CONVEX INEQUALITIES, ISOPERIMETRY AND SPECTRAL GAP

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Abstract. The main idea of these notes is to present the Kannan-Lovász-Simonovits spectral gap conjecture on the correct estimate for the spectral gap of the Laplace-Beltrami operator associated to any log-concave probability on $\mathbb{R}^n$

1. Introduction and notation

These notes are the content of the three lectures explained by the second author in the "VI International Course of Mathematical Analysis in Andalucía", held in Antequera in September, 2014. The authors want to express their gratitude to the organizers of this meeting for giving to one of them the possibility of presenting this quite new theory to young researchers. The content of this course is included in the reference [AB2], where a more detailed and complete information appears.

The main idea of these notes is to present the Kannan-Lovász-Simonovits spectral gap conjecture (KLS). This question was originally posed in relation with some problems in theoretical computer science, but it has a well understood analytic-geometrical meaning: give the correct estimate for the spectral gap (first non trivial eigenvalue) of the Laplace-Beltrami operator associated to any log-concave probability in $\mathbb{R}^n$. The KLS conjecture can also be expressed in terms of a type of Cheeger's isoperimetric inequality and, in this way, is related to Poincare's inequalities and to the concentration of measure phenomenon. In the meanwhile this conjecture is now one of the central points in geometrical asymptotic analysis which is the new branch of functional analysis comming from the geometry of Banach spaces when it interplays with classical convex geometry and probability.

The notes are divided in three parts. In the first chapter we will present Prékopa-Leindler inequality as a certain reverse of classical Hölder's inequality. We will also deduce from it Brunn-Minkowski, isoperimetric inequality and Borell's inequality on concentration of mass for log-concave probabilities on $\mathbb{R}^n$. The second chapter is dedicated to Cheegers-type isoperimetric inequalities, its relation to Poincaré-type inequalities and E. Milman's recent result on the role of convexity in this framework. In the third chapter we will present the KLS conjecture and the main results in this subject, up to now. We will also relate KLS to other conjectures and present our own results on these topics. In the references appears a list of the papers used for the preparation of this work.

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Let us introduce some notation. A convex body K in $\mathbb{R}^n$ is a convex, compact subset of $\mathbb{R}^n$ having the origin in its interior. Given an n-dimensional convex body

K, we will denote by $|K|_n$ (or simply $|K|$) its volume. ($|\widetilde{K}|_n = 1$). The volume of the n-dimensional Euclidean ball will be denoted by ω_n. We also use $|\cdot|$ for the modulus of a real number or the Euclidean norm for a vector in $\mathbb{R}^n$. When we write $a \sim b$, for $a, b > 0$, it means that the quotient of a and b is bounded from above and from below by absolute constants. $O(n)$ and $SO(n)$ will always denote the orthogonal and the symmetric orthogonal group on $\mathbb{R}^n$. The Haar probability on them will be denoted by σ. We will also denote by σ_{n-1} or just σ the Haar probability measure on S^{n-1}.

2. Convex inequalities

In this section we focus on Prékopa-Leindler inequality, presenting it as a kind of reverse of classical Hölder's inequality. Next we present Brunn-Minkowski inequality, which really is the version of Prékopa-Leindler's for charateristic functions. As a consequence we prove the classical isoperimetric inequality in $\mathbb{R}^n$ and Borell's inequality which is the main tool for working with log-concave probabilities.

2.1. **Hölder's and reverse Hölder's inequalities.** It is well known that given two measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$ and $0 \le \lambda \le 1$, Hölder's inequality says that

$$\int_{\mathbb{R}^n} f(x)^{1-\lambda} g(x)^{\lambda} dx \le \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^{\lambda}.$$

In the case that we take characteristic functions $f = \chi_A$, $g = \chi_B$, ($A, B \subseteq \mathbb{R}^n$), we have

$$|A \cap B|_n \le |A|_n^{1-\lambda} |B|_n^{\lambda} \qquad \forall\, 0 \le \lambda \le 1$$

or, equivalently,

$$|A \cap B|_n \le \min\{|A|_n, |B|_n\}.$$

It is also well known that we cannot reverse in general these two inequalities even by adding some constant, i.e., in general it is not true for any constant greater than one that

$$\min\{|A|_n, |B|_n\} \not\le C|A \cap B|_n$$

or

$$\left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda} \not\le C \int_{\mathbb{R}^n} f^{1-\lambda} g^{\lambda}$$

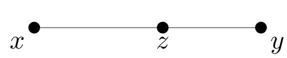
However, we can reverse the inequality if we conider some other expression instead of

$$\int_{\mathbb{R}^n} f^{1-\lambda} g^{\lambda}.$$

For that we use the sup-convolution of these two functions, which is defined in the following way. Given $f, g : \mathbb{R}^n \to \mathbb{R}_+$ and $0 \le \lambda \le 1$ we define

$$f^{1-\lambda} *_{\sup} g^{\lambda}(z) := \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g^{\lambda}(y)$$

(in this definition we consider all possible couples (x, y) such that z is convex combination of x, y for λ,



in particular, we have $f^{1-\lambda}(z)g^{\lambda}(z) \leq f^{1-\lambda} *_{\text{sup}} g^{\lambda}(z))$.

