mathcal{B}_p$, or that (ii) $Ee^{\lambda T} < \infty$ and $\phi^{(1)} \in \mathcal{E}_\alpha$. For any $s \geq t \geq 1$, we have*

$$E\phi(Z(s)) - E\phi(Z(t))$$
$$= \int_t^s \left(E\phi^{(1)}\left(Z(u) + \frac{TS_2}{\sigma\sqrt{u}}\right) - E\phi^{(1)}\left(Z(u) + \frac{TS_1}{\sigma\sqrt{u}}\right)\right) \frac{du}{\sigma\sqrt{u}}. \tag{2.1}$$

If, in addition, $\phi^{(1)}$ *is convex, then*

$$E\phi(Z(t)) \geq E\phi(Z(s)). \tag{2.2}$$

Denote by $x_+ := \max(x, 0)$ and by $x_- := (-x)_+$, $x \in \mathbb{R}$. Concerning truncated moments, we see from (2.2) that, for any $p > 1$ and $a \in \mathbb{R}$, $E(Z(t) - a)_+^p$ decreases and $E(Z(t)-a)_-^p$ increases as $t \to \infty$, provided, of course, that such truncated moments are finite. Other monotonicity properties will be given in Corollary 14.

A basic ingredient in many of the remainder terms in this paper is the following expression

$$R_k(f; u) := \sum_{j=1}^{2} (-1)^{j-1} E(TS_j)^k \left(f\left(Z(u) + \frac{TS_j \beta_k}{\sigma \sqrt{u}} \right) - f(Z) \right), \tag{2.3}$$

where it is assumed that $u \geq 1$, $k \in \mathbb{N}$, $ET^k < \infty$ and f is a certain function for which the expectations in (2.3) are well defined.

An integral expression for $E\phi(Z(t)) - E\phi(Z)$ is obtained from (2.1) by letting $s \to \infty$, provided that we start with a Taylor formula of third order at least for $E\phi(Z(t)) - E\phi(Z(s))$. The standard way to do this is to apply repeatedly the differential calculus to the processes under the integral sign in (2.1) (the derived processes of $\mathbb{Z}$). However, in view of the specific form of such processes, we take a short cut by applying Lemma 5 below. As a result, we give the following.

Theorem 2 *Let* $\phi \in \mathcal{A}^{(3)}$*,* $p \geq 1$*,* $\lambda > 0$ *and* $0 < \alpha < \lambda\sigma$*. Assume either that (i)* $ET^{p+1} < \infty$ *and* $\phi^{(3)} \in \mathcal{B}_{p-1}$*, or that (ii)* $Ee^{\lambda T} < \infty$ *and* $\phi^{(3)} \in \mathcal{E}_\alpha$*. For any* $t \geq 1$*, we have*

$$E\phi(Z(t)) - E\phi(Z) = \frac{ET^2}{6\sigma^3} E\phi^{(3)}(Z) t^{-1/2} + \frac{1}{2\sigma^3} \int_t^\infty R_2\left(\phi^{(3)}; u\right) \frac{du}{u^{3/2}}.$$

Denote by $D(f)$ the set of discontinuity points of the function f. In the setting of Theorem 2, assume further that $D(\phi^{(3)})$ has zero Lebesgue measure. In such a case, it follows from (2.3), Theorem 2 and Billingsley [13, p. 334] that

$$E\phi(Z(t)) - E\phi(Z) = \frac{ET^2}{6\sigma^3} E\phi^{(3)}(Z) t^{-1/2} + o(t^{-1/2}),$$

where the term $o(t^{-1/2})$ possibly depends on $\phi^{(3)}$. A precise estimate of this term can be given under additional requirements on $\phi^{(3)}$, as done in Corollary 3 below. To this end, let $m \in \mathbb{N}$. The usual mth modulus of smoothness of a function f at length $\delta \geq 0$ is defined by

$$w_m(f;\delta) := \sup\left\{|\Delta_h^m f(x)| : \ x \in \mathbb{R}, \ 0 \leq h \leq \delta\right\},$$

where $\Delta_h^m f$ stands for the mth symmetric difference of f with step $h \geq 0$, that is,

$$\Delta_h^m f(x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} f\left(x + \left(\frac{m}{2} - k\right) h\right).$$

It is well known (cf. Ditzian and Totik [22, p. 37]) that

$$w_m(f; a\delta) \leq (1+a)^m w_m(f;\delta), \quad a,\delta \geq 0. \tag{2.4}$$

Assume, in addition that $f \in \mathcal{A}^{(m)}$. Then (cf. Adell and Sangüesa [7])

$$\Delta_h^m f(x) = h^m E f^{(m)}\left(x + \frac{h}{2} W_m\right), \quad x \in \mathbb{R},$$

where $W_m := U_1 + \ldots + U_m$ and $(U_i)_{i\geq 1}$ is a sequence of independent identically distributed random variables having the uniform distribution on $[-1,1]$. We therefore have in this case that

$$w_m(f;\delta) \leq \|f^{(m)}\|\, \delta^m, \quad \delta \geq 0. \tag{2.5}$$

We shall be mainly concerned with moduli of smoothness of first and second order.

Corollary 3 *Assume that $\phi \in \mathcal{AV}^{(3)}$ and that $ET^2 < \infty$. Then,*

$$\begin{aligned} &\left| E\phi(Z(t)) - E\phi(Z) - \frac{ET^2}{6\sigma^3} E\phi^{(3)}(Z) t^{-1/2} \right| \\ &\leq \frac{1}{\sigma^3} V(\phi^{(3)}) \left(ET^2 \Delta(t) + ET^2 w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right)\right) t^{-1/2}, \quad t \geq 1. \end{aligned} \tag{2.6}$$

It follows from (2.8) below that $E(X(t)-t)^3 = tET^2$, $t \geq 0$, thus implying that the third moment of the subordinator $\mathbb{X}$ is always positive. On the other hand, the upper bound in (2.6) depends on ϕ only through $V(\phi^{(3)})$.

Remark 4 *By dominated convergence, the term* $ET^2 w_1(\mathfrak{N}; T/(\sigma\sqrt{t}))$ *tends to 0 as* $t \to \infty$. *If, in addition,* $ET^{2+p} < \infty$ *for some* $0 < p \leq 1$, *such a term is of order* $t^{-p/2}$ *at least, since as follows from* (2.5) *we have that*

$$ET^2 w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right)\left(1_{\{T\leq\sqrt{t}\}} + 1_{\{T>\sqrt{t}\}}\right) \leq \left(\frac{\|\eta\|}{\sigma} + 2\right) ET^{2+p}\, t^{-p/2}.$$

To prove the preceding results, we shall need some auxiliary lemmas. The following technical result will be used throughout the paper (see Alberink [8] for a formula similar to that in (2.7)).

Lemma 5 *Let* X *and* V *be two independent random variables and let* $\phi \in \mathcal{A}^{(n)}$, $n \in \mathbb{N}$. *Assume either that (i)* $\max(E|X|^{n+p}, E|V|^{n+p}) < \infty$ *and* $\phi^{(n)} \in \mathcal{B}_p$, *for some* $p \geq 0$, *or that (ii)* $\max(Ee^{\lambda|X|}, Ee^{\lambda|V|}) < \infty$ *and* $\phi^{(n)} \in \mathcal{E}_\alpha$, *for some* $0 \leq \alpha < \lambda$. *Then,*

$$E\phi(X+V) = \sum_{k=0}^{n-1} \frac{EV^k}{k!}\, E\phi^{(k)}(X) + \frac{1}{n!} EV^n \phi^{(n)}(X + V\beta_n). \tag{2.7}$$

PROOF. Assumptions (i) and (ii), together with Hölder's inequality, guarantee that all of the expectations in (2.7) exist. Formula (2.7) follows by applying Corollary 7.3 in Adell [1] to the function $f(x) := E\phi(X+x)$, $x \in \mathbb{R}$. The proof is complete. $\square$

Let $\mathbb{X}$ be the centred subordinator defined in (1.3) and let ϕ be an absolutely continuous function defined on $[0,\infty)$ such that ϕ and $\phi^{(1)}$ are nonnegative. It follows from Theorem 11.1(b) and Remark 3.9 in Adell [1] that

$$E\phi(X(t)) = \phi(0) + \int_0^t E\phi^{(1)}(X(u) + TS_2)\, du, \quad t \geq 0, \tag{2.8}$$

in the sense that both sides in (2.8) are simultaneously finite or infinite. The relations between the moments of $X(t)$ and those of T are well known (cf. Bose *et al.* [16] or Steutel and van Harn [46, pp. 103–104]). For the sake of completeness, we give an easy proof of the following auxiliary result based on (2.8).

Lemma 6 *Let* $p > 0$, $\lambda > 0$ *and* $0 < \alpha < \lambda\sigma$. *Then,*

(a) $EX(t)^{p+1} < \infty$*, for any* $t \geq 0$*, if and only if* $ET^p < \infty$*.*
(b) $Ee^{\lambda X(t)} < \infty$*, for any* $t \geq 0$*, if and only if* $Ee^{\lambda T S_2} < \infty$*.*
(c) If $Ee^{\lambda T} < \infty$*, the family* $(e^{\alpha |Z(t)|},\ t \geq 1)$ *is uniformly integrable.*

PROOF. Using the inequality

$$(x+y)^p \leq 2^{p-1}(x^p + y^p), \quad x, y \geq 0, \tag{2.9}$$

and applying (2.8) to the function $\phi(x) := x^{p+1}$, $x \geq 0$, we have

$$\begin{aligned} &(p+1)tE(TS_2)^p \leq (p+1)\int_0^t E\left(X(u) + TS_2\right)^p \, du = EX(t)^{p+1} \\ &\leq 2^{p-1}(p+1)t\left(EX(t)^p + E(TS_2)^p\right), \quad t \geq 0. \end{aligned} \tag{2.10}$$

Since $EX(t) = t$, $t \geq 0$, part (a) follows from (2.10) by using induction. Using either (2.8), or (1.3) in conjunction with Lemma 5, we see that

$$Ee^{\lambda X(t)} = \exp\left(\lambda t E e^{\lambda T S_2}\right), \quad t \geq 0, \tag{2.11}$$

which shows part (b). Finally, choose $a \in (\alpha, \lambda\sigma)$. By (2.11) and Lemma 5, we obtain for any $t \geq 1$ that

$$\begin{aligned} Ee^{aZ(t)} &= \exp\left(\frac{a}{\sigma}\sqrt{t}\left(E\exp\left(\frac{a}{\sigma\sqrt{t}}TS_2\right) - 1\right)\right) \\ &= \exp\left(\left(\frac{a}{\sigma}\right)^2 ETS_2 \exp\left(\frac{a}{\sigma\sqrt{t}}TS_2\beta_1\right)\right) \leq \exp\left(\left(\frac{a}{\sigma}\right)^2 ET\exp\left(\frac{aT}{\sigma}\right)\right). \end{aligned}$$

This, together with the inequality $e^{|x|} \leq e^x + e^{-x}$, $x \in \mathbb{R}$, shows part (c). The proof is complete. □

PROOF OF THEOREM 1. Let L be the linear operator represented by $\mathbb{Z}$, as defined in (1.4) and let $j = 1, 2$. We consider the process $\mathbb{V}_j := (V_j(u),\ u \geq 1)$ and the σ-finite measure ν on $[1, \infty)$ defined, respectively, by

$$V_j(u) := Z(u) + \frac{TS_j}{\sigma\sqrt{u}} \quad \text{and} \quad d\nu(u) := \frac{1}{\sigma\sqrt{u}}\, du, \quad u \geq 1. \tag{2.12}$$

Let $\zeta \in \mathbb{R}$. Differentiating with respect to t the function $\psi_\zeta(t) := Ee^{i\zeta Z(t)}$, $t \geq 1$, it can be checked that

$$Ee^{i\zeta Z(t)} - Ee^{i\zeta Z(1)} = i\zeta \int_1^t \left(Ee^{i\zeta V_2(u)} - Ee^{i\zeta V_1(u)} \right) d\nu(u). \tag{2.13}$$

By Theorem 4.1 and Example 4.2 in Adell [1], formula (2.13) means that $\mathbb{V}_j$ is a derived process of L with respect to the integrating measure ν.

