A Monotonicity Property of Euler’s Gamma Function

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Abstract. Let
\[ \Delta(x) = \frac{\log \Gamma(x + 1)}{x} \quad (-1 < x \neq 0), \quad \Delta(0) = -\gamma. \]

For all \( n = 0, 1, 2, \ldots \) and \( x > -1 \), we show that
\[ (-1)^n \Delta^{(n+1)}(x) = (n+1)! \int_{0}^{1} u^{n+1} \zeta(n+2, xu+1) \, du, \]
where \( \zeta \) denotes the Hurwitz zeta function. This representation implies that \( \Delta' \)
is completely monotonic on \( (-1, \infty) \). This extends a result published in 1996 by Grabner, Tichy, and Zimmermann, who proved that \( \Delta \) is increasing and concave on \( (-1, \infty) \).

1. Introduction and Main Results

In this note we are concerned with the function
\[ \Delta(x) = \frac{\log \Gamma(x + 1)}{x} \quad (-1 < x \neq 0), \quad \Delta(0) = -\gamma, \]

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where Γ denotes Euler’s gamma function and γ is Euler’s constant. In the recent past, several authors studied interesting monotonicity properties of Δ as well as other functions defined in terms of Δ (see, for instance, [3, 4, 5, 7, 8, 9, 10]).

Grabner et al. [6] proved that Δ is increasing and concave on \((-1, \infty)\) and used their result to present an upper bound for the permanent of a 0–1 matrix. Vogt and Voigt [11] showed that the function \(x \mapsto \Delta(x) - \log(x + 1) + 1\) is completely monotonic on \((-1, \infty)\).

We recall that a function \(f: I \rightarrow \mathbb{R}\) is said to be completely monotonic, if \(f\) has derivatives of all orders and

\[
0 \leq (-1)^n f^{(n)}(x) \quad (n = 0, 1, 2, \ldots; x \in I).
\]

Completely monotonic functions have important applications in probability and potential theory, in numerical and asymptotic analysis, and in other fields. The main properties of these functions are given in [12, Chapter IV]. In [2] one can find a detailed list of references on this subject.

The result of Grabner et al. yields

\[
0 \leq (-1)^n \Delta^{(n+1)}(x) \quad (n = 0, 1; x > -1),
\]

whereas the monotonicity theorem of Vogt and Voigt leads to the inequality

\[
(-1)^n \Delta^{(n+1)}(x) \leq \frac{n!}{(x + 1)^{n+1}} \quad (n = 0, 1, 2, \ldots; x > -1).
\]

In view of (1.1), it is natural to ask whether \(\Delta'\) is completely monotonic on \((-1, \infty)\). A positive answer to this question is given in the following theorem which provides, in addition, a closed form expression for \((-1)^n \Delta^{(n+1)}(x)\) in terms of the classical Hurwitz zeta function

\[
\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m + a)^s} \quad (s > 1; a > 0).
\]

**Theorem 1.1.** Let \(n\) be a nonnegative integer and let \(x\) be a real number with \(x > -1\). Then,

\[
(-1)^n \Delta^{(n+1)}(x) = (n + 1)! \int_{0}^{1} u^{n+1} \zeta(n + 2, xu + 1) \, du.
\]

As a consequence, \(\Delta'\) is completely monotonic on \((-1, \infty)\).
Remark 1.1. From (1.3), we have for \( n \geq 1 \):

\[
\Delta^{(n)}(0) = (-1)^{n-1} \frac{n!}{n+1} \zeta(n+1),
\]

where \( \zeta \) denotes the Riemann zeta function.

Remark 1.2. If \( h' \) is completely monotonic on \( I \), then \( \exp(-h) \) is also completely monotonic on \( I \). This result can be proved by applying the Leibniz rule and induction. Thus, setting \( h = \Delta \) and \( I = (-1, \infty) \), we conclude from Theorem 1.1 that the function

\[
\Theta(x) = \Gamma(x+1)^{-1/x} \quad (-1 < x \neq 0), \quad \Theta(0) = \exp(\gamma),
\]

is completely monotonic on \( (-1, \infty) \).

A consequence of Theorem 1.1 is that the upper bound in (1.2) is asymptotically sharp, as stated in the following corollary.

