Embedding l_p^n into r-Banach spaces, $0 < r \le p < 2$

by

Jesús Bastero and Julio Bernués*

Departamento de Matemáticas

Facultad de Ciencias

50009 Zaragoza (Spain)

0. Introduction. Previous results.

Given $(X, \|\cdot\|)$, $(Y, |||\cdot|||)$ two quasi-Banach spaces and $0 < \varepsilon < 1$, we say that X $(1+\varepsilon)$-embedds into Y if there is a one-to-one linear map $T\colon X \to Y$ such that $1-\varepsilon \le \dfrac{|||T(x)|||}{\|x\|} \le 1+\varepsilon$. We will denote this fact by the diagram $X \overset{1+\varepsilon}{\hookrightarrow} Y$.

Lately, some authors have been investigating questions from the Local Theory in the context of quasi-Banach spaces. In [2] and [3] the analogue of Dvoretzky's theorem on quasi-Banach spaces is proved. For non-spherical sections we only know answers in particular cases. In [5] the authors show that if $0 < r < p < 2$, $r \le 1$, $\ell_p^k \overset{1+\varepsilon}{\hookrightarrow} \ell_r^n$ provided that $n \ge C(\varepsilon, r, p)\ k$.

In this paper we obtain an analogue of the main results in [13] and give general estimates for the size of ℓ_p^k-sections of any r-Banach space in terms of the stable-type constant. The main ideas of the proofs (use of p-stable random variables, deviation inequalities...) are the same as the ones used in [13]. In some cases Pisier's ideas adapt to the r-Banach case; in some others the extension is not obvious at all. As a corollary we will re-prove the result in [5] quoted above. We do this in sections 1 and 2. In section 3 we study the set $\{p \,|\, \ell_p^n \overset{1+\varepsilon}{\hookrightarrow} X,\ \forall n \in \mathbf{N}, \forall\, 0 < \varepsilon < 1\}$, X an infinite-dimensional r-Banach space. In this way we give a strong version of the Maurey-Pisier

* The two named authors are partially supported by Grant DGICYT PS 87-0059.Ê The second named author is also supported by D.G.A.(Spain)

theorem (see [10]) for the type in r-Banach spaces. An infinite dimensional version of this result has been already proved by N.Kalton [6] using ultrapower techniques. Finally in section 4 we apply the methods used in 1 and 2 to the problem of embedding finite subsets of L_p into ℓ_r^n, $0 < r \le p < 2,\ r \le 1$ (see section 4 for a definition). In [15] the author proves that *any* finite subset $T \subset L_p, 1 \le p < 2$ with card $T = N$ can be $(1+\varepsilon)$-embedded into ℓ_p^n provided that $n \ge C(\varepsilon)\ N\ \log N$, and it is conjectured that the right estimate is some power of $\log N$. In this last section we improve the results in [15] for particular sets T.

In the sequel $(X, \|\cdot\|)$ will denote an r-Banach space, $0 < r < 1$, p will be a real number verifying $0 < p < 2$, $r \le p$.

Definition. *A real-valued random variable θ is called p-stable if its Fourier transform is $\mathbb{E}(e^{it\theta}) = e^{-|t|^p}$.*

An interesting property of p-stable random variables is the following: Let $Z = \sum_{i=1}^n \theta_i x_i$ with $x_i \in X$ and θ_i independent identically distributed (i.i.d.) p-stable random variables. If $(Z_i)_{i=1}^k$ are independent copies of Z then for every $(a_i) \in \mathbf{R}^k$, $\displaystyle\sum_{i=1}^k a_i Z_i \overset{\mathrm{d}}{=} Z \left(\sum_{i=1}^k |a_i|^p \right)^{1/p}$, where $\overset{\mathrm{d}}{=}$ means equality in distribution.

There are only p-stable random variables for $0 < p \le 2$. If $0 < p < 2$, $\displaystyle\mathbb{E}\,\|\sum_{i=1}^n \theta_i x_i\|^s < \infty \iff s < p$. Moreover, for every $0 < t < s < p$, $\displaystyle\left(\mathbb{E}\,\|\sum_{i=1}^n \theta_i x_i\|^s\right)^{1/s} \le C(r,s,t,p) \left(\mathbb{E}\,\|\sum_{i=1}^n \theta_i x_i\|^t\right)^{1/t}$.

Definition. *An r-Banach space X is said to be of stable type p if there is a constant $C > 0$, such that for every $n \in \mathbf{N}$ and any vectors $x_1, \dots, x_n \in X$*

$$\left(\mathbb{E}\,\|\sum_{i=1}^n \theta_i x_i\|^s\right)^{1/s} \le C \left(\sum_{i=1}^n \|x_i\|^p\right)^{1/p}$$

where $s = r$ if $r < p$ and $s = \dfrac{r}{2}$ if $r = p$, and θ_i denote i.i.d. p-stable random variables.

The stable-type constant of X denoted by § is the infimun of the constants C verifying the inequality above.

If we put ε_i i.i.d. Rademacher random variables instead of θ_i, we obtain the definition of Rademacher type of X.

We recall the following properties of stable type. For more details see [14]:

- Every r-Banach space is of stable type s for every $s \in (0,r)$.

- If X is of stable type s, it is of stable type t for every $t < s$.

- $ST_p(\ell_q^n) = C_{p,q} n^{1/q-1/p}$ for $0 < q < p < 2$.

- $ST_p(\ell_p^n) \sim C_p (\log n)^{1/p}$ for $0 < p < 2$.

- The space ℓ_p is of stable type q for every $q < p$, but not of stable type p $(0 < p < 2)$.

We will use the following equivalent definition of §. For a proof of this equivalence follow Proposition 1.2 in [13].

Proposition 0.1. *§ is the infimun of the constants $C > 0$ such that,*

$$\left(I\!\!E \| \sum_{i=1}^{n} \theta_i x_i \|^s \right)^{1/s} \leq C n^{1/p} \sup_{1 \leq i \leq n} \|x_i\|$$

for every $n \in \mathbf{N}$ and any $x_1, \ldots, x_n \in X$, where $s = r$ if $r < p$ and $s = r/2$ if $p = r$.

A more convenient representation for $\sum_{i=1}^n \theta_i x_i$ is known. In order to present it we need to introduce some more notation: Given $x_1, \ldots, x_n \in X$ let Y be the random variable with distribution $\frac{1}{2n} \sum_{i=1}^n (\delta_{x_i} + \delta_{-x_i})$ and let $(Y_j), j \geq 1$ be independent copies of Y. Let Γ_j be the random variable obtained by summing j i.i.d. exponential random variables. The distribution function for Γ_j is known to be

$$I\!\!P(\Gamma_j < t) = \int_0^t \frac{x^{j-1}}{(j-1)!} e^{-x} \, dx$$

Theorem 0.2. ([8,9]). *For every $0 < p < 2$ there is a constant $C_p > 0$ such that*

$$\frac{\sum_{i=1}^{n} \theta_i x_i}{n^{1/p}} \stackrel{\mathrm{d}}{=} C_p \sum_{j=1}^{\infty} \Gamma_j^{-1/p} \, Y_j$$

Notación For every $m \geq 1$, write $S^{(m)} = \sum_{j=1}^{\infty} \Gamma_j^{-1/p} \, Y_j$, $\tilde{S}^{(m)} = \sum_{j=1}^{\infty} j^{-1/p} \, Y_j$, and for $i \geq 1$, $S_i^{(m)} = \sum_{j=1}^{\infty} \Gamma_{ij}^{-1/p} \, Y_{ij}$, $\tilde{S}_i^{(m)} = \sum_{j=1}^{\infty} j^{-1/p} \, Y_{ij}$ where Γ_{ij} and Y_{ij} are independent copies of Γ_j and Y_j respectively.