Suppose first that condition (i) in Theorem 1 is satisfied. Since $\phi \in \mathcal{B}_{p+1}$, we see that ϕ belongs to the domain of L, as follows from Lemma 6(a). By assumption and Lemma 6(a), the expectations under the integral sign in (2.1) are finite. Therefore, (2.1) follows from (2.13), together with Theorem 3.3(i) and Remark 3.6 in Adell [1]. If condition (ii) in Theorem 1 is fulfilled, the proof is similar, by replacing Lemma 6(a) by Lemma 6(b) and (c). Finally, let f be a convex function defined on $[0, 1]$. Clearly,

$$Ef(S_2) \leq E\left(S_2 f(1) + (1 - S_2) f(0)\right) = \frac{1}{2}(f(1) + f(0)) = Ef(S_1). \tag{2.14}$$

Thus, inequality (2.2) follows from (2.1) and (2.14) by setting

$$f(x) := E\phi^{(1)}\left(Z(u) + \frac{T}{\sigma\sqrt{u}}\, x \right), \qquad x \in [0, 1].$$

This completes the proof. $\square$

The following is a slight modification, suitable for our purposes, of Theorem 16.14 in Billingsley [13].

Lemma 7 *Let $(Y(t),\ t \geq 1)$ and Y be integrable random variables such that $Y(t)$ converges in law to Y as $t \to \infty$. Then $EY(t)$ converges to EY as $t \to \infty$, if and only if*

$$\overline{\lim_{a\to\infty}}\ \overline{\lim_{t\to\infty}} \left| EY(t) 1_{(a,\infty)}(|Y(t)|) \right| = 0. \tag{2.15}$$

PROOF. Assume that condition (2.15) is fulfilled. By the Skorohod representation theorem, we can assume that $Y(t)$ converges almost surely to Y as $t \to \infty$. Let $a > 0$ be such that $P(Y \in \{-a, a\}) = 0$. By dominated convergence, we have

$$\lim_{t\to\infty} EY(t)\left(1_{[-a,a]}(|Y(t)|\right) = EY1_{[-a,a]}(|Y|).$$

Hence,

$$\overline{\lim_{t\to\infty}}|EY(t) - EY| \leq \overline{\lim_{t\to\infty}}\left|EY(t)1_{(a,\infty)}(|Y(t)|)\right| + E|Y|1_{(a,\infty)}(|Y|).$$

Therefore, (2.15) implies that $EY(t)$ converges to EY as $t \to \infty$. The converse implication is shown in a similar way, thus completing the proof. □

Remark 8 *Assume that the random variables* $(Y(t),\ t \geq 1)$ *and* Y *belong to* L^p, $p \geq 1$, *and that* $Y(t)$ *converges in law to* Y *as* $t \to \infty$. *Suppose that* $E|Y(t)|^p$ *converges to* $E|Y|^p$ *as* $t \to \infty$. *If* $\phi \in \mathcal{B}_p$ *and* $P(Y \in D(\phi)) = 0$, *it follows from Billingsley* [13, p. 334] and Lemma 7 that $E|\phi(Y(t))|$ converges to $E|\phi(Y)|$ as $t \to \infty$, and, a fortiori, so does $E\phi(Y(t))$ to $E\phi(Y)$.

PROOF OF THEOREM 2. Suppose first that condition (ii) in Theorem 2 is satisfied. Let $1 \leq t \leq u \leq s$. By assumption, it can be seen that the integrability conditions in Lemma 5(ii) are fulfilled. Consequently, we have the Taylor expansion

$$\begin{aligned} E\phi^{(1)}\left(Z(u) + \frac{TS_j}{\sigma\sqrt{u}}\right) &= E\phi^{(1)}(Z(u)) + \frac{\sigma}{\sqrt{u}}ES_jE\phi^{(2)}(Z(u)) \\ &+ \frac{1}{2\sigma^2 u}E(TS_j)^2\phi^{(3)}\left(Z(u) + \frac{TS_j\beta_2}{\sigma\sqrt{u}}\right), \qquad j = 1, 2. \end{aligned} \tag{2.16}$$

On the other hand, $ES_j = 1/2$, $j = 1, 2$. Therefore, by (2.1) and (2.16), we get

$$\begin{aligned} &E\phi(Z(s)) - E\phi(Z(t)) \\ &= \sum_{j=1}^{2}(-1)^j \int_t^s E(TS_j)^2\phi^{(3)}\left(Z(u) + \frac{TS_j\beta_2}{\sigma\sqrt{u}}\right)\frac{du}{2(\sigma\sqrt{u})^3}. \end{aligned} \tag{2.17}$$

Since all of the expectations under the integral sign in (2.17) are uniformly bounded and $(e^{\lambda|Z(t)|},\ t \geq 1)$ is uniformly integrable, as follows from Lemma 6(c), we can take limits in (2.17) as $s \to \infty$, thus showing Theorem 2.

Assume that condition (i) in Theorem 2 is fulfilled. Following a similar pattern as above, we arrive at formula (2.17). To ensure that we can take limits in (2.17) as $s \to \infty$, we shall use Lemma 7 and the statement on monotonic convergence in (2.2). Indeed, let $p \geq 1$. Since $ET^{p+1} < \infty$, Lemma 6(a) implies that $E|Z(t)|^{p+2} < \infty$, $t \geq 1$. Applying (2.17) to the function $\phi(x) := |x|^{p+2}$, $x \in \mathbb{R}$, and taking into account that $EZ(t)^2 = 1$, $t \geq 1$, it can be seen by induction that

$$\sup_{t \geq 1} E|Z(t)|^{p+2} < \infty. \tag{2.18}$$

Let $a > 0$. We consider the function

$$\psi_a(x) := (x+a)_-^{p+2} 1_{(-\infty, -(a+1)]}(x) + (x+a)_- 1_{(-(a+1), -a]}(x). \tag{2.19}$$

For any $r \in (1, p+2)$, the family of random variables $((Z(t)+a)_-^r,\ t \geq 1)$ is uniformly integrable, as follows from (2.18). Therefore, we obtain from (2.2) and (2.19) that

$$\sup_{t \geq 1} E(Z(t)+a)_-^r = E(Z+a)_-^r \leq E\psi_a(Z).$$

By Fatou's lemma, this implies that

$$\sup_{t \geq 1} E(Z(t)+a)_-^{p+2} \leq E\psi_a(Z). \tag{2.20}$$

Using again (2.2), we see that

$$\sup_{t \geq 1} E(Z(t)-a)_+^{p+2} = E(Z(1)-a)_+^{p+2}. \tag{2.21}$$

On the other hand, by inequality (2.9), one can check that

$$\begin{aligned} &|x|^{p+2} 1_{(a,\infty)}(|x|) \\ &\leq 2^{p+1} \left((x-a)_+^{p+2} + (x+a)_-^{p+2} + a^{p+2} 1_{(a,\infty)}(|x|) \right), \quad x \in \mathbb{R}. \end{aligned} \tag{2.22}$$

It therefore follows from (2.20)-(2.22) and the central limit theorem for the process $\mathbb{Z}$ that

$$\overline{\lim_{t\to\infty}} E|Z(t)|^{p+2} 1_{(a,\infty)}(|Z(t)|)$$

$$\leq 2^{p+1}\left(E(Z(1)-a)_+^{p+2} + E\psi_a(Z) + a^{p+2}P(|Z|>a)\right).$$

Letting $a \to \infty$ in the preceding inequality and using dominated convergence, we see that condition (2.15) is fulfilled for $(|Z(t)|^{p+2},\ t \geq 1)$. Therefore, Lemma 7 implies that $E|Z(t)|^{p+2}$ converges to $E|Z|^{p+2}$ as $t \to \infty$. If ϕ is a function satisfying the requirements of Theorem 2(i), then $\phi \in \mathcal{B}_{p+2}$ and, consequently, we obtain from Remark 8 that

$$\lim_{s\to\infty} E\phi(Z(s)) = E\phi(Z). \tag{2.23}$$

On the other hand, by (2.18), the expectations under the integral sign in (2.17) are uniformly bounded. This, together with (2.23), allows us to take limits in (2.17) as $s \to \infty$, thus completing the proof. □

For any $g \in \mathcal{AV}^{(0)}$, we consider the linear operator H_g defined by

$$H_g f(x) := \int_{\mathbb{R}} f(z+x)\, dg(z), \quad x \in \mathbb{R}, \tag{2.24}$$

where f is any function for which the right-hand side in (2.24) exists. The following auxiliary result will be often used.

Lemma 9 *If $f, g \in \mathcal{AV}^{(0)}$, then*

$$w_m(H_g f;\delta) \leq V(f) w_m(g;\delta), \quad \delta \geq 0,\ m \in \mathbb{N}. \tag{2.25}$$

If, in addition, $g(-\infty) = g(\infty) = 0$, then

$$|H_g f(0)| \leq V(f)\|g\|. \tag{2.26}$$

PROOF. Let $m \in \mathbb{N}$. Using induction and Fubini's theorem, we see that

$$\Delta_h^m H_g f(x) = \int_{\mathbb{R}} \Delta_h^m g_-(u-x)\, df(u), \quad x \in \mathbb{R},\ h \geq 0, \tag{2.27}$$

where $g_-(x)$ stands for the left limit of g at x. Since $w_m(g_-;\delta) \leq w_m(g;\delta)$, $\delta \geq 0$, inequality (2.25) follows from (2.27). Inequality (2.26) is shown in a similar way. □

If the function f in (2.3) has bounded variation, we can give the following estimate.

Lemma 10 *Let $u \geq t \geq 1$ and $k \in \mathbb{N}$. If $f \in \mathcal{AV}^{(0)}$ and $ET^k < \infty$, then*

$$|R_k(f;u)| \leq V(f)\left(ET^k\Delta(t) + ET^k w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right)\right).$$

PROOF. Let $u \geq t \geq 1$ and denote indistinctly by S each one of the random variables S_1 and S_2. Using (2.26) with $g = F_u - \mathfrak{N}$ and taking into account (1.5) and Fubini's theorem, we have that

$$\left|E(TS)^k\left(f\left(Z(u) + \frac{TS\beta_k}{\sigma\sqrt{u}}\right) - f\left(Z + \frac{TS\beta_k}{\sigma\sqrt{u}}\right)\right)\right| \leq E(TS)^k V(f)\Delta(t).$$

By (2.25) and Fubini's theorem, we have that

$$\left|E(TS)^k\left(f\left(Z + \frac{TS\beta_k}{\sigma\sqrt{u}}\right) - f(Z)\right)\right| \leq V(f)E(TS)^k w_1\left(\mathfrak{N}; \frac{TS\beta_k}{\sigma\sqrt{u}}\right)$$

$$\leq V(f)ES^kET^k w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right).$$

Since $ES^k \leq 1/2$, $k \in \mathbb{N}$, the conclusion follows from the two previous inequalities. □

PROOF OF COROLLARY 3. The proof is an immediate consequence of Theorem 2 and Lemma 10. □

3 Smooth functions

The purpose of this section is to obtain Edgeworth expansions for $E\phi(Z(t)) - E\phi(Z)$, whenever ϕ is a smooth function and the standardized subordinator $\mathbb{Z}$ satisfies appropriate integrability conditions. Two different kinds of expansions are given, namely, full expansions if ϕ is infinitely differentiable and $\mathbb{Z}$ is exponentially integrable (Theorem 11), and finite expansions if ϕ is in some subset of $\mathcal{AV}^{(3(n-2))}$ and $\mathbb{Z}$ has a finite absolute moment of order n (Theorem 12). As an application, we obtain full expansions when ϕ is a polynomial (Corollary 14). Two main features should be stressed. On the one hand, we provide very simple proofs of the preceding results based on the differential calculus developed in Section 2. On the other, the 'little o' terms appearing in the finite expansions are estimated from above in a recursive way (see Theorem 12 and Remark 13). This is possible because our proofs rely upon certain nested-type formulas which allow us to estimate $E\phi(Z(t)) - E\phi(Z)$ in terms of $E\phi^{(k)}(Z(t)) - E\phi^{(k)}(Z)$ for $k \geq 3$ (see (3.16) and (3.21)). Similar recursive procedures appear when Stein's method is used (cf. Barbour [10] and Rotar [38]). Another interesting fact is that the moments of $Z(t)$ decreasingly converge to the corresponding moments of Z.

