WEIGHTED WEAK BEHAVIOUR OF FOURIER-JACOBI SERIES

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Abstract. Let \( w(x) = (1-x)\alpha(1+x)^\beta \) be a Jacobi weight on the interval \([-1,1]\) and \(1 < p < \infty\). If either \( \alpha > -1/2 \) or \( \beta > -1/2 \) and \( p \) is an endpoint of the interval of mean convergence of the associated Fourier-Jacobi series, we show that the partial sum operators \( S_n \) are uniformly bounded from \( L^p(w) \) to \( L^{p,\infty} \), thus extending a previous result for the case that both \( \alpha, \beta > -1/2 \). For \( \alpha, \beta > -1/2 \), we study the weak and restricted weak \((p,p)\)-type of the weighted operators \( f \rightarrow uS_n(u^{-1}f) \), where \( u \) is also a Jacobi weight.

§1. Introduction and main results.

Let \( w \) be a Jacobi weight on the interval \([-1,1]\), that is,

\[ w(x) = (1-x)\alpha(1+x)^\beta, \quad \alpha, \beta > -1 \]

and let \( 1 < p < \infty \); \( S_n f \) stands for the \( n \)-th partial sum of the Fourier series associated to the Jacobi polynomials, orthonormal on \([-1,1]\) with respect to \( w \). It is well known that \( S_n f \) converges to \( f \) for every \( f \in L^p(w) \) if and only if the partial sum operators \( S_n \) are uniformly bounded in \( L^p(w) \), i.e., there exists a constant \( C > 0 \) such that

\[ ||S_n f||_{L^p(w)} \leq C ||f||_{L^p(w)} \quad \forall n \geq 0, \forall f \in L^p(w) \]  \hspace{1cm} (1)

(throughout this paper, we will denote by \( C \) a constant independent of \( f, n, \) etc., but not necessarily the same at each occurrence). Furthermore, there exists an open interval \((p_0, p_1)\) such that this boundedness holds if and only if \( p \) belongs to \((p_0, p_1)\) (see [6]). The assumption that either \( \alpha > -1/2 \) or \( \beta > -1/2 \) is equivalent to \( 1 < p_0 < p_1 < \infty \). More precisely, in this case (1) holds if and only if

\[ \frac{4(\alpha + 1)}{2\alpha + 3} < p < \frac{4(\alpha + 1)}{2\alpha + 1} \]

when \( \alpha \geq \beta \) (and the analogous inequality with \( \alpha \) replaced by \( \beta \) if \( \beta \geq \alpha \)).

In this paper we examine the behaviour of \( S_n \) at the endpoints of the interval of mean convergence. In order to do this we need some classical definitions and notations. Given
a measure $\mu$ and $1 \leq p < \infty$, the space $L^p_*(\mu) = L^{p,\infty}(\mu)$ is defined to be the space of measurable functions such that

$$\|f\|_{L^p_*(\mu)} = \sup_{y > 0} \left[ \mu(\{x : |f(x)| > y\}) \right]^{1/p} < \infty.$$ 

An operator $T$ is of weak $(p,p)$-type if $T : L^p(\mu) \longrightarrow L^p(\mu)$ is bounded. Now, let $f^*$ be the nonincreasing rearrangement of $f$, given by $f^*(t) = \inf\{s : \lambda(s) \leq t\}$, where $\lambda$ denotes the distribution function of $f$. Then, the Lorentz space $L^{p,r}(\mu)$ is the class of all measurable functions $f$ satisfying

$$\|f\|_{L^{p,r}(\mu)}^r = \left( \frac{r}{p} \int_0^\infty \left[ t^{1/p} f^*(t) \right]^r \frac{dt}{t} \right)^{1/r} < \infty,$$

where $1 \leq p < \infty$, $1 \leq r < \infty$. An operator $T$ is of restricted weak $(p,p)$-type if $T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu)$ is bounded, which is equivalent to $\|T \chi_E\|_{L^p_*(\mu)} \leq C \|\chi_E\|_{L^p(\mu)}$ for all characteristic functions $\chi_E$, with $C > 0$ independent of $E$. We refer the reader to [11] for further information on these topics.

If both $\alpha, \beta > -1/2$, the authors proved (see [2]) that the $n$-th partial sum operators are uniformly of restricted weak $(p,p)$-type but not of weak $(p,p)$-type when $p$ is an endpoint of the interval of mean convergence. In theorems 2 and 3 we extend this result to weighted case $f \longrightarrow uS_n(u^{-1}f)$, where $u$ is also a Jacobi weight, that is,

$$u(x) = (1 - x)^a(1 + x)^b, \quad a, b \in \mathbb{R}.$$ 

Now, the weighted uniform boundedness

$$\|uS_n f\|_{L^p(u^p \mu)} \leq C \|uf\|_{L^p(u^p \mu)} \quad \forall f \in L^p(u^p \mu), \quad \forall n \geq 0$$

holds (see [6]) if and only if

$$|a + (\alpha + 1)(\frac{1}{p} - \frac{1}{2})| < \min\{\frac{1}{4}, \frac{\alpha + 1}{2}\},$$

$$|b + (\beta + 1)(\frac{1}{p} - \frac{1}{2})| < \min\{\frac{1}{4}, \frac{\beta + 1}{2}\}.$$ (2)

Via Pollard’s formula, these operators can be related to the Hilbert transform. Then, the theory of $A_p$ weights is used, as well as some classical dyadic-type decomposition of the interval $[-1,1]$.