We compare the moments of order r of linear combinations of $S_i^{(m)}$ and $\tilde{S}_i^{(m)}$.

Lemma 0.3. *Let $0 < \frac{(4-p)p}{4} < r \leq p < 2$. There is a constant $K_{r,p} > 0$ depending on r and p such that for every $(a_i) \in \mathbf{R}^k$ and $m \in \mathbf{N}$, we have if $r < p$ and $m \geq 1$ or if $r = p$ and $m \geq 2$,*

$$\left| E\| \sum_{i=1}^{k} a_i S_i^{(m)} \|^r - \mathbb{E}\| \sum_{i=1}^{k} a_i \tilde{S}_i^{(m)} \|^r \right| \leq K_{r,p} \sum_{i=1}^{k} |a_i|^r \sup_{1 \leq i \leq n} \|x_i\|^r$$

Demostración:

$$\left| \mathbb{E}\| \sum_{i=1}^{k} a_i S_i^{(m)} \|^r - \mathbb{E}\| \sum_{i=1}^{k} a_i \tilde{S}_i^{(m)} \|^r \right| \leq \mathbb{E}\| \sum_{i=1}^{k} |a_i| \, (S_i^{(m)} - \tilde{S}_i^{(m)}) \, \|^r \leq \sum_{i=1}^{k} |a_i|^r \mathbb{E}\|S_i^{(m)} - \tilde{S}_i^{(m)}\|^r$$

$$\leq \sum_{i=1}^{k} |a_i|^r \mathbb{E}\| \sum_{j=1}^{\infty} \left| \frac{1}{{\Gamma_j}^{1/p}} - \frac{1}{j^{1/p}} \right| Y_{ij} \|^r \leq \sum_{i=1}^{k} |a_i|^r \mathbb{E} \sum_{j=1}^{\infty} \left| \frac{1}{{\Gamma_j}^{1/p}} - \frac{1}{j^{1/p}} \right|^r \|Y_{ij}\|^r$$

It is enough to study the convergence of the series $I_m = \sum_{j=1}^{\infty} \mathbb{E} \left| \Gamma_j^{-1/p} - j^{-1/p} \right|^r$. We know an expression for the distribution function of Γ_j so that we just have to estimate

$$I_m = \int_0^{\infty} \sum_{j=1}^{\infty} |x^{1/p} - j^{1/p}|^r \, \frac{x^{j-1}}{(j-1)!} e^{-x} \, dx$$

By using Stirling's formula and the change of variable $\frac{x}{j} = t$ the formula above reduces to

$$\int_0^{\infty} \left| 1 - t^{1/p} \right|^r \, \frac{1}{t} \sum_{j=1}^{\infty} \left(\frac{t}{e^{t-1}} \right)^j \, j^{1/2 - r/p} \, dt$$

If $r \leq \frac{p}{2}$ the integral above diverges near $t = 1$ for all m.

If $r > \frac{p}{2}$ it always converges near ∞, we have convergence near 0 iff $m > \frac{r}{p}$ and by using Hölder's inequality, converges near $t = 1$ if $\frac{(4-p)p}{4} < r < p$, for any m. Note that $\frac{p}{2} < \frac{(4-p)p}{4}$ and so, $r > \frac{p}{2}$.

///

Observación Some remarks about the number $\frac{(4-p)p}{4}$ will be useful in the sequel:

- $\frac{(4-p)p}{4} < 1$ since $p < 2$.

- It is easy to see that $\frac{p}{2} < \frac{(4-p)p}{4} < p$ and so, $\frac{(4-p)p}{4} < r < p$ implies $1 < \frac{p}{r} < 2$.

- The sequence $(p_n)_{n \geq 1}$ given by the relation $p_{n+1} = \frac{(4-p_n)p_n}{4}, 0 < p_1 < 2$ is strictly monotone and decresing and $\lim_{n \to \infty} p_n = 0$.

Approximation lemmas.

Definition. *Let $\delta > 0$. A subset T of the unit sphere S_X of X is a δ-net if for every $x \in S_X$ there is an element $t \in T$ such that $\|x - t\|^r \leq \delta$.*

Lemma 0.4. *Let X be of dimension n and $\delta > 0$. S_X contains a δ-net of cardinality at most $\exp \dfrac{2n}{r\delta}$.*

The following approximation lemma is an easy consequence of the one used in [2] and [5].

Lemma 0.5. *Let X be r-Banach and Y s-Banach. Let $0 < \varepsilon < 1$ and $\delta = \frac{\varepsilon}{5}$. If a linear operator $T : X \to Y$ verifies*

$$1 - \delta \leq \|Tx\|^r \leq 1 + \delta$$

for every x in a $\delta^{s/r}$-net of S_X then,

$$1 - \varepsilon \leq \|Tx\|^r \leq 1 + \varepsilon$$

for every $x \in S_X$.

We will always work with the function $\|\cdot\|^r$. In order to remove the exponent r at the end of the proofs we need the following easy lemma which is nothing but the Mean Value Theorem applied to the function $t^{1/r}$.

Lemma 0.6. *Let $0 < r, \varepsilon < 1$, $t > 0$ and $\delta = \dfrac{2\varepsilon r}{2^{1/r}}$. Then, $1-\delta \leq t^r \leq 1+\delta \Longrightarrow 1-\varepsilon \leq t \leq 1+\varepsilon$.*

Deviation inequalities.

As in most of the theorems quoted in the introduction, the proof of our main results will rest on the so-called deviation inequalities.

Lemma 0.7. (Deviation inequality). *Let $1 \leq q < 2$. Let (ξ_j) be a sequence of independent random variables with values in X such that $\operatorname{essup} \|\xi_j\| = \lambda_j < \infty$. If $\|(\lambda_j^r)\|_{q,\infty} < \infty$ and $\sum_{j\geq 1} \xi_j$ converges almost surely, (a.s.), to a random variable ξ with $\|\xi\|^r$ integrable then, for every $t > 0$*

$$\mathbb{P}\left\{\left|\|\xi\|^r - \mathbb{E}\|\xi\|^r\right| > t\right\} \leq 2\exp -c_q\left(\frac{t}{\|(\lambda_j^r)\|_{q,\infty}}\right)^{q'} \qquad \text{if} \qquad 1<q<2$$

and

$$\mathbb{P}\left\{\left|\|\xi\|^r - \mathbb{E}\|\xi\|^r\right| > t\right\} \leq K\exp -\exp\left(\frac{ct}{\|(\lambda_j^r)\|_{1,\infty}}\right) \qquad \text{if} \qquad q=1$$

where c, K are positive numerical constants, c_q is a constant depending uniquely on q and q' is such that $q^{-1}+q'^{-1}=1$.