The best possible rate of convergence for $E\phi(Z(t)) - E\phi(Z)$ is $t^{-1/2}$, no matter how much smooth ϕ is. However, this rate may increase for certain functions ϕ. Even, we can find functions ϕ depending on the Hermite polynomials for which the corresponding rate of convergence is faster than any fixed power of $t^{-1/2}$. In the context of standardized sums of independent identically distributed random variables, such properties are well known (see, for instance, Esseen [23], von Bahr [48], Hall [27], and Borisov and Skilyagina [15]).

We shall need the following notations. The Hermite polynomials $H_m(x)$ may be defined in various equivalent ways (cf. Chihara [20, Chap. V]). For instance, by the Rodrigues formula

$$H_m(x) := (-1)^m \frac{\eta^{(m)}(x)}{\eta(x)} = E(x + iZ)^m, \qquad x \in \mathbb{R}, \ \ m \in \mathbb{N} \cup \{0\}, \tag{3.1}$$

where i is the imaginary unit. Apparently, the second equality in (3.1) has been noticed by Withers [49] for the first time. We have the orthogonality property

$$EH_m(Z)H_n(Z) = m!\, \delta_{m,n}, \qquad m, n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

On the other hand, suppose that $\phi \in \mathcal{A}^{(m)}$ with $\phi^{(m)} \in \mathcal{E}_\lambda$, for some $m \in \mathbb{N}$ and $\lambda \geq 0$. Using integration by parts, it can be checked (see also Barbour [10]) that

$$E\phi(Z)H_m(Z) = E\phi^{(k)}(Z)H_{m-k}(Z), \qquad k = 0, \dots, m. \tag{3.3}$$

Let $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$, $i = 1, \dots, m$. We denote by

$$a(k_1) := \frac{ET^{k_1+1}}{(k_1+2)!\,\sigma^{k_1+2}} \tag{3.4}$$

and, for $m \geq 2$,

$$a(k_1, \dots, k_m)$$

$$:= \frac{k_{m-1}}{k_m + k_{m-1}} \frac{k_{m-2}}{k_m + k_{m-1} + k_{m-2}} \cdots \frac{k_1}{k_m + \dots + k_1} a(k_1) \dots a(k_m).$$

Let $m, l \in \mathbb{N}$ with $m \leq l$. We denote by $D_m(l) := \{(k_1, \dots, k_m) \in \mathbb{N}^m : k_1 + \dots + k_m = l\}$ and define the positive constants

$$\alpha_m(l) := \sum_{D_m(l)} a(k_1, \dots, k_m). \tag{3.5}$$

With the preceding notations, we state the following.

Theorem 11 *Let $\lambda > 0$ and $0 < \alpha < \lambda\sigma$. Let $\phi \in \mathcal{A}^{(\infty)}$ be such that $\phi^{(k)} \in \mathcal{E}_\alpha$, $k \in \mathbb{N}$. If $Ee^{\lambda T} < \infty$, then*

$$E\phi(Z(t)) - E\phi(Z) = \sum_{l=1}^{\infty} \gamma(\phi, l) t^{-l/2}, \qquad t \geq 1, \tag{3.6}$$

where

$$\gamma(\phi, l) := \sum_{m=1}^{l} \alpha_m(l) E\phi^{(l+2m)}(Z) = \sum_{m=1}^{l} \alpha_m(l) E\phi(Z) H_{l+2m}(Z). \tag{3.7}$$

Moreover, for any $l \in \mathbb{N}$, we have the recursion formula

$$\gamma(\phi, l) = a(l)E\phi^{(l+2)}(Z) + \frac{1}{l}\sum_{k=1}^{l-1} ka(k)\gamma(\phi^{(k+2)}, l-k). \tag{3.8}$$

In the setting of Theorem 11, we see that the coefficient $\gamma(\phi, l)$ requires the finiteness of ET^{l+1} and involves some derivatives of ϕ up to the order $3l$. Also, if l is odd (resp. even), then $\gamma(\phi, l)$ only depends upon the odd (resp. even) derivatives $\phi^{(l+2)}, \ldots, \phi^{(l+2l)}$, or, alternatively, upon the odd (resp. even) Hermite polynomials $H_{l+2}(x), \ldots, H_{l+2l}(x)$. On the other hand, it follows from (3.4) and (3.5) that the first two coefficients in (3.6) are given by

$$\gamma(\phi, 1) = \frac{ET^2}{6\sigma^3}E\phi^{(3)}(Z), \quad \gamma(\phi, 2) = \frac{ET^3}{24\sigma^4}E\phi^{(4)}(Z) + \frac{1}{2}\left(\frac{ET^2}{6\sigma^3}\right)^2 E\phi^{(6)}(Z).$$

This implies that the rate of convergence in (3.6) is at least t^{-1} for those functions ϕ such that $E\phi^{(3)}(Z) = E\phi(Z)H_3(Z) = 0$, in particular, for symmetric functions ϕ.

The following result is a generalization of Corollary 3.

Theorem 12 *Let $t \geq 1$ and $n = 3, 4, \ldots$. Let $\phi \in \mathcal{AV}^{(3(n-2))}$ be such that $\phi^{(k)} \in \mathcal{AV}^{(0)}$, $k = n, \ldots, 3(n-2)$. If $ET^{n-1} < \infty$, then*

$$E\phi(Z(t)) - E\phi(Z) - \sum_{l=1}^{n-2} \gamma(\phi, l)t^{-l/2} = o_{n-2}(t)V_n^*(\phi)t^{-(n-2)/2}, \tag{3.9}$$

where $V_n^*(\phi) := \max\{V(\phi^{(r)}),\ r = n, \ldots, 3(n-2)\}$ and $o_{n-2}(t)$ satisfies the recurrence relations

$$\begin{aligned} |o_{n-2}(t)| &\leq \sum_{k=1}^{n-3} \frac{ka(k)}{n-2}|o_{n-k-2}(t)| \\ &+ \frac{2}{(n-1)!\,(n-2)\sigma^n}\left(ET^{n-1}\Delta(t) + ET^{n-1}w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right)\right), \end{aligned} \tag{3.10}$$

with

$$|o_1(t)| \leq \frac{1}{\sigma^3}\left(ET^2\Delta(t) + ET^2 w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right)\right). \tag{3.11}$$

The relation between the moment assumption $ET^{n-1} < \infty$ and the number of exact terms in (3.9) is optimal. This is due to the fact that the coefficient $\gamma(\phi, n-2)$ in (3.9) involves ET^{n-1}, as noted after Theorem 11. Inequalities (3.10) and (3.11) give us a relatively simple upper bound for the remainder term, which depends on ϕ only through $V_n^*(\phi)$. The price to pay for it is that the degree of smoothness of ϕ in Theorem 12 is not the best possible (this will be remedied in the following section).

Remark 13 *Since $ET^{n-1} < \infty$, it follows from* (3.10) *and* (3.11) *that all of the terms on the right-hand side in* (3.10) *are of order $t^{-1/2}$ at least, excepting the term*

$$ET^{n-1} w_1 \left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}} \right). \tag{3.12}$$

Actually, we have as in Remark 4 *that*

$$ET^k w_1 \left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}} \right) \leq \left(\frac{\|\eta\|}{\sigma} + 2 \right) ET^{k+1} t^{-1/2}, \quad k = 2, \ldots, n-2.$$

Under the assumption that $ET^{n-1+p} < \infty$, for some $0 < p \leq 1$, the term in (3.12) *is of order $t^{-p/2}$ at least. The proof of the later statement is again similar to that in Remark* 4.

For any real x, denote by $\lfloor x \rfloor$ the integer part of x. We also set $x \wedge y := \min(x, y)$.

Corollary 14 *Let $t \geq 1$ and $n = 3, 4, \ldots$. If $ET^{n-1} < \infty$, then*

$$EZ(t)^n - EZ^n = \sum \gamma(l) t^{-l/2}, \tag{3.13}$$

where the sum is taken over all odd or even integers from 1 *to $n-2$, according to n is odd or even, respectively, and*

$$\gamma(l) := n! \sum_{m=1}^{n(l)} \frac{\alpha_m(l)}{(n-l-2m)!} EZ^{n-l-2m}, \qquad n(l) := \left\lfloor \frac{n-l}{2} \right\rfloor \wedge l. \tag{3.14}$$

Moreover,

$$EH_n(Z(t)) = n! \sum_{m=1}^{\lfloor n/3 \rfloor} \alpha_m(n-2m) t^{-(n-2m)/2}. \tag{3.15}$$

As a consequence, $EZ(t)^n$ *and* $EH_n(Z(t))$ *decrease to* EZ^n *and* 0*, respectively, as* $t \to \infty$*.*

From (3.13), we see that odd moments converge at the rate $t^{-1/2}$, while even moments do at t^{-1}. This property was already shown by von Bahr [48] in the context of normalized sums of independent identically distributed random variables. On the other hand, from (3.15), we see that the rate of convergence for the n-th Hermite polynomial is $t^{-n/2+\lfloor n/3 \rfloor}$ (in this regard, see Esseen [24, p. 5]). Therefore, under the integrability assumptions in Theorem 11, we can find nontrivial functions for which the rate of convergence is faster than any fixed power of $t^{-1/2}$.

PROOF OF THEOREM 11. Let $t \geq 1$. For any $k \in \mathbb{N}$, denote by $g_k(u) := -u^{-k/2}$, $u > 0$. Applying Lemma 5 to the integrand in (2.1) and recalling Lemma 6(c), it is not hard to see that we have the nested-type formula

$$\begin{aligned} E\phi(Z(t)) - E\phi(Z) &= \sum_{k_1=1}^{\infty} a(k_1) E\phi^{(k_1+2)}(Z) t^{-k_1/2} \\ &+ \sum_{k_1=1}^{\infty} a(k_1) \int_t^{\infty} \left(E\phi^{(k_1+2)}(Z(u_1)) - E\phi^{(k_1+2)}(Z) \right) dg_{k_1}(u_1), \end{aligned} \tag{3.16}$$

where $a(k_1)$ is defined in (3.4). Applying successively (3.16) to the integrand in (3.16), we arrive at

$$E\phi(Z(t)) - E\phi(Z) = \sum_{k_1=1}^{\infty} a(k_1) E\phi^{(k_1+2)}(Z) t^{-k_1/2} + \sum_{m=2}^{\infty} A_m(t), \tag{3.17}$$

where

$$A_m(t) := \sum_{k_1,\ldots,k_m=1}^{\infty} a(k_1)\ldots a(k_m) E\phi^{(k_1+\ldots+k_m+2m)}(Z)\times$$

$$\times \int_t^\infty dg_{k_1}(u_1) \int_{u_1}^\infty dg_{k_2}(u_2)\ldots \int_{u_{m-2}}^\infty u_{m-1}^{-k_m/2}\, dg_{k_{m-1}}(u_{m-1}) \tag{3.18}$$

$$= \sum_{k_1,\ldots,k_m=1}^{\infty} a(k_1,\ldots,k_m) E\phi^{(k_1+\ldots+k_m+2m)}(Z) t^{-(k_1+\ldots+k_m)/2}.$$