In the general case $\alpha > -1/2$, $\alpha \geq \beta$ (the case $\beta \geq \alpha$ follows by symmetry), we prove in theorem 1 that the $n$-th partial sum operators are uniformly of restricted weak $(p,p)$-type when $p$ is an endpoint of the interval of mean convergence, thus extending the above cited result (the question of the weak boundedness had already been answered in the negative in [2]). Now, however, uniform bounds are not available for Jacobi polynomials; therefore, a uniform weighted norm inequality is needed for operators of the form $f \longrightarrow u_n H(v_n f)$, where $H$ is the Hilbert transform and $(u_n)$, $(v_n)$ are two sequences of weights involving
Jacobi polynomials or their bounds. This is achieved by studying the $A_p$ constants of the pairs of weights $(u_n, v_n)$, as well as some $L^{p,\infty}$ norms.

Concerning mixed weak norm inequalities for the Hilbert transform, we can state the following property, which can be proved in the same way as theorem 3 of [7] (throughout this paper, the Hilbert transform, as well as $A_p$ classes of weights, are taken on $[-1,1]$): assume that $u_1(x), u_2(x), v(x) \geq 0$, $1 < p < \infty$ and there is a constant $C > 0$ such that

$$
\|u_2Hg\|_{L_p^p(u_1)} \leq C\|g\|_{L^p(v)} \quad \forall g \in L^p(v);
$$

then, there exists another constant $B > 0$ which depends only on $C$, such that for every interval $I$

$$
\|u_2\chi_I\|_{L_p^p(u_1)} \left( \int_{I}^{1} \frac{v(x)^{(1/p-1)}}{(|I| + |x-x_I|)^q} \right)^{1/q} \leq B,
$$

$x_I$ being the centre of $I$.

Let us state the main results of this paper. By symmetry, there is no loss of generality in assuming $\alpha \geq \beta$. Regarding the restricted weak type, by standard arguments it is enough to consider just one of the endpoints of the interval of mean convergence, as we remark below.

**Theorem 1.** Let $\alpha > -1/2, \beta > -1, \alpha \geq \beta$. If $p = \frac{4(\alpha+1)}{2\alpha+1}$, there exists a constant $C > 0$ such that for every measurable set $E$ and for every $n \geq 0$

$$
\|S_n\chi_E\|_{L_p^p(w)} \leq C\|\chi_E\|_{L^p(w)}.
$$

**Theorem 2.** Let $\alpha, \beta \geq -1/2$, $u(x) = (1-x)^a(1+x)^b$, $1 < p < \infty$. If the inequalities

$$
-\frac{1}{4} \leq a + (\alpha + 1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}, \quad -\frac{1}{4} \leq b + (\beta + 1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}
$$

hold, then there exists a constant $C > 0$ such that

$$
\|uS_n(u^{-1}\chi_E)\|_{L_p^p(w)} \leq C\|\chi_E\|_{L^p(w)}
$$

for every $n \geq 0$ and every measurable set $E \subseteq [-1,1]$.

**Remark.** For $1 < p < \infty$ and $1/p + 1/q = 1$, it is easy to see that

$$
\|uS_n(u^{-1}\chi_E)\|_{L_p^p(w)} \leq C\|\chi_E\|_{L^p(w)} \quad \forall n \geq 0, \forall E \subseteq [-1,1]
$$

if and only if

$$
\|u^{-1}S_n(u\chi_E)\|_{L_p^p(w)} \leq C\|\chi_E\|_{L^p(w)} \quad \forall n \geq 0, \forall E \subseteq [-1,1].
$$

This allows us to derive, from theorem 2, the same result for the case

$$
-\frac{1}{4} < a + (\alpha + 1)(\frac{1}{p} - \frac{1}{2}) \leq \frac{1}{4}, \quad -\frac{1}{4} < b + (\beta + 1)(\frac{1}{p} - \frac{1}{2}) \leq \frac{1}{4}
$$
as well as the analog of theorem 1 for \( p = \frac{4(\alpha+1)}{2\alpha+3} \).

**Theorem 3.** Let \( \alpha, \beta \geq -1/2 \), \( u(x) = (1 - x)^a(1 + x)^b \), \( 1 < p < \infty \). If there exists a constant \( C > 0 \) such that for every \( f \in L^p(u^p w) \) and for every \( n \geq 0 \)

\[
\|uS_n f\|_{L^p(w)} \leq C\|uf\|_{L^p(w)},
\]

then the inequalities

\[
|a + (\alpha + 1)\left(\frac{1}{p} - \frac{1}{2}\right)| < \frac{1}{4}, \quad |b + (\beta + 1)\left(\frac{1}{p} - \frac{1}{2}\right)| < \frac{1}{4}
\]

are verified.