Demostración:

Denote by $\mathcal{F}_j$ the $\sigma-$algebra generated by $\{\xi_1,\ldots,\xi_j\}$. Write $d_j = \mathbb{E}(\|\xi\|^r\,|\mathcal{F}_j) - \mathbb{E}(\|\xi\|^r\,|\mathcal{F}_{j-1})$. It is easy to see that $\sum_{j=1}^{\infty} d_j = \|\xi\|^r - \mathbb{E}\|\xi\|^r$, a.s.. Also it is not difficult to prove the analogue of Yurinski's inequality [16] for r-Banach spaces, namely for every $j \geq 1$, $|d_j| \leq \|\xi_j\|^r + \mathbb{E}\|\xi_j\|^r$. Therefore we have essup $|d_j| \leq 2\lambda_j^r$. Conclude by using two well known exponential inequalities for real valued martingales:

For every (d_j) scalar martingale difference sequence such that $\|d_j\|_\infty = \mu_j < \infty$ and $\|(\mu_j)\|_{q,\infty} < \infty$ if $1 \leq q < 2$ we have $\forall\ t > 0$,

$$\mathbb{P}\left\{ |\sum_{j=1}^{\infty} d_j| > t \right\} \leq 2 \exp -c_q \left(\frac{t}{\|(\mu_j)\|_{q,\infty}} \right)^{q'} \qquad \text{if} \qquad 1 < q < 2$$

and

$$\mathbb{P}\{|\sum_{j=1}^{\infty} d_j| > t\} \leq K \exp - \left(\exp \frac{c\, t}{\|(\mu_j)\|_{1,\infty}} \right) \qquad \text{if} \qquad q = 1$$

We refer to [5] for more information on the former and to [13] for the latter. ///

Observación By Yurinski's inequality and the property of orthogonality of martingale differences it is not difficult to check that $\mathbb{E}\left|\, \|\xi\|^r - \mathbb{E}\|\xi\|^r \,\right|^2 = \mathbb{E}\left| \sum_{j=1}^{\infty} d_j \right|^2 \leq 4 \sum_{j=1}^{\infty} \lambda_j^{2r}$.

1. The case $r < p$.

Now we are in position to state the main result of the section,

Theorem 1.1. *Let $r, p \in \mathbf{R}$ such that $0 < \frac{(4-p)p}{4} < r < p < 2$. There exists a constant $C(r,p) > 0$ such that for every $0 < \varepsilon < 1$ and every r-Banach space X, $\ell_p^k \overset{1+\varepsilon}{\hookrightarrow} X$ as long as*

$$k < C(r,p)\, \varepsilon^{\frac{p^2}{r(p-r)}} \left(ST_p(X) \right)^{\frac{1}{\frac{1}{r} - \frac{1}{p}}}$$

Demostración: Fix $0 < \varepsilon < 1$. By Proposition 0.1. pick $x_1, \ldots, x_n$ in the unit ball of X such that

$$\left(\mathbb{E}\, \| \sum_{i=1}^{n} \theta_i x_i \|^r \right)^{1/r} \geq \frac{1}{2} (ST_p(X)) n^{1/p}$$

It follows from Theorem 0.2 that $I\!\!E\|S^{(1)}\|^r \geq \left(\frac{1}{2C_p}\right)^r (ST_p(X))^r$

Let $k \in \mathbf{N}$ to be fixed, and let $(a_i) \in \mathbf{R}^k$ such that $\sum_{i=1}^k |a_i|^p = 1$.

Denote $\xi_{ij} = a_i Y_{ij} j^{-1/p}$. For such a sequence of random variables Lemma 0.7. particularizes as follows:

$$I\!\!P\left\{\left|\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\right| > t\right\} \leq 2\exp -c_q\, t^{q'}$$

with $q = \frac{p}{r}$, $1 < q < 2$ (the proof of this fact reduces to the same computations as in the Banach space setting; see [12] for the details). Also Lemma 0.3. applied to the same sequence yields to the inequality

$$\left|I\!\!E\|\sum_{i=1}^k a_i S_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\right| \leq K_{r,p}\, k^{1-r/p}$$

Let $\delta = \dfrac{2\,\varepsilon\, r}{5.\, 2^{1/r}}$ and $\delta' = \dfrac{\delta}{1 + (2C_p)^r}$. Then,

$$I\!\!P\left\{\left|\,\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i S_i^{(1)}\|^r\,\right| > \delta\, I\!\!E\|S^{(1)}\|^r\right\}$$

$$\leq I\!\!P\left\{\left|\,\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\,\right| + \right.$$

$$\left. + \left|\, I\!\!E\|S^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\,\right| > \delta'\, I\!\!E\|S^{(1)}\|^r + \delta'\,(2\,C_p)^r\, I\!\!E\|S^{(1)}\|^r\right\}$$

$$\leq I\!\!P\left\{\left|\,\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\,\right| + \right.$$

$$\left. + \left|\, I\!\!E\|S^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\,\right| > \delta'\, I\!\!E\|S^{(1)}\|^r + \delta'\, ST_p(X)^r\right\}$$

Now if we choose k such that $K_{r,p}\, k^{1-r/p} \leq \delta'\, ST_p(X)^r$ we have by Lemma 0.3.,

$$I\!\!P\left\{\left|\,\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i S_i^{(1)}\|^r\,\right| > \delta\, I\!\!E\|S^{(1)}\|^r\right\} \leq$$

$$\leq I\!\!P\left\{\left|\,\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r - I\!\!E\|\sum_{i=1}^k a_i \tilde{S}_i^{(1)}\|^r\,\right| > \delta' I\!\!E\|S^{(1)}\|^r\right\} \leq 2\exp -C_{p,r}\, \delta'^{q'}\left(I\!\!E\|S^{(1)}\|^r\right)^{q'} \leq$$

$$\leq 2\exp -C_{p,r}\, \delta'^{q'}\left(ST_p(X)\right)^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}$$

It is straightforward to check that the restriction on k is the same as $k \leq C(\varepsilon, r, p)\big(ST_p(X)\big)^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}$. The rest of the proof is standard. We have already estimated the probability

$$\mathbb{P}\left\{\left|\|\sum_{i=1}^{k} a_i \tilde{S}_i^{(1)}\|^r - \mathbb{E}\|S^{(1)}\|^r\right| > \delta \mathbb{E}\|S^{(1)}\|^r\right\} \leq 2\exp -C_{p,r}\,\delta'^{q'}\ \big(ST_p(X)\big)^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}$$