Thus, it suffices to rearrange the terms in (3.17) and (3.18) in powers of $t^{-1/2}$ to show formula (3.6). On the other hand, let $u \geq t$ and $k \in \mathbb{N}$. Since

$$E\phi^{(k+2)}(Z(u)) - E\phi^{(k+2)}(Z) = \sum_{r=1}^{\infty} \gamma(\phi^{(k+2)}, r)\, u^{-r/2},$$

as follows from (3.6), we have from (3.16) that

$$E\phi(Z(t)) - E\phi(Z) = \sum_{k=1}^{\infty} a(k) E\phi^{(k+2)}(Z)\, t^{-k/2}$$

$$+ \sum_{k=1}^{\infty}\sum_{r=1}^{\infty} \frac{k}{k+r} a(k)\gamma(\phi^{(k+2)}, r)\, t^{-(k+r)/2}. \tag{3.19}$$

Formula (3.8) follows by equating the coefficient of $t^{-l/2}$ in (3.6) and (3.19). This completes the proof. □

PROOF OF THEOREM 12. Let $u \geq t \geq 1$, $n = 3, 4, \ldots$ and let g_k be as in the previous proof. Since $\phi^{(n)} \in \mathcal{AV}^{(0)}$, then $\phi \in \mathcal{B}_n$ and therefore $E\phi(Z(t))$ exists thanks to Lemma 6(a). Applying Lemma 5 to the integrand in (2.1), we obtain

$$E\phi^{(1)}\left(Z(u)+\frac{TS_1}{\sigma\sqrt{u}}\right)-E\phi^{(1)}\left(Z(u)+\frac{TS_2}{\sigma\sqrt{u}}\right)$$

$$\begin{aligned}&=\sum_{k=1}^{n-2}\frac{k\sigma}{2}\,a(k)E\phi^{(k+2)}(Z)u^{-(k+1)/2}\\&+\sum_{k=1}^{n-3}\frac{k\sigma}{2}\,a(k)\left(E\phi^{(k+2)}(Z(u))-E\phi^{(k+2)}(Z)\right)u^{-(k+1)/2}\\&+\frac{R_{n-1}(\phi^{(n)};u)}{(n-1)!\,\sigma^{n-1}}\,u^{-(n-1)/2},\end{aligned}\tag{3.20}$$

where $R_{n-1}(\phi^{(n)};u)$ was defined in (2.3). Therefore, by Remark 8, (2.1) and (3.20), we have the nested-type formula

$$\begin{aligned}E\phi(Z(t))-E\phi(Z)&=\sum_{k=1}^{n-2}a(k)E\phi^{(k+2)}(Z)t^{-k/2}\\&+\sum_{k=1}^{n-3}a(k)\int_t^{\infty}\left(E\phi^{(k+2)}(Z(u))-E\phi^{(k+2)}(Z)\right)\,dg_k(u)\\&+\frac{1}{(n-1)!\,\sigma^n}\int_t^{\infty}R_{n-1}(\phi^{(n)};u)\,u^{-n/2}du.\end{aligned}\tag{3.21}$$

By Lemma 10, the last term on the right-hand side in (3.21) is bounded above by

$$\frac{2V(\phi^{(n)})}{(n-1)!(n-2)\sigma^n}\left(ET^{n-1}\Delta(t)+ET^{n-1}w_1\left(\mathfrak{N};\frac{T}{\sigma\sqrt{t}}\right)\right)t^{-(n-2)/2}.\tag{3.22}$$

To show Theorem 12, we shall use induction in (3.21). Let $k=1,\dots,n-3$. By induction hypothesis, we have that

$$
E\phi^{(k+2)}(Z(u)) - E\phi^{(k+2)}(Z) = \sum_{l=1}^{n-k-2} \gamma(\phi^{(k+2)}, l)\, u^{-l/2}
$$

$$
+\, o_{n-k-2}(u) V^*_{n-k}\left(\phi^{(k+2)}\right)\, u^{-(n-k-2)/2}, \qquad u \geq t. \tag{3.23}
$$

Observe that

$$
V^*_{n-k}\left(\phi^{(k+2)}\right) = \max\left\{V(\phi^{(k+2+r)}),\ r = n-k, \ldots, 3(n-k-2)\right\}
$$

$$
= \max\left\{V(\phi^{(r)}),\ r = n+2, \ldots, 3(n-2)+2-2k\right\} \leq V^*_n(\phi), \tag{3.24}
$$

and

$$
\int_t^\infty u^{-(n-k-2)/2}\, dg_k(u) = \frac{k}{n-2}\, t^{-(n-2)/2}. \tag{3.25}
$$

Therefore, Theorem 12 follows from (3.8) and (3.21)–(3.25). The proof is complete. □

PROOF OF COROLLARY 14. Formulae (3.13) and (3.14) readily follow from Theorem 12 and (3.7). Again by Theorem 12, (3.7) and (3.2), we have that

$$
EH_n(Z(t)) = \sum_{l=1}^{n-2} \gamma(H_n, l) t^{-l/2}, \qquad t \geq 1, \tag{3.26}
$$

where

$$
\gamma(H_n, l) := \sum_{m=1}^{\lfloor (n-l)/2 \rfloor \wedge l} \alpha_m(l) EH_n(Z) H_{l+2m}(Z)
$$

$$
= n! \sum_{m=1}^{\lfloor (n-l)/2 \rfloor \wedge l} \alpha_m(l) \delta_{n,l+2m}. \tag{3.27}
$$

Observe that $\{(l,m) \in \mathbb{N}^2 : 1 \le l \le n-2, 1 \le m \le \lfloor (n-l)/2 \rfloor \wedge l, l+2m = n\} = \{(l,m) \in \mathbb{N}^2 : 1 \le m \le \lfloor n/3 \rfloor, l = n-2m\}$. Hence, we have from (3.26) and (3.27) that

$$EH_n(Z(t)) = n! \sum_{m=1}^{\lfloor n/3 \rfloor} \alpha_m(n-2m) t^{-(n-2m)/2}, \quad t \ge 1,$$

thus showing (3.15). Finally, the statement on monotonic convergence is due to the fact that each coefficient $\alpha_m(l)$ is positive, as follows from (3.5). The proof is complete. □

4 Moderately smooth functions

In this section, we give finite expansions for $E\phi(Z(t)) - E\phi(Z)$ when the standardized subordinator $\mathbb{Z}$ has a finite absolute moment of order $n \ge 3$ and ϕ is in the set $\mathcal{AV}^{(j)}$, with $n-2 \le j \le n$. Since ϕ is not differentiable enough, we cannot directly apply the differential calculus developed in the preceding section and therefore need to consider some smooth approximants of ϕ. We consider approximants of the form $P_h\phi(x) = E\phi(x+hY)$ built up from a random variable Y satisfying nice integrability properties and whose characteristic function vanishes outside $[-1,1]$. This kind of approximants have been widely used in the literature (see, for instance, Esseen [23], Bhattacharya and Ranga Rao [11, Chap. 2], and Götze and Hipp [26]). As in Theorem 12, the upper bounds for the remainder terms are given in a recursive way. We provide in Theorem 15 specific upper bounds when $n = 3$ and $j = 1,2,3$. For $n \ge 4$, the upper bounds for the remainders are given in terms of those corresponding to smaller values of n. This is done in Theorem 16 and also in Lemma 20, which is the main auxiliary result in this section.

Together with the notations in the previous section, we shall use here the following. Let $t \ge 1$ and $j \in \mathbb{N}$. If $ET^{j+1} < \infty$, we consider the distribution function $\mathfrak{N}_{t,j}$ of the finite signed measure on $\mathbb{R}$ whose density is given by

$$\eta_{t,j}(z) := \left(1 + \sum_{l=1}^{j} \left(\sum_{m=1}^{l} \alpha_m(l) H_{l+2m}(z)\right) t^{-l/2}\right) \eta(z), \quad z \in \mathbb{R}. \tag{4.1}$$

To unify the notation, we also set $\mathfrak{N}_{t,0} := \mathfrak{N}$. Define

$$\varepsilon := \varepsilon(t) = \frac{b}{\sigma\sqrt{t}}, \qquad b := \frac{2ET^2}{3\sigma^2}. \tag{4.2}$$

For any $N \in \mathbb{N}$, let Y_N be a random variable with probability density

$$\rho_N(z) := K_N \left(\frac{2N}{z} \sin \frac{z}{2N} \right)^{2N}, \qquad z \in \mathbb{R}, \tag{4.3}$$

where K_N is an appropriate positive constant. Finally, denote by $\tilde{e}(x) := e^{ix}$, $x \in \mathbb{R}$. We enunciate the following.

Theorem 15 *Assume that $ET^2 < \infty$. For any $t \geq 1$, we have*

(a) If $\phi \in \mathcal{AV}^{(j)}$, $j = 1, 2$, then

$$\left| E\phi(Z(t)) - E\phi(Z) - \frac{ET^2}{6\sigma^3} E\phi(Z) H_3(Z) t^{-1/2} \right| \leq V(\phi^{(j)}) r_j(t),$$

where

$$\begin{aligned} r_j(t) &:= \frac{1}{\sigma^3 \sqrt{2\pi t}} \left(\frac{a(1) 2^{(6-j)/2} E|Z|^{5-j} ET^2}{\sqrt{t}} \right. \\ &\left. + (1 + E|Z|) ET^2 w_1 \left(\tilde{e}; \frac{T}{\sigma \sqrt{t}} \right) \right) + \frac{E|Y_3|^j}{j!} \| F_t - \mathfrak{N}_{t,1} \| \varepsilon^j. \end{aligned} \tag{4.4}$$

(b) If $\phi \in \mathcal{AV}^{(3)}$, then

$$\begin{aligned} &\left| E\phi(Z(t)) - E\phi(Z) - \frac{ET^2}{6\sigma^3} E\phi(Z) H_3(Z) t^{-1/2} \right| \\ &\leq \frac{V(\phi^{(3)})}{\sigma^3 \sqrt{t}} \left(ET^2 \Delta(t) + ET^2 w_1 \left(\mathfrak{N}; \frac{T}{\sigma \sqrt{t}} \right) \right) \\ &+ \frac{V(\phi^{(3)}) EY_3^2}{2} r_1(t) \varepsilon^2 + \frac{V(\phi^{(3)}) E|Y_3|^3}{6} \| F_t - \mathfrak{N}_{t,1} \| \varepsilon^3, \end{aligned}$$

where $r_1(t)$ is defined in (4.4).

Observe that the upper bounds in Theorem 15 depend on ϕ only through $V(\phi^{(j)})$. Every term in such upper bounds is of order t^{-1} at least, excepting those involving the expression $ET^2 w_1(f; T/(\sigma\sqrt{t}))$, where $f = \tilde{e}$ or $\mathfrak{N}$. The order of such terms is $t^{-(p+1)/2}$ at least, provided that $ET^{2+p} < \infty$, $0 < p \leq 1$ (see Remark 4).

For a certain moderately smooth function f, suppose that $Ef(Z(t)) - Ef(Z)$ is expressible as a sum of r exact terms and one remainder term. In accordance with the notations in Theorem 12 and recalling (3.7) and (4.1), we write in this case

$$Ef(Z(t)) - Ef(Z) - \sum_{l=1}^{r} \gamma(f,l) t^{-l/2} = \int_{\mathbb{R}} f(z)\, d(F_t - \mathfrak{N}_{t,r})(z) \tag{4.5}$$
$$= o_r(f;t) t^{-r/2}.$$

On the other hand, for any $h > 0$ and $N \in \mathbb{N}$, we consider the approximant

$$P_h f(x) := Ef(x + hY_N), \qquad x \in \mathbb{R}, \qquad f \in \mathcal{B}_{2(N-1)}. \tag{4.6}$$

Finally, denote by $\lceil x \rceil$ the ceiling of x, that is, the smallest integer not less than x. With the preceding notations, we give the following.