§2. Preliminary lemmas.

A basic tool in the study of Fourier series on the interval \([-1, 1]\) is Pollard’s decomposition of the kernels \( K_n(x, t) \) (see [9], [6]): if \( \{P_n\}_{n \geq 0} \) is the sequence of polynomials orthonormal with respect to \( w(x)dx \) and \( \{Q_n\}_{n \geq 0} \) is the sequence of polynomials associated to \( (1 - x^2)w(x)dx \), then

\[
K_n(x, t) = r_nT_{1,n}(x, t) + s_nT_{2,n}(x, t) + s_nT_{3,n}(x, t),
\]

where

\[
T_{1,n}(x, t) = P_{n+1}(x)P_{n+1}(t),
\]

\[
T_{2,n}(x, t) = (1 - t^2)\frac{P_{n+1}(x)Q_n(t)}{x - t},
\]

\[
T_{3,n}(x, t) = (1 - x^2)\frac{P_{n+1}(t)Q_n(x)}{t - x}
\]

and \( \{r_n\}, \{s_n\} \) are bounded sequences. In fact, for any measure \( \mu \) on \([-1, 1]\) with \( \mu' > 0 \) a.e. (in particular, for \( w(x)dx \)),

\[
\lim_{n \to \infty} r_n = -1/2, \quad \lim_{n \to \infty} s_n = 1/2
\]

(this can be deduced from [9] and [10] or [4]). Therefore, we can write

\[
S_nf = r_nW_{1,n}f + s_nW_{2,n}f - s_nW_{3,n}f,
\]

where

\[
W_{1,n}f(x) = P_{n+1}(x)\int_{-1}^{1} P_{n+1}(t)f(t)w(t)dt,
\]

\[
W_{2,n}f(x) = P_{n+1}(x)H((1 - t^2)Q_n(t)f(t)w(t), x)
\]

and

\[
W_{3,n}f(x) = (1 - x^2)Q_n(x)H(P_{n+1}(t)f(t)w(t), x),
\]

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**Lemma 4.** Let \( \{x_n\} \) be a sequence of positive numbers with \( \lim_{n \to \infty} x_n = 0 \). Let \( 1 < p < \infty \), \( r, s, R, S \in \mathbb{R} \). Then,

\[
(|x|^r(|x| + x_n)^s, |x|^R(|x| + x_n)^S) \in A_p([-1,1]) \quad \text{uniformly}
\]

if and only if

\[
\begin{align*}
-1 &< r; \\
-1 &< r + s; \\
R &< p - 1; \\
R &\leq r; \\
R + S &< p - 1; \\
R + S &\leq r + s.
\end{align*}
\]

**Proof.** According to its definition,

\[
(|x|^r(|x| + x_n)^s, |x|^R(|x| + x_n)^S) \in A_p([-1,1]) \quad \text{uniformly}
\]

if and only if there exists a constant \( C > 0 \) such that

\[
\int_a^b |x|^r(|x| + x_n)^s \, dx \left( \int_a^b |x|^R(|x| + x_n)^S \right)^{-1/(p-1)} \, dx)^{p-1} \leq C(b - a)^p
\]

for all \(-1 \leq a < b \leq 1\), \( \forall n \geq 1 \). Integrability conditions imply the above inequalities. In turn, if those inequalities hold we can easily deduce (6) from the estimate

\[
\int_a^b x^\gamma(x + c)^\mu \, dx \leq \begin{cases} 
Kb^{\gamma+\mu}(b - a) & \text{if } c \leq b \\
Kb^{\gamma+\mu}(b - a) & \text{if } b \leq c 
\end{cases}
\]
valid for $0 \leq a < b \leq 1, 0 \leq c \leq 1, \gamma > -1, \gamma + \mu > -1$, with a constant $K$ which depends only on $\gamma, \mu$. □

The same property holds if we replace $x$ by $x - a$, with $a \in [-1, 1]$. Even more, it is not difficult to show that in order to see whether a finite product of this type of expressions is uniformly in $A_p$, we only need to check the above inequalities for each factor of the form

$$|(x - a)^r (|x - a| + x_n)^s, |x - a|^R (|x - a| + x_n)^S)$$

separately.

We will eventually need to show that some of the operators are not of strong or weak type. In this sense, the following lemma (see [5]) will be used:

**Lemma 5.** Let $\text{supp } d\alpha = [-1, 1], \alpha' > 0$ a.e. in $[-1, 1]$, and $0 < p \leq \infty$. There exists a constant $C > 0$ such that if $g$ is a Lebesgue-measurable function on $[-1, 1]$, then

$$\|\alpha'(x)^{-1/2} (1 - x^2)^{-1/4} \|_{L^p(|g|^p \, dx)} \leq C \liminf_{n \to \infty} \| P_n \|_{L^p(|g|^p \, dx)} .$$

There is a weak version of this property: it is a consequence of Kolmogorov’s condition (see [1], lemma V.2.8, p. 485) and the previous lemma.