Let $\delta_1 = \delta^{\min(1,p)/r}$. Let N_{δ_1} be the cardinality of a δ_1-net T_{δ_1} in the unit ball of ℓ_p^n. It follows that

$$\mathbb{P}\left\{\left|\|\sum_{i=1}^{k} a_i \tilde{S}_i^{(1)}\|^r - \mathbb{E}\|S^{(1)}\|^r\right| \leq \delta\, \mathbb{E}\|S^{(1)}\|^r \ \mid\ \forall (a_i) \in T_{\delta_1}\right\}$$

$$\geq 1 - N_{\delta_1}\, 2\exp -C_{p,r}\,\delta'^{q'}\ \big(ST_p(X)\big)^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}$$

If we oblige the second part of the inequality to be strictly positive then there will exist an element $\omega = \omega(\varepsilon)$ in the probability space such that $\left|\|\sum_{i=1}^{k} a_i \tilde{S}_i^{(1)}(\omega)\|^r - \mathbb{E}\|S^{(1)}\|^r\right| \leq \delta\, \mathbb{E}\|S^{(1)}\|^r$ holds for every $(a_i) \in T_{\delta_1}$. This is achieved, in view of Lemma 0.4., if

$$2\ \exp \frac{2k}{\min(1,p)\delta_1} \exp -C_{p,r}\,\delta^{q'}\ \big(ST_p(X)\big)^{\frac{1}{\frac{1}{r}-\frac{1}{p}}} < 1$$

which is a consequence of the condition $k < C(r,p)\,\varepsilon^{\frac{p^2}{r(p-r)}}\ \big(ST_p(X)\big)^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}$. Finally use Lemma 0.6. to remove the exponent r and get $1-\varepsilon \leq \|\sum_{i=1}^{k} a_i \frac{\tilde{S}_i^{(1)}}{\mathbb{E}\|S^{(1)}\|}\| \leq 1+\varepsilon$.

///

As announced we deduce the main result in [5]:

Corollary 1.2. *If $X = \ell_r^n\,(0 < r < 1)$, and $r < p < 2$ then for every $0 < \varepsilon < 1$ there is a constant $C = C(\varepsilon, r, p)$ such that $\ell_p^k \overset{1+\varepsilon}{\hookrightarrow} \ell_r^n$ for every $k \leq Cn$.*

Demostración: Recall that $ST_p(\ell_r^n) = C_{p,r} n^{1/r-1/p}$ for $0 < r < p < 2$ and $\frac{(4-p)p}{4} < 1$. Theorem 1.1 tells us that $\ell_p^k \overset{1+\varepsilon}{\hookrightarrow} \ell_s^n$ whenever $\frac{(4-p)p}{4} < s < p, s < 1$ for every $k \leq Cn$. By iteration we get the result.

2. The case $r = p$.

The difference from the previous case is that the moment of order r of $\|S^{(1)}\|$ does not exist and that is the reason why we will have to truncate it and consider $S^{(m)}$ and $\tilde{S}^{(m)}$, $m \geq 2$. As before it will be important to compare the moments of certain variables.

Lemma 2.1.. *Let $\delta > 0, 0 < r < 1$. There exists functions $m = m(\delta, r), C(\delta, r)$ and $\varphi(\delta, r)$ with $\varphi(\delta, r) \to 0$ as $\delta \to 0$ for fixed r, such that for every $k \in \mathbf{N}$ such that*

$$\log k \leq C(\delta, r) \left(ST_r(X)\right)^r$$

and every $(a_i) \in \mathbf{R}^k$ such that $\sum_{i=1}^k |a_i|^r = 1$, we have

$$\left| I\!\!E \| \sum_{i=1}^k a_i \tilde{S}_i^{(m)} \|^r - M^r \right| < M^r \varphi(\delta, r)$$

where $M = \left(I\!\!E \| \sum_{i=1}^k a_i S_i^{(1)} \|^{r/2} \right)^{2/r} = \left(I\!\!E \|S^{(1)}\|^{r/2} \right)^{2/r}.$

Denote $\Phi_m = \| \sum_{i=1}^k a_i \tilde{S}_i^{(m)} \|$ and $\Psi_m = \| \sum_{i=1}^k a_i S_i^{(m)} \|$.

We will prove 2.1. later; now we will state the main theorem of the section:

Theorem 2.2. *Let $0 < r < 1$. For every $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon, r) > 0$ such that for every r-Banach space X, $\ell_r^k \overset{1+\varepsilon}{\hookrightarrow} X$ as long as*

$$\log k < C(\varepsilon, r) \big(ST_r(X)\big)^r$$

Demostración: Fix $0 < \varepsilon < 1$. Let $\delta = \frac{2\varepsilon r}{5.2^{1/r}}$ and $m = m(\delta, r) \geq 2$ given by Lemma 1.2. Let $k \in \mathbf{N}$ and $(a_i) \in \mathbf{R}^k$ with $\sum_{i=1}^k |a_i|^r = 1$. Choose vectors $x_1 \ldots x_n \in B_X$ such that

$$\frac{1}{n^{1/r}} \left(I\!\!E \| \sum_{i=1}^n \theta_i x_i \|^{r/2} \right)^{2/r} \geq \frac{1}{2} ST_r(X)$$

By 0.2., $M = \left(\mathbb{E}\,\|\sum_{i=1}^{k} a_i S_i^{(1)}\|^{r/2}\right)^{2/r} = \frac{1}{n^{1/r}C_r}\left(\mathbb{E}\,\|\sum_{i=1}^{k}\theta_i x_i\|^{r/2}\right)^{2/r} \geq \frac{1}{2C_r} ST_r(X)$.

By Lemma 0.7. and proceeding as in the case $r < p$ we have for every $m \geq 2$,

$$\mathbb{P}\left\{\left|\|\sum_{i=1}^{k} a_i \tilde{S}_i^{(m)}\|^r - \mathbb{E}\|\sum_{i=1}^{k} a_i \tilde{S}_i^{(m)}\|^r\right| > t\right\} \leq K \exp -(\exp c\, t)$$

With the notation of Lemma 2.1. define $\delta' = \delta'(\varepsilon, r)$ such that $\delta \geq \varphi(\delta', r) + \delta'$. By using triangle inequality (and again Lemma 2.1.) it is easy to show that

$$\mathbb{P}\left\{\left|\|\sum_{i=1}^{k} a_i \tilde{S}_i^{(m)}\|^r - M^r\right| > \delta M^r\right\} \leq K \exp -(\exp c\delta' M^r)$$

and the result now follows by using again standard density arguments.

///

Proof of Lemma 2.1. We have to prove $|\,\mathbb{E}(\Phi_m^r) - M^r\,| \leq M^r\,\varphi(\delta, r)$.

Paso 1. Given $\delta > 0$ by Lemma 0.2. we can pick $m = m(\delta, r)$ such that $\left|\mathbb{E}(\Phi_m^r) - \mathbb{E}(\Psi_m^r)\right| < \delta$.