Theorem 16 *Let $n = 4, 5, \ldots$, $n - 2 \leq j \leq n$, $N := \lceil 1 + n/2 \rceil$ and let ε be as in (4.2). Assume that $\phi \in \mathcal{AV}^{(j)}$ and that $ET^{n-1} < \infty$. For any $t \geq 1$, we have*

$$\left| \int_{\mathbb{R}} \phi(z)\, d(F_t - \mathfrak{N}_{t,n-2})(z) \right| \leq |o_{n-2}(P_\varepsilon \phi; t)|\, t^{-(n-2)/2}$$
$$+ \sum_{k=1}^{j-1} \frac{\varepsilon^k |EY_N^k|}{k!} \left| \int_{\mathbb{R}} \phi^{(k)}(z)\, d(F_t - \mathfrak{N}_{t,n-k-2})(z) \right|$$
$$+ \sum_{k=1}^{j-1} \frac{\varepsilon^k |EY_N^k|}{k!} \int_{\mathbb{R}} |\phi^{(k)}(z)|\, |\eta_{t,n-2}(z) - \eta_{t,n-k-2}(z)|\, dz$$
$$+ \frac{V(\phi^{(j)}) E|Y_N|^j}{j!} \|F_t - \mathfrak{N}_{t,n-2}\| \varepsilon^j,$$

where the term $|o_{n-2}(P_\varepsilon \phi; t)|$ is recursively bounded in Lemma 20.

Concerning Theorem 16, some remarks are in order. First, it is known that the relation between the integrability assumptions on $\mathbb{Z}$ and the degree of smoothness ϕ, that is, $n-2 \leq j \leq n$, cannot be improved (cf. Götze and Hipp [26] and Borisov *et al.* [14]). Indeed, if $ET^{n-1} < \infty$, it follows from Lemma 6(a) that $\mathbb{Z}$ has a finite absolute moment of order n, and this requires that $j \leq n$, since otherwise the existence of $E\phi(Z(t))$ is not guaranteed. Also, if $\phi \in \mathcal{AV}^{(j)}$ for some $j \in \mathbb{N}$, it follows from Lemma 5 and (4.6) that

$$P_\varepsilon \phi(z) = \sum_{k=0}^{j-1} \frac{\phi^{(k)}(z)}{k!} E(\varepsilon Y_N)^k + \frac{\varepsilon^j}{j!} EY_N^j \phi^{(j)}(z + \varepsilon Y_N \beta_j), \quad z \in \mathbb{R}, \tag{4.7}$$

where $\phi^{(j)} \in \mathcal{AV}^{(0)}$. Looking at the remainder term in (4.7), we see that, in order to obtain $n-2$ exact terms in the expansion of $E\phi(Z(t)) - E\phi(Z)$ it is required that $n-2 \leq j$ (see also the proof of Theorem 16).

In second place, let $k = 1, \ldots, j-1$. Note that $EY_N^k = 0$ if k is odd, because Y_N is symmetric. Taking into account (4.1) and the fact that $\phi^{(k)} \in \mathcal{AV}^{(j-k)}$, the last two upper bounds in Theorem 16 can be explicitly computed in terms of appropriate moments of Y_N and Z. Such bounds are of order $t^{-(n-1)/2}$ at least.

Finally, in the terminology of (4.5), Theorem 16 gives a recursive procedure to estimate the remainder term $o_{n-2}(\phi; t)$ when $\phi \in \mathcal{AV}^{(j)}$, $n-2 \leq j \leq n$. This is achieved by estimating $o_{n-2}(P_\varepsilon \phi; t)$ and $o_{n-k-2}(\phi^{(k)}; t)$ for all even integers k between 1 and $j-1$ (see the second upper bound in Theorem 16). In turn, it is shown in Lemma 20 below that $o_{n-2}(P_\varepsilon \phi; t)$ can be recursively estimated in terms of $o_{n-k-2}((P_\varepsilon \phi)^{k+2}; t)$, $k = 1, \ldots, n-3$. As a result, $o_{n-2}(\phi; t)$ depends on ϕ only through $V(\phi^{(j)})$, as we shall see in Lemma 20. Obviously, the upper bounds for the remainders in Theorem 16 are more cumbersome than those in Theorem 12. This is the price to be paid if we wish to obtain Edgeworth expansions with optimal correlation between the moment assumptions on $\mathbb{Z}$ and the degree of differentiability of ϕ.

To prove the preceding theorems, some auxiliary results will be needed. If $g \in \mathcal{AV}^{(0)}$, we denote by $\hat{g}$ its Fourier-Stieltjes transform. The approximants in (4.6) satisfy the following.

Lemma 17 *Let $h > 0$, $N \in \mathbb{N}$ and let $f \in \mathcal{AV}^{(r)}$ for some $r = 0, \ldots, 2(N-1)$. Then,*

$$(P_h f)^{(r)}(x) = Ef^{(r)}(x + hY_N), \qquad x \in \mathbb{R}. \tag{4.8}$$

Moreover, for any $x, y \in \mathbb{R}$ and $m = r+1, r+2, \ldots,$ we have

$$\begin{aligned}&(P_h f)^{(m)}(y) - (P_h f)^{(m)}(x)\\ &= \frac{1}{2\pi}\int_{-1/h}^{1/h} \frac{e^{i\zeta y} - e^{i\zeta x}}{i\zeta}(i\zeta)^{m-r}\widehat{f^{(r)}}(-\zeta) E e^{i\zeta h Y_N}\, d\zeta.\end{aligned} \tag{4.9}$$

PROOF. Since $f \in \mathcal{AV}^{(r)}$, then $f^{(k)} \in \mathcal{B}_{2(N-1)}$, $k = 0, \ldots, r$. Therefore, (4.8) follows by differentiating under the expectation sign in (4.6). Since the characteristic function of Y_N vanishes outside $[-1, 1]$, we have from Fourier inversion (cf. Feller [25, p. 509]) that

$$\rho_N^{(n)}(z) = \frac{1}{2\pi}\int_{-1}^{1} e^{-i\zeta z}(-i\zeta)^n E e^{i\zeta Y_N}\, d\zeta, \qquad z \in \mathbb{R}, \quad n \in \mathbb{N} \cup \{0\}. \tag{4.10}$$

Let $m = r+1, r+2, \cdots$ and denote by $g := f^{(r)} \in \mathcal{AV}^{(0)}$. The function $z^{2N}\rho_N^{(n)}(z)$, $z \in \mathbb{R}$, is bounded for any $n \in \mathbb{N}$. Indeed, since (4.3) can be written as $z^{2N}\rho_N(z) =: h(z)$, where $h^{(n)}(z)$ is bounded for any n, the assertion follows by differentiating repeatedly in the equality $\rho_N^{(1)}(z) = h^{(1)}(z) z^{-2N} - 2N\rho_N(z) z^{-1}$. Therefore, we can differentiate $m - r$ times in (4.8) to obtain

$$(P_h f)^{(m)}(x) = \frac{(-1)^{m-r}}{h^{m-r}} \int_{\mathbb{R}} g(x + hz)\rho_N^{(m-r)}(z)\, dz, \qquad x \in \mathbb{R}. \tag{4.11}$$

Let $x, y \in \mathbb{R}$ with $x \leq y$. By (4.11), Fubini's theorem and (4.10), we get

$$(P_h f)^{(m)}(y) - (P_h f)^{(m)}(x) = \frac{(-1)^{m-r}}{h^{m-r}} \int_{\mathbb{R}} \rho_N^{(m-r)}(z)\, dz \int_{(x+hz, y+hz]} dg(u)$$

$$= \frac{(-1)^{m-r}}{h^{m-r}} \int_{\mathbb{R}} dg(u) \int_{(u-y)/h}^{(u-x)/h} \rho_N^{(m-r)}(z)\, dz$$

$$= \frac{1}{2\pi h^{m-r}} \int_{\mathbb{R}} dg(u) \int_{-1}^{1} (i\zeta)^{m-r} E e^{i\zeta Y_N}\, d\zeta \int_{(u-y)/h}^{(u-x)/h} e^{-i\zeta z}\, dz$$

$$= \frac{1}{2\pi}\int_{-1/h}^{1/h} \frac{e^{i\zeta y} - e^{i\zeta x}}{i\zeta}(i\zeta)^{m-r}\hat{g}(-\zeta) E e^{i\zeta h Y_N}\, d\zeta,$$

thus completing the proof. $\square$

In the following auxiliary result, we bound remainder terms of the form (2.3) involving the derivatives of the approximants $P_h f$.

Lemma 18 *Let $h > 0$, $N \in \mathbb{N}$, $u \geq t \geq 1$ and $k \in \mathbb{N}$. Assume that $\phi \in \mathcal{AV}^{(r)}$ for some $r = 0, \ldots, 2(N-1)$ and that $ET^k < \infty$. Then,*

$$|R_k((P_h\phi)^{(r)}; u)| \leq V(\phi^{(r)}) \left(ET^k \Delta(t) + ET^k w_1 \left(\Phi; \frac{T}{\sigma\sqrt{t}} \right) \right)$$

and, for $m = r+1, r+2, \ldots$,

$$|R_k((P_h\phi)^{(m)}; u)| \leq \frac{V(\phi^{(r)})}{2\pi} \left(ET^k \int_{-1/h}^{1/h} |\zeta|^{m-r-1} |Ee^{i\zeta Z(u)} - Ee^{i\zeta Z}| \, d\zeta \right.$$

$$\left. + \sqrt{2\pi} E(1+|Z|)|Z|^{m-r-1} ET^k w_1 \left(\tilde{e}; \frac{T}{\sigma\sqrt{t}} \right) \right).$$

PROOF. By (4.8), $V((P_h\phi)^{(r)}) \leq V(\phi^{(r)})$. Hence, the first upper bound in Lemma 18 follows as in the proof of Lemma 10.

Let $u \geq t \geq 1$ and $m = r+1, r+2, \ldots$. Set $g := \phi^{(r)} \in \mathcal{AV}^{(0)}$ and denote indistinctly by S each one of the random variables S_1 and S_2. By (4.9) and Fubini's theorem we have that

$$\left| E(TS)^k \left((P_h\phi)^{(m)} \left(Z(u) + \frac{TS\beta_k}{\sigma\sqrt{u}} \right) - (P_h\phi)^{(m)} \left(Z + \frac{TS\beta_k}{\sigma\sqrt{u}} \right) \right) \right|$$

$$\leq \frac{1}{2\pi} \int_{-1/h}^{1/h} |\zeta|^{m-r-1} \left| Ee^{i\zeta Z(u)} - Ee^{i\zeta Z} \right| \left| \hat{g}(-\zeta) E(TS)^k \exp\left(\frac{i\zeta TS\beta_k}{\sigma\sqrt{u}} \right) \right| d\zeta$$

$$\leq \frac{V(g)}{2\pi} E(TS)^k \int_{-1/h}^{1/h} |\zeta|^{m-r-1} |Ee^{i\zeta Z(u)} - Ee^{i\zeta Z}| \, d\zeta,$$

where we have used that $|\hat{g}(-\zeta)| \leq V(g)$, $\zeta \in \mathbb{R}$. Again by (4.9), Fubini's theorem and (2.4), we obtain

$$
\left| E(TS)^k \left((P_h\phi)^{(m)} \left(Z + \frac{TS\beta_k}{\sigma\sqrt{u}} \right) - (P_h\phi)^{(m)}(Z) \right) \right|
$$

$$
\leq \frac{1}{2\pi} \int_{-1/h}^{1/h} E(TS)^k \left| \exp\left(\frac{i\zeta TS\beta_k}{\sigma\sqrt{u}} \right) - 1 \right| \left| \zeta^{m-r-1} E e^{i\zeta Z} \right| |\hat{g}(-\zeta)| \, d\zeta
$$

$$
\leq \frac{V(g)}{2\pi} E(TS)^k w_1 \left(\tilde{e}; \frac{TS\beta_k}{\sigma\sqrt{u}} \right) \int_{-1/h}^{1/h} (1+|\zeta|)|\zeta|^{m-r-1} e^{-\zeta^2/2} \, d\zeta
$$

$$
\leq \frac{V(g)}{\sqrt{2\pi}} ES^k ET^k w_1 \left(\tilde{e}; \frac{T}{\sigma\sqrt{t}} \right) E(1+|Z|)|Z|^{m-r-1}.
$$