**Lemma 6.** Let $\text{supp } d\alpha = [-1, 1], \alpha' > 0$ a.e. in $[-1, 1]$, and $0 < p < \infty$. There exists a constant $C > 0$ such that if $g, h$ are Lebesgue-measurable functions on $[-1, 1]$, then

$$\|\alpha'(x)^{-1/2} (1 - x^2)^{-1/4} g(x) \|_{L^p(|h|^p \, dx)} \leq C \liminf_{n \to \infty} \| P_n g \|_{L^p(|h|^p \, dx)} .$$

The following lemma will be useful to estimate some weighted $L^p_s$ norms:

**Lemma 7.** Let $1 \leq p < \infty, r, s \in \mathbb{R}, a > 0$. Then,

$$\chi_{(0, a)}(x) x^r \in L^p_s(x^s \, dx) \iff pr + s + 1 \geq 0, \quad (r, s) \neq (0, -1).$$

Moreover, in this case there is a constant $K$ depending on $r, s, p$ such that

$$\|\chi_{(0, a)}(x) x^r \|_{L^p_s(x^s \, dx)} = Ka^{r+(s+1)/p} ,$$

**Proof.** Since

$$\|\chi_{(0, a)}(x) x^r \|_{L^p_s(x^s \, dx)}^p = \sup_{y > 0} y^p \int_A x^s \, dx$$

with $A = \{ x; 0 < x < a, x^r > y \}$, the proof is reduced to a simple calculation of that integral, depending on the sign of $pr + s + 1$ and $r$. □

Finally, this lemma will be used in the study of the operator $W_{2,n}$.
Lemma 8. Let $\alpha > -1$, $1 < p < \infty$, $1/p + 1/q = 1$, $0 < r < 1$, $n \in \mathbb{N}$. If $(\alpha + 1)(1/p - 1/2) < 1/4$, then there exists a constant $C$, independent of $r$ and $n$, such that

$$
\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.
$$

Proof. a) Case $\alpha \geq -1/2$. Since $-(2\alpha+3)/4 < 0$,

$$
\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq \|(1-t)^{(1-2\alpha)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})}.
$$

By lemma 7 and taking into account that $(2\alpha +1)/4 \geq 0$, we have

$$
\|(1-t)^{(1-2\alpha)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq C(1-r)^{1-2\alpha/4+(\alpha+1)/q} \leq C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.
$$

b) Case $1 - r \leq n^{-2}$. Then, by lemma 7 and the inequality $1-r+n^{-2} \leq 2n^{-2}$, we obtain

$$
\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq (n^{-2})^{-(2\alpha+3)/4}\|(1-t)\chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq C(n^{-2})^{-(2\alpha+3)/4}(1-r)^{1+(\alpha+1)/q} \leq C(1-r)^{1-\alpha(1+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.
$$

c) Case $\alpha < -1/2$ and $n^{-2} \leq 1 - r$. Then $1 - 2\alpha \geq 0$ and $1 - r + n^{-2} \leq 2(1-r)$. Thus,

$$
\|(1-t)(1-t+n^{-2})^{-(2\alpha+3)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq \|(1-t+n^{-2})^{(1-2\alpha)/4} \chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq (1-r+n^{-2})^{(1-2\alpha)/4}\|\chi_{(r,1)}(t)\|_{L^q_2((1-t)^{\alpha})} \leq C(1-r+n^{-2})^{(1-2\alpha)/4}(1-r)^{(\alpha+1)/q} = C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}(1-r+n^{-2})^{-\alpha}(1-r)^{\alpha} \leq C(1-r)^{1-(\alpha+1)/p}(1-r+n^{-2})^{(2\alpha+1)/4}.
$$

§3. Proof of theorems 1 and 2.

The proof of theorem 1 consists of lemmas 9, 10 and 11 below. In order to prove theorem 2, analogous weighted lemmas can be shown using that, in the case $\alpha, \beta \geq -1/2$, not only the polynomials $Q_n$ but also the $P_n$ satisfy an uniform estimate similar to (5).
Lemma 9. Under the hypothesis of theorem 1, there exists a constant \( C \) such that
\[
\| W_{1,n} f \|_{L^p_w} \leq C \| f \|_{L^p(w)} \quad \forall f \in L^p(w), \ \forall n \in \mathbb{N}.
\]

Lemma 10. Under the hypothesis of theorem 1, there exists a constant \( C \) such that
\[
\| W_{3,n} f \|_{L^p(w)} \leq C \| f \|_{L^p(w)} \quad \forall f \in L^p(w), \ \forall n \in \mathbb{N}.
\]

Lemma 11. Under the hypothesis of theorem 1, there exists a constant \( C \) such that for every measurable set \( E \subseteq [-1,1] \) and for every \( n \geq 0 \)
\[
\| W_{2,n} \chi_E \|_{L^p_w} \leq C \| \chi_E \|_{L^p(w)} \quad (7)
\]

Proof of lemma 9. From its definition, we have
\[
\| W_{1,n} f \|_{L^p_w} \leq \| P_{n+1} \|_{L^p_w} \| P_{n+1} \|_{L^q_w} \| f \|_{L^p(w)},
\]
where \( 1/p + 1/q = 1 \). So, we only need to prove
\[
\| P_n \|_{L^p(w)} \leq C \quad \forall n \in \mathbb{N}
\]
and
\[
\| P_n \|_{L^q(w)} \leq C \quad \forall n \in \mathbb{N},
\]
which follows from lemma 7, (4) and the dominate convergence theorem.