Paso 2. By $\mathbb{E}_Y$ we mean that we are fixing Γ_{ij} and integrating with respect to Y_{ij} and analagously $\mathbb{E}_\Gamma$. With this notation $\mathbb{E}_\Gamma \mathbb{E}_Y = \mathbb{E}_Y \mathbb{E}_\Gamma = \mathbb{E}$. Then

$$\begin{aligned}\left|\mathbb{E}(\Psi_m^{r/2}) - (\mathbb{E}(\Phi_m^r))^{1/2}\right| &\leq \mathbb{E}\left|\Psi_m^{r/2} - (\mathbb{E}(\Phi_m^r))^{1/2}\right| \leq \mathbb{E}\left|\Psi_m^r - \mathbb{E}(\Phi_m^r)\right|^{1/2} \\ &\leq \mathbb{E}\left|\Psi_m^r - \mathbb{E}_Y(\Psi_m^r)\right|^{1/2} + \mathbb{E}\left|\mathbb{E}_Y(\Psi_m^r) - \mathbb{E}(\Phi_m^r)\right|^{1/2} \\ &= \mathbb{E}\left|\Psi_m^r - \mathbb{E}_Y(\Psi_m^r)\right|^{1/2} + \mathbb{E}_\Gamma\left|\mathbb{E}_Y(\Psi_m^r) - \mathbb{E}_Y(\Phi_m^r)\right|^{1/2}\end{aligned}$$

We have to estimate the two summands,

Paso 3. $\left(\mathbb{E}_\Gamma\left|\mathbb{E}_Y(\Psi_m^r)-\mathbb{E}_Y(\Phi_m^r)\right|^{1/2}\right)^2 \leq \mathbb{E}_\Gamma\left|\mathbb{E}_Y(\Psi_m^r)-\mathbb{E}_Y(\Phi_m^r)\right| \leq \mathbb{E}\left|\Psi_m^r - \Phi_m^r\right| \leq \delta$ (the last inequality is Step 1).

Paso 4. For m big enough,

$$\left(\mathbb{E}\left|\Psi_m^r-\mathbb{E}_Y(\Psi_m^r)\right|^{1/2}\right)^2 \leq \left(\mathbb{E}\left|\Psi_m^r-\mathbb{E}_Y(\Psi_m^r)\right|^2\right)^{1/2}$$

$$= \left(\mathbb{E}_\Gamma\mathbb{E}_Y\left|\Psi_m^r-\mathbb{E}_Y(\Psi_m^r)\right|^2\right)^{1/2} \leq 2\left(\sum_{j\geq m}\mathbb{E}_\Gamma\Gamma_j^{-2}\right)^{1/2} \leq \delta$$

For the second inequality we first use the remark after 0.7. to obtain

$$\mathbb{E}_Y\left|\Psi_m^r-\mathbb{E}_Y(\Psi_m^r)\right|^2 \leq 4\sum_{i,j}|a_i|^{2r}\Gamma_{ij}^{-2}$$

then take espectation respect to Γ_{ij}. Finally, the third inequality follows from the known fact that $\mathbb{E}(\Gamma_j^{-2}) \sim j^{-2}$ (see [9]).

Paso 5. Let m be the maximun of the values needed in steps 1 and 4. Let $Z_i = \sum_{j\leq m}\Gamma_{ij}^{-1}$. We want to find estimates of $\|\sum_{i=1}^k|a_i|^r Z_i\|_{1/2}$ which will be applied next. In order to do so we need three lemmas. The first one is a straightforward computation; the second one can be found in [12] and the third one is sufficiently known.

Lemma 2.3. $\mathbb{P}\{\sum_{j\leq m}\Gamma_{ij}^{-1} > t\} \leq \dfrac{m^2}{t}$

Lemma 2.4. ([12]). *Let* (Z_i) *be a sequence of independent positive random variables. Let the function* $\omega \to \|Z_i(\omega)\|_{q,\infty}$. *For every* $0 < q < \infty$

$$\|\ \|Z_i(\omega)\|_{q,\infty}\|_{q,\infty}^q \leq 2e\sup_{t>0} t^q\sum_i \mathbb{P}(Z_i > t)$$

Lemma 2.5. *Let* $0 < q$,

(i) For every $(a_i) \in \mathbf{R}^n, n > 1$, $\|(a_i)\|_{q,\infty} \leq \|(a_i)\|_q \leq c_q\ (\log n)^{1/q}\ \|(a_i)\|_{q,\infty}$

(ii) For any $0 < s < q$ there is a positive constant $C_{q,s}$ such that for any measurable function f defined on a probability space, $C_{q,s}\|f\|_s \leq \|f\|_{q,\infty} \leq \|f\|_q$.

Hence,

$$\|\sum_{i=1}^{k} |a_i|^r Z_i\|_{1/2} \leq C\|\sum_{i=1}^{k} |a_i|^r Z_i\|_{1,\infty} \leq C \| \quad \| |a_i|^r Z_i(\omega)\|_{1,\infty}\|_{1,\infty} \log k$$

$$\leq C \log k \ \sup_{t>0} t \sum_{i=1}^{k} I\!P\{|a_i|^r Z_i > t\} \leq C \log k \sup_{t>0} t \sum_{i=1}^{k} \frac{m^2}{t} |a_i|^r = B(\delta, r) \log k \sum_{i=1}^{k} |a_i|^r$$

for some function B of δ and r.

Paso 6.

$$\left| I\!E\Psi_m^{r/2} - M^{r/2}\right|^2 = \left| I\!E\Psi_m^{r/2} - I\!E\| \sum_{i=1}^{k} a_i S_i^{(1)} \|^{r/2}\right|^2 \leq \left(I\!E\left|\Psi_m^{r/2} - \| \sum_{i=1}^{k} a_i S_i^{(1)} \|^{r/2}\right|\right)^2$$

$$\leq \left(I\!E\left|\Psi_m^{r} - \| \sum_{i=1}^{k} a_i S_i^{(1)} \|^{r}\right|^{1/2}\right)^2 \leq \left(I\!E\| \sum_{i=1}^{k} a_i (\tilde{S}_i^{(m)} - S_i^{(1)}) \|^{r/2}\right)^2$$

$$\leq \left(I\!E\left| \sum_{i=1}^{k} |a_i|^r \| \sum_{j\leq m} \Gamma_{ij}^{-1/r}\, Y_{ij} \|^r\right|^{1/2}\right)^2 \leq \left(I\!E\left| \sum_{i=1}^{k} |a_i|^r \sum_{j\leq m} \Gamma_{ij}^{-1}\right|^{1/2}\right)^2$$

$$\leq \ \text{(by Step 5)} \ \leq B(\delta, r) \log k$$

That is, for a certain function $C'(\delta, r)$ we have proved that $\left|I\!E(\Psi_m^{r/2}) - M^{r/2}\right| \leq (B(\delta, r) \log k)^{1/2} \leq C'(\delta, r)\, M^{r/2}$.