Since $ES^k \leq 1/2$, $k \in \mathbb{N}$, the second upper bound in Lemma 18 follows from (2.3) and the two preceding inequalities. The proof is complete. □

Lemma 19 *Let $u \geq 1$ and let $a(1)$ and b be as in* (3.4) *and* (4.2), *respectively. Asume that $ET^2 < \infty$. Whenever $|b\zeta| \leq \sigma\sqrt{u}$, we have*

$$
\left| E e^{i\zeta Z(u)} - E e^{i\zeta Z} \right| \leq \frac{a(1)}{\sqrt{u}} |\zeta|^3 e^{-\zeta^2/4}.
$$

PROOF. By Lemma 5, we have that

$$
E\frac{e^{i\zeta T} - 1}{T} = i\zeta - \frac{\zeta^2}{2} ET + \frac{(i\zeta)^3}{6} ET^2 e^{i\zeta T\beta_3},
$$

thus implying, by virtue of (1.3) and (1.4), that

$$
E e^{i\zeta Z(u)} = \exp\left(-\frac{\zeta^2}{2} + \frac{(i\zeta)^3}{6\sigma^3\sqrt{u}} ET^2 \exp\left(\frac{i\zeta T\beta_3}{\sigma\sqrt{u}} \right) \right). \tag{4.12}
$$

This, together with the inequality $|e^z - 1| \leq |z|e^{|z|}$, $z \in \mathbb{C}$, completes the proof. □

The following lemma is crucial. We give in it analogous statements to those in Theorem 12 for the smooth approximants $P_\varepsilon\phi$.

Lemma 20 *Let* $n = 3, 4, \ldots$, $N = \lceil 1 + n/2 \rceil$ *and let* ε *be as in* (4.2). *Assume that* $\phi \in \mathcal{AV}^{(r)}$ *for some* $r = 0, \ldots, n$ *and that* $ET^{n-1} < \infty$. *For any* $t \geq 1$, *we have*

$$EP_\varepsilon\phi(Z(t)) - EP_\varepsilon\phi(Z) - \sum_{l=1}^{n-2} \gamma(P_\varepsilon\phi, l) t^{-l/2} = o_{n-2}(P_\varepsilon\phi; t) t^{-(n-2)/2}, \quad (4.13)$$

where

$$\begin{aligned} |o_{n-2}(P_\varepsilon\phi; t)| \leq & \sum_{k=1}^{n-3} \frac{ka(k)}{n-2} \left| o_{n-k-2}\left((P_\varepsilon\phi)^{(k+2)}; t \right) \right| \\ & + \frac{2V(\phi^{(r)})}{(n-1)!(n-2)\sigma^n} c_{n,r}(t), \end{aligned} \quad (4.14)$$

where

$$c_{n,n}(t) := ET^{n-1}\Delta(t) + ET^{n-1} w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}} \right) \quad (4.15)$$

and, for $r < n$,

$$\begin{aligned} c_{n,r}(t) :=& \frac{a(1) 2^{(n-r+3)/2} E|Z|^{n-r+2} ET^{n-1}}{\sqrt{2\pi t}} \\ & + \frac{E(1+|Z|)|Z|^{n-r-1}}{\sqrt{2\pi}} ET^{n-1} w_1\left(\tilde{e}; \frac{T}{\sigma\sqrt{t}} \right). \end{aligned} \quad (4.16)$$

PROOF. Let $t \geq 1$. By assumption, $\phi \in \mathcal{B}_{n-r}$ and therefore $P_\varepsilon\phi$ and $EP_\varepsilon\phi(Z(t))$ are well defined, as follows from (4.6). We proceed as in the proof of Theorem 12 and use induction on n. Let $g_k(u) := -u^{-k/2}$, $u > 0$, $k \in \mathbb{N}$. As in (3.21), we have that

$$EP_\varepsilon\phi(Z(t)) - EP_\varepsilon\phi(Z) - \sum_{k=1}^{n-2} a(k)E(P_\varepsilon\phi)^{(k+2)}(Z)t^{-k/2}$$

$$- \sum_{k=1}^{n-3} a(k) \int_t^\infty \left(E(P_\varepsilon\phi)^{(k+2)}(Z(u)) - E(P_\varepsilon\phi)^{(k+2)}(Z)\right) dg_k(u) \tag{4.17}$$

$$= \frac{1}{(n-1)!\sigma^n} \int_t^\infty R_{n-1}\left((P_\varepsilon\phi)^{(n)}; u\right) u^{-n/2}\, du.$$

Let $u \geq t$. If $r = n$, it follows from Lemma 18 that

$$\left|R_{n-1}\left((P_\varepsilon\phi)^{(n)}; u\right)\right|$$
$$\leq V(\phi^{(r)}) \left(ET^{n-1}\Delta(t) + ET^{n-1}w_1\left(\mathfrak{N}; \frac{T}{\sigma\sqrt{t}}\right)\right). \tag{4.18}$$

If $r < n$, it can be checked from Lemmas 18 and 19 that

$$\left|R_{n-1}\left((P_\varepsilon\phi)^{(n)}; u\right)\right| \leq \frac{V(\phi^{(r)})}{\sqrt{2\pi}} \left(\frac{a(1)2^{(n-r+3)/2}E|Z|^{n-r+2}ET^{n-1}}{\sqrt{t}}\right.$$
$$\left. + E(1+|Z|)|Z|^{n-r-1}ET^{n-1}w_1\left(\tilde{e}; \frac{T}{\sigma\sqrt{t}}\right)\right). \tag{4.19}$$

Thus, the second term on the right-hand side in (4.14) and formulae (4.15) and (4.16) follow from (4.17)-(4.19).

By induction hypothesis, we have that

$$E(P_\varepsilon\phi)^{(k+2)}(Z(u)) - E(P_\varepsilon\phi)^{(k+2)}(Z)$$

$$= \sum_{l=1}^{n-k-2} \gamma\left((P_\varepsilon\phi)^{(k+2)}, l\right) u^{-l/2} \tag{4.20}$$

$$+ o_{n-k-2}\left((P_\varepsilon\phi)^{(k+2)}; u\right) u^{-(n-k-2)/2}.$$

Therefore, formulae (4.13) and (4.14) follow from (3.8), (4.17) and (4.20). The proof is complete. □

PROOF OF THEOREM 15. *Part (a).* Let $j = 1, 2$, $\phi \in \mathcal{AV}^{(j)}$ and let ε be as in (4.2). Since $EY_3 = 0$, we have from (4.7) that

$$P_\varepsilon\phi(z) = \phi(z) + \frac{\varepsilon^j}{j!} f_j(z), \qquad z \in \mathbb{R}, \tag{4.21}$$

where $f_j \in \mathcal{AV}^{(0)}$ is given by

$$f_j(z) := EY_3^j \phi^{(j)}(z + \varepsilon Y_3 \beta_j), \qquad z \in \mathbb{R}. \tag{4.22}$$

From (4.21), we see that

$$\int_{\mathbb{R}} \phi(z)\, d(F_t - \mathfrak{N}_{t,1})(z) = \int_{\mathbb{R}} P_\varepsilon\phi(z)\, d(F_t - \mathfrak{N}_{t,1})(z)$$
$$- \frac{\varepsilon^j}{j!} \int_{\mathbb{R}} f_j(z)\, d(F_t - \mathfrak{N}_{t,1})(z). \tag{4.23}$$

The first term on the right-hand side in (4.23) is bounded above by $V(\phi^{(j)})$ times the first term on the right-hand side in (4.4). This follows by applying Lemma 20 with $n = 3$. By (4.22), we have that $V(f_j) \le V(\phi^{(j)}) E|Y_3|^j$. Therefore, (2.26) implies that the second term on the right-hand side in (4.23) is bounded by $V(\phi^{(j)})$ times the corresponding term in (4.4). This completes the proof of part (a).

Part (b). The proof follows a similar pattern as that of part (a). From (4.7), we have that

$$\int_{\mathbb{R}} \phi(z)\, d(F_t - \mathfrak{N}_{t,1})(z) = \int_{\mathbb{R}} P_\varepsilon \phi(z)\, d(F_t - \mathfrak{N}_{t,1})(z)$$

$$- \frac{\varepsilon^2 E Y_3^2}{2} \int_{\mathbb{R}} \phi^{(2)}(z)\, d(F_t - \mathfrak{N}_{t,1})(z) - \frac{\varepsilon^3}{6} \int_{\mathbb{R}} f_3(z)\, d(F_t - \mathfrak{N}_{t,1})(z), \tag{4.24}$$

where $f_3 \in \mathcal{AV}^{(0)}$ is given by

$$f_3(z) := E Y_3^3 \phi^{(3)}(z + \varepsilon Y_3 \beta_3), \qquad z \in \mathbb{R}.$$

By Lemma 20 with $n = 3$, the first term on the right-hand side in (4.24) is bounded by

$$\frac{V(\phi^{(3)})}{\sigma^3 \sqrt{t}} \left(E T^2 \Delta(t) + E T^2 w_1 \left(\Phi; \frac{T}{\sigma \sqrt{t}} \right) \right).$$

Since $\phi^{(2)} \in \mathcal{AV}^{(1)}$, the second term on the right-hand side in (4.24) is bounded as in part (a) with $j = 1$. The third term is also bounded as in part (a). The proof is complete. □

PROOF OF THEOREM 16. The proof follows along the lines of that in Theorem 15 by observing that, thanks to (4.7), we can write

$$\int_{\mathbb{R}} \phi(z)\, d(F_t - \mathfrak{N}_{t,n-2})(z) = \int_{\mathbb{R}} P_\varepsilon \phi(z)\, d(F_t - \mathfrak{N}_{t,n-2})(z)$$

$$- \sum_{k=1}^{j-1} \frac{\varepsilon^k E Y_N^k}{k!} \int_{\mathbb{R}} \phi^{(k)}(z)\, d(F_t - \mathfrak{N}_{t,n-k-2} + \mathfrak{N}_{t,n-k-2} - \mathfrak{N}_{t,n-2})(z)$$

$$- \frac{\varepsilon^j}{j!} \int_{\mathbb{R}} f_j(z)\, d(F_t - \mathfrak{N}_{t,n-2})(z),$$

where $f_j \in \mathcal{AV}^{(0)}$ is given by

$$f_j(z) := E Y_N^j \phi^{(j)}(z + \varepsilon Y_N \beta_j), \qquad z \in \mathbb{R}.$$

The proof is complete. □

5 Functions of bounded variation

In this section, we consider the problem of estimating $E\phi(Z(t)) - E\phi(Z)$ when $\mathbb{Z}$ has a finite third moment and ϕ is a function of bounded variation. As is well known, such a problem heavily depends on whether or not F_t is a lattice distribution. In the nonlattice case, we give in Theorem 21 a one term Edgeworth expansion for $E\phi(Z(t))-E\phi(Z)$, providing at the same time an upper bound for the remainder term, which involves $V(\phi)$, $w_1(\tilde{e};\cdot)$ and a certain exponential term coming from the nonlattice character of the random variable T. In the lattice case, we give explicit lower and upper bounds of the same order of magnitude for $\|F_t - \mathfrak{N}\|_I$, where I is an arbitrary open interval and $\|\cdot\|_I$ is the usual supremum norm on I (Theorem 22). Both bounds contain the maximum jump of F_t over the interval I and are shown to be asymptotically sharp. This could be considered as an alternative to the classical approach to the lattice case. Methodologically, the second modulus of smoothness plays a key role.