Proof of lemma 10. It is clear that
\[
\| W_{3,n} f \|_{L^p(w)} \leq C \| f \|_{L^p(w)}
\]
for every \( f \in L^p(w) \) if and only if
\[
\| H g \|_{L^p((1-x^2)^p |Q_n|^p w^{1-p})} \leq C \| g \|_{L^p((P_{n+1}^{-p} w^{1-p}))}
\]
for every \( g \in L^p([P_{n+1}^{-p} w^{1-p}]) \). Using again (4) and its analogous for \( Q_n \), it is enough to obtain
\[
\| H g \|_{L^p(v_n)} \leq C \| g \|_{L^p(v_n)} \quad \forall n, \ \forall g \in L^p(v_n),
\]
with
\[
u_n(x) = (1-x)^{p+\alpha}(1-x+n^{-2})^{-p(2\alpha+3)/4}(1+x)^{p+\beta}(1+x+n^{-2})^{-p(2\beta+3)/4}
\]
and
\[
u_n(x) = (1-x)^{\alpha(1-p)}(1-x+n^{-2})^{p(2\alpha+1)/4}(1+x)^{\beta(1-p)}(1+x+n^{-2})^{p(2\beta+1)/4}.
\]
Now, we only need to prove that
\[(1 - x)^{(p + \alpha)\delta} (1 - x + n^{-2})^{-p(2\alpha + 3)/4}, (1 - x)\alpha(1-p)\delta (1 - x + n^{-2})^{p(2\alpha + 1)/4} \in A_p\]
and
\[(1 + x)^{(p + \beta)\delta} (1 + x + n^{-2})^{-p(2\beta + 3)/4}, (1 + x)\beta(1-p)\delta (1 + x + n^{-2})^{p(2\beta + 1)/4} \in A_p\]
uniformly in \(n\), for some \(\delta > 1\). This can be deduced from lemma 4. 

**Proof of lemma 11.** From \(p = \frac{4(\alpha + 1)}{2\alpha + 1}\) and \(\alpha \geq \beta\), it follows
\[-\frac{1}{4} \leq (\alpha + 1)\left(\frac{1}{p} - \frac{1}{2}\right) < \frac{1}{4}\]  
(8)
and
\[-\frac{1}{4} \leq (\beta + 1)\left(\frac{1}{p} - \frac{1}{2}\right) < \frac{1}{4}.\]  
(9)
We will prove that (8) and (9) imply (7). By the symmetry of these inequalities, we can consider only the case \(E \subseteq [0,1]\).

a) We will show first that there exists a constant \(C > 0\) such that
\[\|\chi_{[-3/4,3/4]} W_{2,n}(\chi E)\|_{L^p(w)} \leq C\|\chi E\|_{L^p(w)}\]
for all \(n \in \mathbb{N}\) and every measurable set \(E \subseteq [0,1]\). Since \(1 - x\) and \(1 + x\) are bounded away from 0 and \(\infty\) on \([-3/4,3/4]\), from (4) and the definition of \(W_{2,n}\) we get
\[\|\chi_{[-3/4,3/4]} W_{2,n}(\chi E)\|_{L^p(w)} \leq C\|\chi_{[-3/4,3/4]} H((1 - t^2)\chi \chi Q_n w)\|_{L^p((1-x)^r(1+x)^s)}\]
for any previously fixed \(r, s\). If we find \(r, s\) such that
\[H : L^p((1 - x)^{\alpha-p(2\alpha+1)/4}(1 + x)^{\beta-p(2\beta+1)/4}) \longrightarrow L^p((1 - x)^r(1 + x)^s)\]  
(10)
is bounded, then from (5) it would follow
\[\|\chi_{[-3/4,3/4]} H((1 - t^2)\chi \chi Q_n w)\|_{L^p((1-x)^r(1+x)^s)} \leq \|H((1 - t^2)\chi \chi Q_n w)\|_{L^p((1-x)^r(1+x)^s)} \leq C\|\chi E\|_{L^p(w)},\]
as we want to show. In order to get (10), it is enough to have
\[((1 - x)^{r\delta}(1 + x)^{s\delta}, (1 - x)^{\delta(\alpha-p(2\alpha+1)/4}(1 + x)^{\delta(\beta-p(2\beta+1)/4}) \in A_p\]
for some \(\delta > 1\). This is equivalent, by lemma 4, to the following conditions:
\[-1 < r;\]  
\[\alpha - p(2\alpha + 1)/4 < p - 1;\]  
\[\alpha - p(2\alpha + 1)/4 \leq r;\]  
\[-1 < s;\]  
\[\beta - p(2\beta + 1)/4 < p - 1;\]  
\[\beta - p(2\beta + 1)/4 \leq s.\]
It is easy to see that the second row inequalities hold, while for the others we only need to take \( r \) and \( s \) large enough.

b) Now, we are going to prove that there exists a constant \( C > 0 \) such that

\[
\|\chi_{[-1, -3/4]} W_{2,n}(\chi E)\|_{L^r_w(u)} \leq C \|\chi E\|_{L^r(w)}
\]

for all \( n \in \mathbb{N} \) and every measurable set \( E \subseteq [0, 1] \).