Final. Joining steps 2 and 6,

$$\left| (I\!E\Phi_m^r)^{1/2} - M^{r/2}\right| \leq \left| (I\!E\Phi_m^r)^{1/2} - I\!E(\Psi_m^{r/2})\right| + \left|I\!E(\Psi_m^{r/2}) - M^{r/2}\right| \leq \varphi'(\delta, r) M^{r/2}$$

and by using the Mean Value Theorem to remove the exponent $\frac{1}{2}$ we get the desired result with $\varphi = 2\varphi'(1 + \varphi')$. ///

3. The Maurey-Pisier theorem for the type for r-Banach spaces.

Notation.

$$p(X) = \inf\{p \mid \ell_p^n \overset{1+\varepsilon}{\hookrightarrow} X,\ \forall n \in \mathbf{N},\ \forall\, 0 < \varepsilon < 1\}$$

$$\tilde{p}(X) = \sup\{p \mid X \,\text{is of stable type}\, p\}$$

In order to prove the result we need to recall some relations between stable type and Rademacher type (type for short).

Lemma 3.2. ([4,14]). *For any r-Banach space X,*

(i) If X is of type p then is of stable type q for every $q < p$.

(ii) If X is of stable type p then is of type p.

Theorem 3.3. ([7]).

(i) If X is an r-Banach space of type p for $1 < p \leq 2$ then X is a Banach space. (i.e. there is an equivalent norm in X such that X turns to be Banach).

(ii) If X is an r-Banach space of type p for $0 < r < p < 1$ then X a p-Banach space. (i.e. there is an equivalent p-norm in X such that X turns to be p-Banach).

(iii) If X is an r-Banach space of type 1 for then X a p-Banach space for every $p < 1$. (i.e. there is an equivalent p-norm in X such that X turns to be p-Banach).

Theorem 3.1. *Let X be an infinite dimensional r-Banach space. Then*

i) $p(X) = \tilde{p}(X)$.

ii) $\ell^n_{p(X)} \overset{1+\varepsilon}{\hookrightarrow} X \ \forall n \in \mathbf{N}, \forall\ 0 < \varepsilon < 1$.

Demostración: i) Standard arguments taken from the Banach space context show that $r \leq \tilde{p}(X) \leq p(X) \leq 2$. The non-trivial part is to see $\tilde{p}(X) = p(X)$. Suppose $\tilde{p}(X) < p(X)$. By definition, X is of stable type q for every $q < \tilde{p}(X)$ and so is of type q-Rademacher and can be renormed to be a q-Banach space. Now choose q_1 such that $\tilde{p}(X) < q_1 < p(X)$ and $\frac{(4-q_1)q_1}{4} < q < q_1$. Since $ST_{q_1}(X) = \infty$, Theorem 1.1. tells us that $\ell^n_{q_1} \overset{1+\varepsilon}{\hookrightarrow} X$ which means $p(X) \leq q$, contradiction.

ii) also follows by standard arguments. ///

Observación Again, proceeding as in the Banach space context one can show

i) $[\tilde{p}(X), 2] = \{p \mid \ell_p^n \overset{1+\varepsilon}{\hookrightarrow} X \ \ \forall\, n \in \mathbf{N} \ \ \forall\, 0 < \varepsilon < 1\}$.

ii) $\{p \,|\, X$ is of stable type $p\}$ is an open interval.

4. Embedding subsets of L_p into ℓ_r^n, $0 < r \leq p < 2, r \leq 1$.

Given $(X, \|\cdot\|)$, $(Y, |||\cdot|||)$ two quasi-Banach spaces, $0 < \varepsilon < 1$ and a set $T \subset X$ we say that T $(1+\varepsilon)$-embedds into Y (and we will denote this by the diagram $T \overset{1+\varepsilon}{\hookrightarrow} Y$) if there is a one-to-one map $f \colon T \to Y$ such that $1 - \varepsilon \leq \dfrac{|||f(x) - f(y)|||}{\|x - y\|} \leq 1 + \varepsilon$, $\forall\, x, y \in T$.

Observación Since simple functions are dense in L_p an approximation argument shows that for every $0 < \varepsilon < 1$ and any finite set $T \subset L_p$ there is $T' \subset \ell_p$ and a one-to-one map f from T onto T' such that $1 - \varepsilon \leq \dfrac{\|f(x) - f(y)\|_p}{\|x - y\|_p} \leq 1 + \varepsilon$. Moreover if two functions $x, y \in T \subset L_p$ have disjoint support so do the corresponding images in T'. For notation reasons all the theorems are stated supposing $T \subset \ell_p$ but they can be re-written considering T contained in L_p.

Notación Let r and p be as above. Given $T \subset \ell_r$ denote $D_{(r,p)} = \sup\limits_{t,s \in T} \dfrac{\|t - s\|_r}{\|t - s\|_p}$.

Theorem 4.1. *For every r, p such that $0 < r < p < 2$, $0 < r \leq 1$ there are constants $C, C', C'' > 0$ uniquely dependent on r and p such that for every $0 < \varepsilon < 1$ and any finite set $T \subset \ell_p$ of cardinality* card $T = N$, *the diagram $T \overset{1+\varepsilon}{\hookrightarrow} \ell_r^n$ holds*

i) If $0 < \dfrac{p(4-p)}{4} < r < p$, as long as

$$D_{(r,p)}^{q'r} + \log N < C \varepsilon^{q'}\, n$$

ii) If $r \leq \dfrac{p(4-p)}{4}$, as long as

$$D_{(r,p)}^{q'r} + \log N < C' \varepsilon^{C''}\, n$$

where $q = \dfrac{p}{r}$ and q' the conjugate exponent of q.

Demostración: It is enough to prove the first statement since ii) is consequence of i), Corollary 1.2. and the fact that for every $0 < r < s$, $D_{(s,p)} \leq D_{(r,p)}$. Throughout the proof any constant depending on p and r will be denoted with the same letter C. For every $n \in \mathbf{N}$ let $Y\colon \Omega \to \ell_r^n$ be the random variable with distribution function $\dfrac{1}{2n}\displaystyle\sum_{i=1}^{n}(\delta_{e_i}+\delta_{-e_i})$, where e_i is the canonical basis of ℓ_r^n. With the notation introduced above define for every $t = (t_i)_1^\infty \in T$, $\Theta_t = \displaystyle\sum_{i=1}^{\infty} t_i S_i^{(1)}$ and $\tilde\Theta_t = \displaystyle\sum_{i=1}^{\infty} t_i \tilde S_i^{(1)}$. For every $t, s \in T$ consider $\Theta_t - \Theta_s$ and $\tilde\Theta_t - \tilde\Theta_s$. By the fundamental property of p-stable random variables we have $\|\Theta_t - \Theta_s\|_r \overset{d}{=} \|t-s\|_p \, \|S^{(1)}\|_r$ and $I\!E\Big(\|\Theta_t - \Theta_s\|_r^r \Big) = \|t-s\|_p^r \, C \, n^{\frac{1}{q'}}$. Also Lemma 0.3. yields in this case to $\Big| I\!E\Big(\|\Theta_t - \Theta_s\|_r^r \Big) - I\!E\Big(\|\tilde\Theta_t - \tilde\Theta_s\|_r^r \Big) \Big| \leq C \, \|t-s\|_r^r$. That is

$$\frac{1}{\|t-s\|_p^r}\Big| I\!E\Big(\|\Theta_t - \Theta_s\|_r^r \Big) - I\!E\Big(\|\tilde\Theta_t - \tilde\Theta_s\|_r^r \Big) \Big| \leq C \, D_{(r,p)}^r$$