The following notations will be used. We shall consider the approximants $P_h f(x) = Ef(x+hY_2)$ built up from a random variable Y_2 whose probability density is given by

$$\rho_2(z) := \frac{3}{8\pi}\left(\frac{4}{z}\sin\left(\frac{z}{4}\right)\right)^4, \qquad z\in\mathbb{R}. \tag{5.1}$$

We shall assume that $ET^2 < \infty$. In the comments following Corollary 3, it was noted that $\mu_3 := E(X(1)-1)^3 = ET^2$, thus implying that μ_3 is always positive. We simply denote by $\mathfrak{N}_t := \mathfrak{N}_{t,1}$, as defined in (4.1), that is,

$$d\mathfrak{N}_t(z) := \left(1+\frac{ET^2}{6\sigma^3}\,H_3(z)t^{-1/2}\right)\eta(z)\,dz, \qquad z\in\mathbb{R}. \tag{5.2}$$

It is readily seen from (1.3) and (1.4) that

$$\left|Ee^{i\zeta Z(u)}\right| = \exp\left(-uE\frac{1-\cos(\zeta T/(\sigma\sqrt{u}))}{T}\right), \qquad \zeta\in\mathbb{R}, \quad u\geq 1. \tag{5.3}$$

This implies (cf. Feller [25, p. 501]) that $Z(u)$ has a lattice distribution with maximal span $s/(\sigma\sqrt{u}) > 0$ for any $u \geq 1$, if and only if T has a lattice distribution with maximal span $s>0$. Denote by

$$q_\tau := \min\left\{E\frac{1-\cos(xT)}{T} : \; b^{-1} \le x \le \tau^{-1}\right\}, \qquad 0 < \tau \le b, \tag{5.4}$$

where b is defined in (4.2). Observe that if T has a nonlattice distribution, then $q_\tau > 0$ for any $0 < \tau \le b$.

With these notations, we state the following.

Theorem 21 *Let $\phi \in \mathcal{AV}^{(0)}$ and $t \ge 1$. Assume that $ET^2 < \infty$ and that T has a nonlattice distribution. For any arbitrary small $0 < \tau \le b$, we have*

$$\int_{\mathbb{R}} \phi(z)\, d(F_t - \mathfrak{N}_t)(z) = V(\phi)\left(\frac{o_\tau(t)}{\pi\sigma^3} + \tau O(t)\right) t^{-1/2}, \tag{5.5}$$

where

$$\begin{aligned} |o_\tau(t)| \le{}& \frac{(ET^2)^2}{\sigma^3}\left(2^7 + \frac{1}{3}\left(\frac{\sigma\sqrt{t}}{\tau}\right)^6 e^{-q_\tau t}\right) t^{-1/2} \\ &+ (4+\sqrt{2\pi})ET^2 w_1\left(\tilde{e}; \frac{T}{\sigma\sqrt{t}}\right) \end{aligned} \tag{5.6}$$

and

$$|O(t)| \le \frac{6}{\sigma\sqrt{2\pi}}\left(1 + \frac{ET^2}{6\sigma^3}\, t^{-1/2}\right). \tag{5.7}$$

In view of (5.5) and (5.6), it would be desirable to find a function $\tau := \tau(t)$ converging to zero as $t \to \infty$, in such a way that $(\tau^{-1}(t)\sqrt{t})^6 \exp(-tq_{\tau(t)})$ remains bounded. We do not know how to construct such a function $\tau(t)$ for a general random variable T satisfying the requirements in Theorem 21. This seems possible, however, for specific random variables T, but will not be considered here. In any case, Theorem 21 appears more informative than the usual results stating that

$$\int_{\mathbb{R}} \phi(z)\, d(F_t - \mathfrak{N}_t)(z) = o(t),$$

without any further knowledge on the function $o(t)$.

To deal with the lattice case, we shall need the following. Let I be a real interval. For any function f, we consider the following local second modulus of smoothness of f at length $r \geq 0$

$$w_2(f; I; r) := \sup \{|f(x+h) - 2f(x) + f(x-h)| : x \in I,\ 0 \leq h \leq r\}.$$

If f has right and left limits at each point $x \in I$, respectively denoted by $f_+(x)$ and $f_-(x)$, we define

$$D(f, I) := \sup \{|f_+(x) - f_-(x)| :\ x \in I\}.$$

For any $r \geq 0$, we denote by $I(r) := \{x \in \mathbb{R} :\ d(x, I) \leq r\}$, where $d(\cdot, \cdot)$ is the usual euclidean distance. We also denote by

$$K(I) := \sup_{x \in I} \left| E 1_{[x,\infty)}(Z) H_3(Z) \right| = \sup_{x \in I} |\eta(x) H_2(x)|. \tag{5.8}$$

Suppose that T has a lattice distribution with maximal span $s > 0$. Since $X(0) = 0$, the random variables $(X(t))_{t \geq 0}$ and T take on values in the set $s\mathbb{N} \cup \{0\}$. In such a case, we define

$$l(r) := \min \left\{ E \frac{1 - \cos(\zeta T/s)}{T} :\ r \leq |\zeta| \leq \pi \right\}, \quad 0 < r \leq \pi. \tag{5.9}$$

Observe that $l(r) > 0$ for $0 < r \leq \pi$. Concerning the lattice case, we give the following.

Theorem 22 *Let I be an open interval and let ε and b as in* (4.2). *Assume that $ET^2 < \infty$ and that T has a lattice distribution with maximal span $s > 0$. For any $t \geq 1$, we have*

$$\begin{aligned} \frac{1}{2} D(F_t, I) \leq \|F_t - \mathfrak{N}\|_I \leq \frac{1}{2} D\left(F_t, I(s/(\sigma\sqrt{t}))\right) \\ + \frac{ET^2}{6\sigma^3} K(I) t^{-1/2} + \frac{|\tilde{o}(t)|}{2\pi\sigma^3} t^{-1/2} + (|O_1(t)| + |O_2(t)|)\varepsilon^2, \end{aligned} \tag{5.10}$$

where

$$|\tilde{o}(t)| \leq \frac{2^{10}a(1)ET^2}{\sqrt{t}} + (4+\sqrt{2\pi})ET^2 w_1\left(\tilde{e}; \frac{T}{\sigma\sqrt{t}}\right), \tag{5.11}$$

$$|O_1(t)| \leq \frac{6}{\sqrt{2\pi}}\left(e^{-1/2} + \frac{ET^2}{2\sigma^3}\, t^{-1/2}\right) \tag{5.12}$$

and

$$|O_2(t)| \leq \frac{18}{\pi}\left(4 + \left(\frac{\pi\sigma\sqrt{t}}{s}\right)^2 e^{-tl((s/b)\wedge\pi)}\right). \tag{5.13}$$

In view of (5.11)-(5.13), the remainder terms in Theorem 22 are of order t^{-1} at least, excepting that containing $ET^2 w_1(\tilde{e}; T/(\sigma\sqrt{t}))$. We can apply to this term the considerations made in Remark 4. By the local limit theorem (cf. Petrov [34, Chap. VII]), the order of $D(F_t, I)$ is $t^{-1/2}$. Therefore, the lower and the upper bounds in (5.10) have the same order of magnitude. In addition, if I is a sufficiently small interval centered at ± 1 (the zeros of $H_2(x)$), we see from (5.8) and (5.10) that $\|F_t - \mathfrak{N}\|_I$ is asymptotically as close as we wish to $D(F_t, I)/2$. In this sense, the bounds in (5.10) are asymptotically sharp. In turn, the term $D(F_t, I)$ can be further estimated either using the local limit theorem or by means of the concentration function of F_t, for which different upper bounds are available (see, for instance, Petrov [34, Chap. III], Salikhov [39], and Adell and Lekuona[5]).

In order to prove Theorem 21, denote by

$$\delta := \delta(t) = \frac{\tau}{\sigma\sqrt{t}}, \qquad 0 < \tau \leq b, \quad t \geq 1. \tag{5.14}$$

Lemma 23 *Let δ be as in* (5.14). *In the setting of Theorem* 21, *we have*

$$\int_{\mathbb{R}} P_\delta \phi(z)\, d(F_t - \mathfrak{N}_t)(z) = \frac{V(\phi)}{2\pi\sigma^3}\, o_\tau(t) t^{-1/2},$$

where $o_\tau(t)$ is estimated in (5.6).

PROOF. Let $t \geq 1$. Applying (4.17) with $n = 3$ and ε replaced by δ, we have that

$$\int_{\mathbb{R}} P_\delta \phi(z)\, d(F_t - \mathfrak{N}_t)(z) = \frac{1}{2\sigma^3} \int_t^\infty R_2\left((P_\delta\phi)^{(3)}; u\right) u^{-3/2}\, du, \tag{5.15}$$

where, as follows from Lemma 18 with $m = 3$ and $r = 0$, we have the bound

$$\left|R_2\left((P_\delta\phi)^{(3)}; u\right)\right| \leq \frac{V(\phi)}{2\pi}\left(ET^2\int_{-1/\delta}^{1/\delta}|\zeta|^2\left|Ee^{i\zeta Z(u)} - Ee^{i\zeta Z}\right|d\zeta\right.$$
$$\left.+\sqrt{2\pi}E(1+|Z|)|Z|^2ET^2w_1\left(\tilde{e}; \frac{T}{\sigma\sqrt{t}}\right)\right), \tag{5.16}$$

provided that $u \geq t$. We claim that

$$\int_{-1/\delta}^{1/\delta}|\zeta|^2\left|Ee^{i\zeta Z(u)} - Ee^{i\zeta Z}\right|d\zeta$$
$$\leq \frac{ET^2}{\sigma^3}\left(8\sqrt{2\pi}E|Z|^5 + \frac{1}{3}\left(\frac{\sigma\sqrt{t}}{\tau}\right)^6 e^{-q_\tau t}\right), \quad u \geq t, \tag{5.17}$$

where q_τ is defined in (5.4). To this end, let $u \geq t$ and $\zeta \in \mathbb{R}$. Applying (2.17) to the function $\phi_\zeta(x) := e^{i\zeta x}$, $x \in \mathbb{R}$, it is not hard to see that

$$\left|Ee^{i\zeta Z(u)} - Ee^{i\zeta Z}\right| \leq \frac{ET^2}{2\sigma^3}|\zeta|^3\int_u^\infty\left|Ee^{i\zeta Z(v)}\right|\frac{dv}{v^{3/2}}.$$

This, together with Fubini's theorem, implies that the left-hand side in (5.17) is bounded above by

$$\frac{ET^2}{2\sigma^3}\int_u^\infty\frac{dv}{v^{3/2}}\int_{-1/\delta}^{1/\delta}|\zeta|^5\left|Ee^{i\zeta Z(v)}\right|d\zeta. \tag{5.18}$$

On the other hand, from (5.3) and (5.4) we have for any $v \geq u$ that

$$\left|Ee^{i\zeta Z(v)}\right| \leq e^{-q_\tau v}, \qquad b^{-1} \leq \left|\frac{\zeta}{\sigma\sqrt{v}}\right| \leq \tau^{-1}.$$

This, in conjunction with (4.12), implies that

$$\int_{-1/\delta}^{1/\delta} |\zeta|^5 \left| E e^{i\zeta Z(v)} \right| d\zeta \leq \int_{|b\zeta|\leq \sigma\sqrt{v}} |\zeta|^5 e^{-\zeta^2/4} \, d\zeta$$

$$+ \int_{\sigma\sqrt{v}<|b\zeta|\leq b/\delta} |\zeta|^5 e^{-q_\tau t} \, d\zeta \leq 8\sqrt{2\pi} E|Z|^5 + \frac{1}{3} \left(\frac{\sigma\sqrt{t}}{\tau} \right)^6 e^{-q_\tau t}. \tag{5.19}$$

Claim (5.17) follows from (5.18) and (5.19). Finally, we have by calculus that

$$E|Z|^{2m+1} = \frac{2^{m+1} m!}{\sqrt{2\pi}}, \qquad m = 0, 1, \ldots \tag{5.20}$$