Corollary 1.2. For any nonnegative integer \( n \), we have

\[
\lim_{x \to \infty} \frac{(x+1)^{n+1}}{n!} (-1)^{n} \Delta^{(n+1)}(x) = 1.
\]

In order to prove Theorem 1.1 we need a lemma. It states that a certain function, defined in terms of the exponential function, is completely monotonic on \( \mathbb{R} \).

Lemma 1.3. Let \( N \geq 0 \) be an integer and

\[
g_N(x) = \left[ 1 - e^{-x} \sum_{m=0}^{N} \frac{x^m}{m!} \right] x^{-N-1} \quad (x \neq 0), \quad g_N(0) = \frac{1}{(N+1)!}.
\]

Then we have for \( n \geq 0 \) and \( x \in \mathbb{R} \):

\[
(-1)^n g_N^{(n)}(x) = \frac{1}{N!} \int_0^1 e^{-xa} x^{n+N} \, du.
\]

In particular, \( g_N \) is completely monotonic on \( \mathbb{R} \).
2. The Proofs

Proof of Lemma 1.3. We get
\[ g_N(x) = \frac{x^{-N-1}}{N!} \int_0^x e^{-t} t^N dt = \frac{1}{N!} \int_0^1 e^{-xu} u^N du. \]
Differentiation leads to (1.6). \[ \square \]

Proof of Theorem 1.1. Let \( x > -1, t > 0, \) and \( n \geq 0. \) We obtain
\[ \Delta'(x) = \frac{\psi(x + 1)}{x} - \frac{\log \Gamma(x + 1)}{x^2} (x \neq 0), \quad \Delta'(0) = \frac{\pi^2}{12}, \]
where \( \psi = \Gamma'/\Gamma \) denotes the digamma function. Using the integral formulas
\[ \log \Gamma(z) = \int_0^\infty (z - 1)e^{-t} - \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} \frac{dt}{t} \quad (z > 0) \]
and
\[ \psi(z) = \int_0^\infty \left[ e^{-t} - \frac{e^{-zt}}{1 - e^{-t}} \right] dt \quad (z > 0) \]
(see [1, pp. 258, 259]), we get
\[ (2.1) \Delta'(x) = \int_0^\infty \frac{te^{-t}}{1 - e^{-t}} g_1(xt) dt, \]
where \( g_1 \) is given in (1.5). Applying (1.6) with \( N = 1 \) and
\[ \frac{1}{1 - e^{-t}} = \sum_{m=0}^\infty e^{-mt} \]
we obtain from (2.1) and Fubini’s theorem
\[ (-1)^n \Delta^{(n+1)}(x) = \int_0^\infty \frac{t^{n+1} e^{-t}}{1 - e^{-t}} (-1)^n g_1^{(n)}(xt) dt \]
\[ = \int_0^\infty t^{n+1} e^{-t} \sum_{m=0}^\infty e^{-mt} \int_0^1 e^{-xtu} u^{n+1} du \frac{dt}{t} \]
\[ = \int_0^1 u^{n+1} \sum_{m=0}^\infty \int_0^\infty t^{n+1} e^{-t(m+ux+1)} dt \frac{du}{u} \]
\[ = \int_0^1 u^{n+1} \sum_{m=0}^\infty \frac{\Gamma(n+2)}{(m+ux+1)^{n+2}} du \]
\[ = (n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1) du. \]
This completes the proof of Theorem 1.1. □

**Proof of Corollary 1.2.** Let \( n \geq 0 \) and \( x > -1 \). We make use of (1.3) and get

\[
(-1)^n \Delta^{(n+1)}(x) = (n+1)! \int_0^1 u^{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+ xu + 1)^{n+2}} du \geq (n+1)! \int_0^1 u^{n+1} \int_0^{\infty} \frac{1}{(t + xu + 1)^{n+2}} dt du \\
= n! \int_0^1 \frac{u^{n+1}}{(xu + 1)^{n+1}} du.
\]

(2.2)

From (2.2) and (1.2) we obtain

\[
\int_0^1 \left( \frac{xu + u}{xu + 1} \right)^{n+1} du \leq \frac{(x + 1)^{n+1}}{n!} (-1)^n \Delta^{(n+1)}(x) \leq 1.
\]

Applying the dominated convergence theorem we conclude that (1.4) holds. □

**References**


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