As \( E \subseteq [0, 1] \), we can drop the denominator \( x - t \) in \( W_{2,n}(\chi E) \) and, using the inequality (5), we get

\[
|\chi_{[-1, -3/4]}(x)W_{2,n}(\chi E, x)| \leq \\
\leq C\chi_{[-1, -3/4]}(x)|P_{n+1}(x)| \int_{-1}^{1} (1 - t)^{1-(2\alpha+3)/4} \chi E(t)w(t)dt \leq \\
\leq C\chi_{[-1, -3/4]}(x)|P_{n+1}(x)| \|(1 - t)^{(1-2\alpha)/4}\|_{L^r(w)}\|\chi E\|_{L^r(w)} \leq \\
\leq C\|\chi E\|_{L^r(w)}\chi_{[-1, -3/4]}(x)|P_{n+1}(x)|.
\]

Therefore

\[
\|\chi_{[-1, -3/4]} W_{2,n}(\chi E)\|_{L^r_w(u)} \leq \\
\leq C\|\chi E\|_{L^r(w)}\|\chi_{[-1, -3/4]}(1 + x + n^{-2})^{-2(\beta+1)/4}\|_{L^r_w(u)} \leq C\|\chi E\|_{L^r(w)},
\]

by the dominate convergence and lemma 7.

c) We must show now that there exists a constant \( C > 0 \) such that

\[
\|\chi_{[3/4, 1]} W_{2,n}(\chi E)\|_{L^r_w(u)} \leq C\|\chi E\|_{L^r(w)}
\]

for all \( n \in \mathbb{N} \) and every measurable set \( E \subseteq [0, 1] \). Let us define, for \( k = 2, 3, \ldots \) the sets

\[
I_k = [1 - 2^{-k}, 1 - 2^{-k-1}], \\
J_{k1} = [0, 1 - 2^{-k+1}], \quad J_{k2} = [1 - 2^{-k+1}, 1 - 2^{-k-2}], \quad J_{k3} = [1 - 2^{-k-2}, 1].
\]

For each \( k \geq 2 \), \( J_{ki} \) \( (i = 1, 2, 3) \) are non-overlapping sets such that \( [0, 1] = J_{k1} \cup J_{k2} \cup J_{k3} \).

The sets \( I_k \) are also disjoint and \( \bigcup_{k \geq 2} I_k = [3/4, 1) \). The following properties are easy to check:

\[
\forall k \geq 2, \forall x \in I_k, \quad 2^{-k-1} \leq 1 - x \leq 2^{-k}; \quad (11)
\]

\[
\forall k \geq 2, \forall t \in J_{k2}, \quad 2^{-k-2} \leq 1 - t \leq 2^{-k+1}; \quad (12)
\]

\[
\forall k \geq 2, \forall x \in I_k, \forall t \in J_{k1}, \quad 2^{-k+1} \leq 1 - t \leq 2(x - t) \leq 2(1 - t); \quad (13)
\]

\[
\forall k \geq 2, \forall x \in I_k, \forall t \in J_{k3}, \quad 1 - t \leq 2^{-k-2} \leq t - x \leq 2^{-k}. \quad (14)
\]

We can write

\[
\chi_{[3/4, 1]} W_{2,n}(\chi E) = \sum_{k \geq 2} \chi_{I_k} W_{2,n}(\chi E\chi_{J_{k1}}) + \sum_{k \geq 2} \chi_{I_k} W_{2,n}(\chi E\chi_{J_{k2}}) + \sum_{k \geq 2} \chi_{I_k} W_{2,n}(\chi E\chi_{J_{k3}}).
\]
We prove that each term is bounded.

(c1) If \( x \in I_k \), from (13) and Hölder’s inequality for Lorentz spaces it follows

\[
|H((1-t^2)\chi_E \chi_{J_k} Q_n w, x)| \leq C \int_{-1}^{1} \chi_E \chi_{J_k} |Q_n w| \leq C \|\chi_E\|_{L^p(w)} \|\chi_{J_k} Q_n\|_{L^q(w)}.
\]

From the estimates (4) for \( Q_n \), property (8), lemma 7 and using that \( 1 \leq 1 + t \leq 2 \) for \( t \in J_{k1} \), we obtain

\[
\|\chi_{J_k} Q_n\|_{L^q(w)} \leq C \|\chi_{J_k} (1 - t + n^{-2})^{(\alpha+1)/q - (2\alpha+3)/4} (1 - t + n^{-2})^{-(\alpha+1)/q}\|_{L^q(w)} \leq C (1 - x + n^{-2})^{\alpha+1/q - (2\alpha+3)/4}.
\]