Proceeding as in the proof of Theorem 1.1. we have,

$$I\!P\Big\{ \Big| \frac{\|\tilde\Theta_t - \tilde\Theta_s\|_r^r}{\|t-s\|_p^r} - \frac{I\!E\Big(\|\Theta_t - \Theta_s\|_r^r \Big)}{\|t-s\|_p^r} \Big| > \varepsilon \, \frac{I\!E\Big(\|\Theta_t - \Theta_s\|_r^r \Big)}{\|t-s\|_p^r} \Big\}$$
$$\leq I\!P\Big\{ \Big| \frac{\|\tilde\Theta_t - \tilde\Theta_s\|_r^r - I\!E\Big(\|\tilde\Theta_t - \tilde\Theta_s\|_r^r \Big)}{\|t-s\|_p^r} \Big| > C\,(\varepsilon n^{1/q'} - D_{r,p}^r) \Big\} \leq 2\, \exp -C\big(\varepsilon n^{1/q'} - D_{r,p}^r\big)^{q'}$$

We have estimated the probability

$$I\!P\Big\{ \Big| \frac{\|\tilde\Theta_t - \tilde\Theta_s\|_r^r}{\|t-s\|_p^r} - I\!E\big(\|S^{(1)}\|_r^r \big) \Big| \leq \varepsilon \, I\!E\big(\|S^{(1)}\|_r^r \big) \mid \forall\, t, s \in T \Big\} \geq 1 - \binom{N}{2} 2 \exp -C\big(\varepsilon n^{1/q'} - D_{r,p}^r\big)^{q'}$$

If this probability is strictly positive there will be an element ω in the probability space such that,

$$I\!E\big(\|S^{(1)}\|_r^r \big)\,(1-\varepsilon) \leq \frac{\|\tilde\Theta_t(\omega) - \tilde\Theta_s(\omega)\|_r^r}{\|t-s\|_p^r} \leq (1+\varepsilon) I\!E\big(\|S^{(1)}\|_r^r \big) \qquad \forall\ t, s \in T$$

Conclude using Lemma 0.6. in order to remove the exponent r. ///

Observe that the smaller the constant $D_{(r,p)}$ is the better estimate will be obtained. This is the case for instance when we consider subsets T formed by points of mutually disjoint support.

Observación Let $T \subset \ell_p$ be a set of points with mutually disjoint support. The map $f\colon T \to \ell_p$ defined by $f(t_i) = \|t_i\|_p \, e_i$ is an isometry. Indeed, for every pair $t_i, t_j \in T$,

$$\|t_i - t_j\|_p^p = \sum_{k=1}^{\infty} |t_i(k)|^p + |t_j(k)|^p = \| \; \|t_i\|_p e_i - \|t_j\|_p e_j \; \|_p^p = \|f(t_i) - f(t_j)\|_p^p$$

Observación If $T = \{\lambda_1 e_1, \dots, \lambda_N e_N\}$, then $\displaystyle\sup_{1 \le i \ne j \le N} \frac{\|\lambda_i e_i - \lambda_j e_j\|_r}{\|\lambda_i e_i - \lambda_j e_j\|_p} \le 2^{\frac{1}{r} - \frac{1}{p}}$.

Corollary 4.2. *For every r, p such that $0 < r < p < 2$, $r \le 1$ there are constants $C, C', C'' > 0$ uniquely dependent on r and p such that for every $0 < \varepsilon < 1$ and any finite set $T \subset \ell_p$ of points of mutually disjoint support and cardinality* card $T = N$, *the diagram $T \overset{1+\varepsilon}{\hookrightarrow} \ell_r^n$ holds*

i) If $0 < \dfrac{(4-p)p}{4} < r < p$, as long as

$$n > \frac{C}{\varepsilon^{q'}} \log N$$

ii) If $r \le \dfrac{(4-p)p}{4}$, as long as

$$n > \frac{C'}{\varepsilon^{C''}} \log N$$

donde $q = \dfrac{p}{r}$ y $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

Demostración: By the remarks above we can assume that T is of the form $T = \{\lambda_1 e_1, \dots \lambda_N e_N\} \subset \ell_p^N$ and $D_{(r,p)} \le 2^{1/q'}$. Now use Theorem 4.1. ///

Observación An standard volumetric argument shows that the relation between n and N in Corollary 4.2. is the best posible.

The same techniques can be used to embedd subsets of ℓ_p into ℓ_p^n.

Proposition 4.3.

i) For every $1 < p < 2$ there is a constant $C = C(p) > 0$ such that for every $0 < \varepsilon < 1$ and any finite set $T \subset \ell_p$ with cardinality card $T = N$,*the diagram $T \overset{1+\varepsilon}{\hookrightarrow} \ell_p^n$ holds provided that*

$$D^p_{(1,p)} + (\log N)^{\frac{p}{p'}} < C\varepsilon^p \log n$$

ii) For every $0 < p \leq 1$ and $0 < \delta < \dfrac{p}{4-p}$, there is a constant $C = C(p,\delta) > 0$ such that for every $0 < \varepsilon < 1$ and any finite subset $T \subset \ell_p$ of cardinality card $T = N$*, the diagram $T \overset{1+\varepsilon}{\hookrightarrow} \ell_p^n$ holds provided that*

$$D^p_{(\frac{p}{1+\delta},p)} + (\log N)^{\delta} < C\varepsilon^{1+\delta} \log n$$

The idea of the proof, that we omit, is to consider ℓ_p as an r-Banach space for appropiate r and and proceed exactly as in Theorem 4.1. However the estimates are not as satisfactory as before. As a corollary we study the situation in the case of sets of poits of mutually disjoint support.

Corollary 4.4. *For every $1 < p < 2$. There is a constant $C = C(p) > 0$ such that, for every $0 < \varepsilon < 1$ and any finite set $T \subset \ell_p$ of points of mutually disjoint support and cardinality* card $T = N$*, the diagram $T \overset{1+\varepsilon}{\hookrightarrow} \ell_p^n$ holds provided that*

$$\log n > \frac{C}{\varepsilon^p}(\log N)^{\frac{p}{p'}}$$

Demostración: Assume that T is of the form $T = \{\lambda_1 e_1, \ldots \lambda_N e_N\} \subset \ell_p^N$; for such T, $D_{1,p} \leq 2^{1/q'}$. Use Proposition 4.3. ///

Observación Write $f(N) = \exp\left[(\log N)^{\frac{p}{p'}}\right]$. It is straightforward to check that $(\log N)^a \ll f(N) \ll N^b$ for all $a, b > 0$ and so the relation given by $f(N)$ is sharper than $N \log N$, (i.e. the one achieved by Schechtman in [15] for any T) although it is worse than the one conjectured by himself in the same paper, a power of $\log N$.