Hence, the conclusion follows from (5.15)-(5.17) and (5.20). □

Let $t \geq 1$ and let δ be as in (5.14). It follows from the reformulation of Esseen's smoothing lemma shown in Adell and Lekuona [4, Lemma 2.2] that

$$\|F_t - \mathfrak{N}_t\| \leq \frac{1}{2p(r)-1} \|P_\delta F_t - P_\delta \mathfrak{N}_t\| + \frac{rp(r)}{2p(r)-1} \|\mathfrak{N}_t^{(1)}\| \delta, \tag{5.21}$$

where $p(r) := P(|Y_2| \leq r) > 1/2$ and the random variable Y_2 is defined in (5.1). By calculus, we have that $\|\eta H_3\| \leq 1/\sqrt{2\pi}$. Choosing $r = 4$, it can be seen from (5.1) that $p(r) \geq 3/4$. We therefore have from (5.2) and (5.21) that

$$\|F_t - \mathfrak{N}_t\| \leq 2\|P_\delta F_t - P_\delta \mathfrak{N}_t\| + \frac{6}{\sqrt{2\pi}} \left(1 + \frac{ET^2}{6\sigma^3} t^{-1/2} \right) \delta. \tag{5.22}$$

PROOF OF THEOREM 21. By (2.26), we have that

$$\left| \int_{\mathbb{R}} \phi(z) \, d(F_t - \mathfrak{N}_t)(z) \right| \leq V(\phi) \|F_t - \mathfrak{N}_t\|. \tag{5.23}$$

Let δ be as in (5.14). Since the random variable Y_2 is continuous and symmetric, we have that

$$|P_\delta F_t(x) - P_\delta \mathfrak{N}_t(x)| = \left| \int_{\mathbb{R}} P_\delta 1_{[x,\infty)}(z)\, d(F_t - \mathfrak{N}_t)(z) \right|$$
$$\leq \frac{1}{2\pi\sigma^3} |o_\tau(t)| t^{-1/2}, \quad x \in \mathbb{R}, \tag{5.24}$$

as follows from Lemma 23. The conclusion follows from (5.22)-(5.24). □

To prove Theorem 22, we shall need the following three auxiliary results.

Lemma 24 *Let $\phi \in \mathcal{AV}^{(0)}$ and let ε be as in* (4.2)*. For any $t \geq 1$, we have*

$$\left| \int_{\mathbb{R}} (P_\varepsilon \phi(z) - \phi(z))\, d\mathfrak{N}_t(z) \right| \leq V(\phi) |O_1(t)| \varepsilon^2,$$

where $|O_1(t)|$ is estimated in (5.12)*.*

PROOF. Since the random variable Y_2 is symmetric, we have from (2.25) and (2.5) that

$$\left| \int_{\mathbb{R}} (P_\varepsilon \phi(z) - \phi(z))\, d\mathfrak{N}_t(z) \right|$$
$$= \frac{1}{2} \left| \int_{\mathbb{R}} (E\phi(z + \varepsilon Y_2) - 2\phi(z) + E\phi(z - \varepsilon Y_2))\, d\mathfrak{N}_t(z) \right|$$
$$\leq \frac{V(\phi)}{2} E w_2(\mathfrak{N}_t; \varepsilon |Y_2|) \leq \frac{V(\phi)}{2} E Y_2^2 \|\mathfrak{N}_t^{(2)}\| \varepsilon^2.$$

It can be checked that $EY_2^2 = 12$, $\|\eta H_1\| = 1/\sqrt{2\pi e}$ and $\|\eta H_4\| = 3/\sqrt{2\pi}$. Therefore, the conclusion follows from (5.2) and the preceding inequalities. □

The following is a slight extension of Lemma 5.1 in Adell and Lekuona [5].

Lemma 25 *Let X be a random variable having a lattice distribution F with maximal span $s > 0$. For any real interval I and any $r \geq 0$, we have*

$$w_2(F;I;r) \le D(F,I(s)) + \frac{3r^2}{\pi}\int_{|\zeta|\le\pi/s}\left|\zeta E e^{i\zeta X}\right| d\zeta.$$

PROOF. Suppose that X takes on values in the set $c + s\mathbb{I}$, where $c \in \mathbb{R}$ and $\mathbb{I}$ is the set of integers. By considering, if necessary, the random variable $(X-c)/s$, we can assume without loss of generality that $c = 0$ and $s = 1$. If $r < 1$, we obviously have that $w_2(F;I;r) \le D(F;I(1))$. Assume that $r \ge 1$. Let $x \in \mathbb{R}$ and $1 \le h \le r$. Denote by $A_1 := (\lfloor x\rfloor, \lfloor x+h\rfloor] \cap \mathbb{I}$ and by $A_2 := (\lfloor x-h\rfloor, \lfloor x\rfloor] \cap \mathbb{I}$. It can be verified that $|\#A_1 - \#A_2| \le 1$. For the sake of concreteness, assume that $\#A_2 = \#A_1 + 1$, otherwise the proof is similar. Using Fourier inversion (cf. Feller [25, p. 511]), we have that

$$\begin{aligned}
&|F(x+h) - 2F(x) + F(x-h)| \le P(X = \lfloor x\rfloor) \\
&+\left|\sum_{k=1}^{\lfloor x+h\rfloor-\lfloor x\rfloor}\left(P(X=\lfloor x\rfloor+k) - P(X=\lfloor x\rfloor-k)\right)\right| \\
&\le D(F,I(1)) + \frac{1}{2\pi}\left|\sum_{k=1}^{\lfloor x+h\rfloor-\lfloor x\rfloor}\int_{-\pi}^{\pi} e^{-i\zeta\lfloor x\rfloor}\left(e^{-i\zeta k} - e^{i\zeta k}\right) E e^{i\zeta X}\, d\zeta\right| \\
&\le D(F,I(1)) + \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\zeta E e^{i\zeta X}\right| d\zeta \sum_{k=1}^{\lfloor x+h\rfloor-\lfloor x\rfloor} k \\
&\le D(F,I(1)) + \frac{3r^2}{\pi}\int_{-\pi}^{\pi}\left|\zeta E e^{i\zeta X}\right| d\zeta,
\end{aligned}$$

the last inequality because $\lfloor x+h\rfloor - \lfloor x\rfloor \le \lfloor h\rfloor + 1 \le 2h$. The proof is complete. □

Lemma 26 *In the setting of Theorem* 22*, we have*

$$\int_{|\zeta|\le\pi\sigma\sqrt{t}/s}\left|\zeta E e^{i\zeta Z(t)}\right| d\zeta \le 4 + \left(\frac{\pi\sigma\sqrt{t}}{s}\right)^2 e^{-tl((s/b)\wedge\pi)}$$

where $l(\cdot)$ *is defined in* (5.9).

PROOF. It follows from (4.2), (4.12) and (5.20) that

$$\int_{|\zeta|\le\sigma\sqrt{t}/b}\left|\zeta E e^{i\zeta Z(t)}\right|\,d\zeta \le \int_{\mathbb{R}}|\zeta|e^{-\zeta^2/4}\,d\zeta = 4. \tag{5.25}$$

On the other hand, it follows from (1.3) and (5.9) that

$$\left|Ee^{i\zeta(X(t)-t)/s}\right| = \exp\left(-tE\frac{1-\cos(\zeta T/s)}{T}\right) \le e^{-tl(r)}, \quad 0<r\le|\zeta|\le\pi.$$

Hence,

$$\begin{aligned}&\int_{\sigma\sqrt{t}/b\le|\zeta|\le\pi\sigma\sqrt{t}/s}\left|\zeta E e^{i\zeta Z(t)}\right|\,d\zeta\\&=\left(\frac{\sigma\sqrt{t}}{s}\right)^2\int_{s/b\le|\zeta|\le\pi}|\zeta E e^{i\zeta(X(t)-t)/s}|\,d\zeta \le \left(\frac{\pi\sigma\sqrt{t}}{s}\right)^2 e^{-tl((s/b)\wedge\pi)}.\end{aligned} \tag{5.26}$$

The conclusion follows from (5.25) and (5.26). □

PROOF OF THEOREM 22. Let $x\in I$ and let $h>0$ be such that $x+h\in I$. By the triangular inequality, we have that

$$\frac{1}{2}\,|F_t(x+h)-F_t(x)| \le \|F_t-\mathfrak{N}\|_I + \frac{1}{2}|\mathfrak{N}(x+h)-\mathfrak{N}(x)|.$$

Thus, the lower bound in (5.10) follows by letting $h\to 0$ in the preceding inequality. Let $\phi\in\mathcal{AV}^{(0)}$ and let ε be as in (4.2). By (5.2), we have that

$$\begin{aligned}&\int_{\mathbb{R}}\phi(z)\,d(F_t-\mathfrak{N})(z)\\&=\frac{ET^2}{6\sigma^3}E\phi(Z)H_3(Z)t^{-1/2}+\int_{\mathbb{R}}\phi(z)\,d(F_t-\mathfrak{N}_t)(z)\\&=\frac{ET^2}{6\sigma^3}E\phi(Z)H_3(Z)t^{-1/2}+\int_{\mathbb{R}}(\phi(z)-P_\varepsilon\phi(z))\,dF_t(z)\\&+\int_{\mathbb{R}}P_\varepsilon\phi(z)\,d(F_t-\mathfrak{N}_t)(z)+\int_{\mathbb{R}}(P_\varepsilon\phi(z)-\phi(z))\,d\mathfrak{N}_t(z).\end{aligned} \tag{5.27}$$

Let $x \in I$. From now on, set $\phi := 1_{[x,\infty)}$ in (5.27). Lemma 24 provides the upper bound $|O_1(t)|\varepsilon^2$ for the last term on the right-hand side in (5.27). Applying Lemma 20 with $n = 3$ and $r = 0$, and recalling (5.20), we get

$$\left| \int_{\mathbb{R}} P_\varepsilon \phi(z) \, d(F_t - \mathfrak{N}_t)(z) \right|$$

$$\leq \frac{V(\phi)}{\sigma^3 \sqrt{2\pi}} \left(\frac{2^6 a(1) E|Z|^5 ET^2}{\sqrt{t}} + E(1 + |Z|)|Z|^2 ET^2 w_1 \left(\tilde{e}; \frac{T}{\sigma \sqrt{t}} \right) \right) t^{-1/2}$$

$$\leq \frac{V(\phi)}{2\pi\sigma^3} |\tilde{o}(t)| t^{-1/2},$$

where $|\tilde{o}(t)|$ is estimated in (5.11). The preceding inequality gives us the third term of the upper bound in (5.10).

Denote by $\bar{F}_t(y) := P(Z(t) \geq y)$, $y \in \mathbb{R}$, the survival function of $Z(t)$. Since Y_2 is continuous and symmetric, we have that

$$\left| \int_{\mathbb{R}} (\phi(z) - P_\varepsilon \phi(z)) \, dF_t(z) \right| = \left| E\bar{F}_t(x + \varepsilon Y_2) - \bar{F}_t(x) \right|$$

$$= \frac{1}{2} \left| E \left(\bar{F}_t(x + \varepsilon Y_2) - 2\bar{F}_t(x) + \bar{F}_t(x - \varepsilon Y_2) \right) \right|$$

$$\leq \frac{1}{2} E w_2 \left(\bar{F}_t; I; \varepsilon |Y_2| \right) \leq \frac{1}{2} D(F_t, I(s/(\sigma\sqrt{t}))) + |O_2(t)|\varepsilon^2,$$

where $|O_2(t)|$ is estimated in (5.13). In the last inequality, we have used Lemmas 25 and 26 and the facts that $EY_2^2 = 12$ and $D(\bar{F}_t, I) = D(F_t, I)$ for any interval I.

Finally, it follows from (5.8) that the first term on the right-hand side in (5.27) is bounded by the second term on the right-hand side in (5.10). This concludes the proof. □

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