Therefore, if \( x \in I_k \) then

\[
|H((1-t^2)\chi_E \chi_{J_k} Q_n w, x)| \leq C \|\chi_E\|_{L^p(w)} (1 - x + n^{-2})^{\alpha+1/q - (2\alpha+3)/4},
\]

where the constant \( C \) does not depend on \( k \). Since the \( I_k \) are disjoint, this implies

\[
\left| \sum_{k \geq 2} \chi_{I_k}(x) W_{2, n}(\chi_E \chi_{J_{k1}} x) \right| \leq C (1 - x + n^{-2})^{-(\alpha+1)/q - (2\alpha+3)/4} = C (1 - x + n^{-2})^{-(\alpha+1)/p} \left\| \chi_E \right\|_{L^p(w)} \leq C (1 - x)^{-(\alpha+1)/p} \left\| \chi_E \right\|_{L^p(w)}.
\]

Then, by lemma 7,

\[
\left\| \sum_{k \geq 2} \chi_{I_k} W_{2, n}(\chi_E \chi_{J_{k1}}) \right\|_{L^p(w)} \leq C \|\chi_E\|_{L^p(w)}.
\]

c2) Let \( k \geq 2 \). By (4) and (11),

\[
\left\| \chi_{I_k} W_{2, n}(\chi_E \chi_{J_{k2}}) \right\|_{L^p(w)} = C \left\| \chi_{I_k} P_{n+1} H((1-t^2)\chi_E \chi_{J_{k2}} Q_n w) \right\|_{L^p(w)} \leq C (2^{-k} + n^{-2})^{-(\alpha+1)/q} (2^{-k})^{\alpha/p} \left\| \chi_{I_k} H((1-t^2)\chi_E \chi_{J_{k2}} Q_n w) \right\|_{L^p(w)} \leq C (2^{-k} + n^{-2})^{-(\alpha+1)/q} (2^{-k})^{\alpha/p} \left\| H((1-t^2)\chi_E \chi_{J_{k2}} Q_n w) \right\|_{L^p(w)}.
\]

Since the Hilbert transform is bounded in \( L^p(dx) \), this expression can be bounded, using (4) and (12), by

\[
C (2^{-k} + n^{-2})^{-(\alpha+1)/q} (2^{-k})^{\alpha/p} (1 - x^2) \chi_E \chi_{J_{k2}} Q_n w \|_{L^p(dx)} \leq \]
Now, as the functions $\chi_{I_k} W_{2,n} (\chi_E \chi_{J_{k_2}})$ have non-overlapping support and $\sum_{k \geq 2} \chi_{J_{k_2}} \leq 3$, we get
\[
\left\| \sum_{k \geq 2} \chi_{I_k} W_{2,n} (\chi_E \chi_{J_{k_2}}) \right\|_{L^p_{\alpha}(w)} \leq \sum_{k \geq 2} \left\| \chi_{I_k} W_{2,n} (\chi_E \chi_{J_{k_2}}) \right\|_{L^p_{\alpha}(w)} \leq C \sum_{k \geq 2} \left\| \chi_E \chi_{J_{k_2}} \right\|_{L^p(w)} \leq C \left\| \chi_E \right\|_{L^p(w)}.
\]
That is,
\[
\left\| \sum_{k \geq 2} \chi_{I_k} W_{2,n} (\chi_E \chi_{J_{k_2}}) \right\|_{L^p_{\alpha}(w)} \leq C \left\| \chi_E \right\|_{L^p(w)}.
\]

c3) Let $k \geq 2$ and $x \in I_k$. By (14), Hölder’s inequality for Lorentz spaces and (4), it follows
\[
|H((1 - t^2)\chi_E \chi_{J_{k_3}} Q_n w, x)| \leq C 2^k \int_{-1}^1 (1 - t^2)\chi_E (t) \chi_{J_{k_3}} (t) Q_n (t) w(t) dt \leq C 2^k \left\| (1 - t)\chi_E \chi_{J_{k_3}} Q_n \right\|_{L^1(w)} \leq C 2^k \left\| \chi_E \right\|_{L^p(w)} \left\| (1 - t)(1 - t + n^{-2})^{-(2\alpha+3)/4} \chi_{J_{k_3}} \right\|_{L^2((1-t)^\alpha)}.
\]
By lemma 8 and (8),
\[
\left\| (1 - t)(1 - t + n^{-2})^{-(2\alpha+3)/4} \chi_{J_{k_3}} \right\|_{L^2((1-t)^\alpha)} \leq C (2^{-k})^{(1-(\alpha+1)/p)} (2^{-k} + n^{-2})^{(2\alpha+1)/4},
\]
what, together with (11), implies
\[
|H((1 - t^2)\chi_E \chi_{J_{k_3}} Q_n w, x)| \leq C (1 - x)^{-(\alpha+1)/p} (1 - x + n^{-2})^{(2\alpha+1)/4} \left\| \chi_E \right\|_{L^p(w)}
\]
if $x \in I_k$, with a constant $C$ which does not depend on $x$, $E$, $k$, $n$. Since the $I_k$ are non-overlapping, we have
\[
|\sum_{k \geq 2} \chi_{I_k} (x) W_{2,n} (\chi_E \chi_{J_{k_3}}, x)| \leq C \chi_{[3/4,1]} (1 - x)^{-(\alpha+1)/p} \left\| \chi_E \right\|_{L^p(w)}
\]
and, by lemma 7,
\[
\left\| \sum_{k \geq 2} \chi_{I_k} W_{2,n} (\chi_E \chi_{J_{k_3}}) \right\|_{L^p_{\alpha}(w)} \leq C \left\| \chi_E \right\|_{L^p(w)}.
\]
This concludes the proof of the lemma. ■