If $0 < p \leq 1$, the relation obtained by applying Proposition 4.3. to a set T of points of support mutually disjoint is $\log n > \dfrac{C}{\varepsilon^{1+\delta}}(\log N)^{\delta}, \;\forall\, \delta > 0$ and it can be substantially improved (and actually reach a power of logarithm estimate) by using the techniques of Theorem 2.2.

Theorem 4.5. *For every $0 < p \leq 1$ and $0 < \varepsilon < 1$, there are constants $C, C' > 0$ depending on p and ε such that, for any set $T \subset \ell_p$ of points with mutually disjoint support and cardinality* card $T = N$*, the diagram $T \overset{1+\varepsilon}{\hookrightarrow} \ell_p^n$ holds provided that*

$$n > C\,(\log N)^{C'}$$

Demostración: Suppose $T = \{\lambda_1 e_1, \dots \lambda_N e_N\} \subset \ell_p^N$. Recall that $ST_p(\ell_p^n) \sim C_p(\log n)^{1/p}$. There are vectors $x_1, \dots, x_k \in B_{\ell_p^n}$ such that $M = \mathbb{E}(\|S^{(1)}\|_p^{p/2})^{2/p} = \mathbb{E}\left(\|\sum_{i=1}^{k}\theta_i\, x_i\,\|_p^{p/2}\right)^{2/p} k^{-1/p}C_p^{-1} \geq (2C_p)^{-1}(\log n)^{1/p}$.

For every $1 \leq i \leq N$ and $m \in \mathbf{N}$ denote $\Theta_i^{(m)} = \lambda_i\,\tilde{S}_i^{(m)}$. For any $1 \leq i \neq j \leq N$ consider $\Theta_i^{(m)} - \Theta_j^{(m)}$. The two main ingredients in the proof of Theorem 2.2. (deviation inequality and Lemma 2.1.) particularize respectively as follows:

$$\mathbb{P}\left\{\left|\frac{\|\Theta_i^{(m)} - \Theta_j^{(m)}\|_p^p}{\|\lambda_i\, e_i - \lambda_j\, e_j\|_p^p} - \frac{\mathbb{E}\left(\|\Theta_i^{(m)} - \Theta_j^{(m)}\|_p^p\right)}{\|\lambda_i\, e_i - \lambda_j\, e_j\|_p^p}\right| > t\right\} \leq K\,\exp -(\exp ct) \qquad \forall\, t > 0$$

and

"Let $\delta > 0$, $0 < p \leq 1$.There are functions $m(\delta,p)$, $C(\delta,p)$ and $\varphi(\delta,p)$ with $\varphi(\delta,p) \to 0$ as $\delta \to 0$ and fixed p, such that if $\log 2 \leq C(\delta,p)\,\log n$, then

$$\left|\frac{\mathbb{E}\left(\|\Theta_i^{(m)} - \Theta_j^{(m)}\|_p^p\right)}{\|\lambda_i\, e_i - \lambda_j e_j\|_p^p} - M^p\right| < M^p\varphi(\delta,p) \quad \text{"}$$

Now for every $0 < \varepsilon < 1$ let $\delta = \delta(\varepsilon) > 0$ and $\delta' = \delta'(\varepsilon,p)$ such that $\delta \geq \varphi(\delta',p) + \delta'$. If $\log 2 \leq C(\delta',p)\,\log n$, then for every $1 \leq i \neq j \leq N$ we have

$$\mathbb{P}\left\{\left|\frac{\|\Theta_i^{(m)} - \Theta_j^{(m)}\|_p^p}{\|\lambda_i\, e_i - \lambda_j e_j\|_p^p} - M^p\right| > \delta\, M^p\right\} \leq K\exp -(\exp C\delta'\,\log n) = K\exp -n^{C\delta'}$$

and so

$$\mathbb{P}\left\{ \left| \frac{\|\Theta_i^{(m)} - \Theta_j^{(m)}\|_p^p}{\|\lambda_i\, e_i - \lambda_j e_j\|_p^p} - M^p \right| \leq \delta\, M^p \mid \forall\ 1 \leq i \neq j \leq N \right\} \geq 1 - K \binom{N}{2} \exp -n^{C\delta'}$$

Conclude as in all the results above. Observe that, by choosing appropiately the constant C, the restriction $\log 2 \leq C\ \log n$ is not such.

///

References.

1. J. BERNUES, "Inclusiones asintóticas del ℓ_∞^n cubo y de ℓ_p^n, $0 < p < 2$ en espacios de dimensión finita," Ph.D. Thesis, University of Zaragoza, June 1991.

2. S. J. DILWORTH, "The dimension of Euclidean subspaces of quasi-normed spaces," *Math. Proc. Cambridge Phil. Soc.* **97** (1985), 311-320.

3. Y. GORDON and D. R. LEWIS, "Dvoretzky's theorem on quasi-normed spaces," To appear in Ill. Jour. of Math..

4. J. HOFFMAN-JORGENSEN, "Sums of independent Banach random variables," *Studia Math.* **52** (1974), 159-185.

5. W. B. JOHNSON and G. SCHECHTMAN, "Embedding ℓ_p^m into ℓ_1^n," *Acta Math.* **49** (1982), 71-85.

6. N. J. KALTON, "The convexity type of quasi-Banach spaces," Unpublished.

7. N. J. KALTON, "Convexity, type and the three space problem," *Studia Math.* **69** (1980-81), 247-287.

8. M. LEDOUX and M. TALAGRAND, "Probability in Banach Spaces," Springer-Verlag, 1991.

9. M. MARCUS and G. PISIER, "Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes," *Acta Math.* **152** (1984), 245-301.

10. B. MAUREY and G. PISIER, "Séries de variables aléatoires vectorielles indépendantes et géométrie des espaces de Banach," *Studia Math.* **58** (1976), 45-90.

11. V.D. MILMAN and G. SCHECHTMAN, "Asymtotic theory of finite dimensional normed spaces," Lecture Notes in Math. **1200**. Springer-Verlag, 1986.

12. G. PISIER, "Probabilistic Methods in the Geometry of Banach Spaces," Lecture Notes in Math. **1206**, 167-242. Springer-Verlag 1986.

13. G. PISIER, "On the dimension of the ℓ_p^n subspaces of Banach spaces, for $1 \leq p < 2$," *Trans. Amer.Math.Soc.* **276** (1983), 201-211.

14. G. PISIER, "Type des espaces normes," Seminaire Maurey-Schwartz 1973-74. Exposé III.

15. G. SCHECHTMAN, "More on embedding subspaces of L_p in ℓ_r," *Compositio Math.* **61** (1987), 159-169.

16. V. YURINSKI, "Exponential bounds for large deviations," *Theor. Probab. Appl.* **19** (1974), 154-155.