The weak boundedness
\[ \|u S_n f\|_{L^p_w} \leq C \|u f\|_{L^p(w)} \]
implies the following conditions (see [2], theorem 1, with the appropriate changes):
\[ u \in L^p_w(w) \]
\[ u^{-1} \in L^q(w) \]
\[ u(x)w(x)^{-1/2}(1-x^2)^{-1/4} \in L^p_w(w) \]
\[ u(x)^{-1}w(x)^{-1/2}(1-x^2)^{-1/4} \in L^q(w). \]

With the weight \( u(x) = (1-x)^a(1+x)^b \) and having in mind that \( \alpha, \beta \geq -1/2 \), this means
\[ -\frac{1}{4} \leq a + (\alpha + 1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}, \]
\[ -\frac{1}{4} \leq b + (\beta + 1)(\frac{1}{p} - \frac{1}{2}) < \frac{1}{4}. \]

Therefore, we only need to show that the equality cannot occur in the left hand side of these equations. Assume, for example,
\[ -\frac{1}{4} = a + (\alpha + 1)(\frac{1}{p} - \frac{1}{2}). \]

Let us consider again Pollard’s decomposition of the partial sums \( S_n f \). As we mentioned at the beginning of the previous section, the proofs of lemmas 9 and 10 essentially show that there exists a constant \( C \) such that for all \( f \in L^p(w) \) and every \( n \in \mathbb{N} \)
\[ \|u W_{1,n} f\|_{L^p_w(w)} \leq C \|u f\|_{L^p(w)} \]
and
\[ \|u W_{3,n} f\|_{L^p(w)} \leq C \|u f\|_{L^p(w)} \]
(notice that in the case \( \alpha, \beta \geq -1/2 \), the polynomials \( P_n \) satisfy the estimate analogous to (5), what simplifies the proofs). Under our hypothesis, this implies that there exists also a constant \( C \) such that for all \( f \in L^p(u^p w) \) and every \( n \in \mathbb{N} \)
\[ \|u W_{2,n} f\|_{L^p_{1,w}(w)} \leq C \|u f\|_{L^p(w)}, \]
that is,
\[ \|u P_{n+1} H g\|_{L^p_{1,w}(w)} \leq C \|u(x)(1-x^2)^{-1}Q_n(x)^{-1}w(x)^{-1}g\|_{L^p(w)}. \]

Applying (3), we have
\[ \|u P_{n+1} \chi_I\|_{L^p_{1,w}(w)} \left( \int_{-1}^{1} \frac{u(x)^{-q}(1-x^2)^{q}|Q_n(x)|^q w(x)}{(|I| + |x-x_I|)^q} dx \right)^{1/q} \leq C \]
for every interval $I \subseteq [-1, 1]$, with a constant $C > 0$ independent of $n$ and $I$; now, by lemma 6
\[
\|u(x)(1 - x^2)^{-1/4}w(x)^{-1/2}\chi_I\|_{L^p(w)} \left(\int_{-1}^{1} \frac{u(x)^{-q}(1 - x^2)^{q/4}w(x)^{-q/2}}{|I| + |x - x_I|} \, dx\right)^{1/q} \leq C.
\]
Taking $I = [1 - \varepsilon, 1]$, it follows
\[
\|x^{a-\alpha/2-1/4}\chi_{[0,\varepsilon]}\|_{L^p(w)} \left(\int_{0}^{1} \frac{x^{aq+q/4+\alpha(1-q/2)}}{\varepsilon + |x - \varepsilon/2|} \, dx\right)^{1/q} \leq C. 
\] (16)
Now, by lemma 7 and (15)
\[
\|x^{a-\alpha/2-1/4}\chi_{[0,\varepsilon]}\|_{L^p(w)} = K 
\] (17) and
\[
\int_{0}^{1} \frac{x^{-aq+q/4+\alpha(1-q/2)}}{\varepsilon + |x - \varepsilon/2|} \, dx = \int_{0}^{1} \frac{x^{1/(p-1)}}{\varepsilon + |x - \varepsilon/2|} \, dx \geq C \int_{\varepsilon}^{1} x^{1/(p-1)-q} \, dx = C |\log \varepsilon|,
\]
which, together with (17), leads to a contradiction in (16). Therefore, (15) cannot be true and the theorem is proved.

**REFERENCES**