This function is not necessarily measurable, but we can consider its exterior Lebesgue integral defined by:

$$\int_{\mathbb{R}^n}^{*} f^{1-\lambda} *_{\text{sup}} g^{\lambda}(z)dz = \inf\left\{\int_{\mathbb{R}^n} h(z)dz : f^{1-\lambda} *_{\text{sup}} g^{\lambda}(z) \leq h(z)\right\}$$

and we have the following result:

Theorem 2.1. (Prékopa-Leindler's inequality)

Let $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ three measurable functions such that, for some $0 \leq \lambda \leq 1$,

$$f(x)^{1-\lambda} g(y)^{\lambda} \leq h((1-\lambda)x + \lambda y) \qquad \forall\, x, y \in \mathbb{R}^n$$

Then

$$\left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)dy\right)^{\lambda} \leq \int_{\mathbb{R}^n} h(z)dz.$$

Moreover, we have the reverse Hölder's inequality

$$\left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)dy\right)^{\lambda} \leq \int_{\mathbb{R}^n}^{*} f^{1-\lambda} *_{sup} g^{\lambda}(z)dz.$$

Proof. Dimension $n = 1$.

Let $A, B \subseteq \mathbb{R}$ be non-empty compact sets. Then we have

$$|A + B|_1 \geq |A|_1 + |B|_1$$

(this inequality is the one-dimensional case of Brunn-Minkowski inequality, which we will present later). Indeed,

$$A + B \supseteq (\min A + B) \cup (A + \max B)$$

and

$$(\min A + B) \cap (A + \max B) = \min A + \max B,$$

what implies

$$|A + B|_1 \geq |\min A + B|_1 + |A + \max B|_1 = |A|_1 + |B|_1.$$

For the rest of Borel sets we can use an approximation argument.

Given two bounded Borel measurable functions f, g we can assume without loss of generality that $\|f\|_\infty = \|g\|_\infty = 1$.

For any $0 \leq t < 1$, since

$$\{x \in \mathbb{R} : h(x) \geq t\} \supseteq (1-\lambda)\{x \in \mathbb{R} : f(x) \geq t\} + \lambda\{x \in \mathbb{R} : g(x) \geq t\},$$

we have that

$$\begin{aligned}\int_{\mathbb{R}} h(x)dx &\geq \int_0^1 |\{h \geq t\}|dt \geq (1-\lambda)\int_0^1 |\{f \geq t\}|dt + \lambda \int_0^1 |\{g \geq t\}|dt \\ &\geq \text{ (by the arithmetic-geometric mean inequality)} \\ &\geq \left(\int_{\mathbb{R}} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}} g(x)dx\right)^{\lambda}\end{aligned}$$

For general measurable functions we apply approximation arguments and the monotone convergence theorem.

Induction for $n > 1$.

Fix $x_1 \in \mathbb{R}$, let $f_{x_1} : \mathbb{R}^{n-1} \to [0, \infty)$ be $f_{x_1}(x_2, \dots, x_n) = f(x_1, \dots, x_n)$. Whenever $z_1 = (1-\lambda)x_1 + \lambda y_1$, we have

$$h_{z_1}((1-\lambda)(x_2, \dots, x_n) + \lambda(y_2, \dots, y_n)) \geq f_{x_1}(x_2, \dots, x_n)^{1-\lambda} g_{y_1}(y_2, \dots, y_n)^{\lambda}$$

for any $(x_2, \dots, x_n), (y_2, \dots, y_n) \in \mathbb{R}^{n-1}$.

By the induction hypothesis

$$\int_{\mathbb{R}^{n-1}} h_{z_1}(\overline{z}) d\,\overline{z} \geq \left(\int_{\mathbb{R}^{n-1}} f_{x_1}(\overline{x}) d\,\overline{x} \right)^{1-\lambda} \left(\int_{\mathbb{R}^{n-1}} g_{y_1}(\overline{y}) d\,\overline{y} \right)^{\lambda}.$$

Applying again the inequality for $n = 1$ and Fubini's theorem we obtain the result.

□

If we consider characteristic functions, we have $f = \chi_A$, $g = \chi_B$, $A, B \subseteq \mathbb{R}^n$

Corollary 2.1. (Brunn-Minkowski inequality)

Let A, B two Borel sets in $\mathbb{R}^n$. For any $0 \leq \lambda \leq 1$

$$|A|^{1-\lambda} |B|^{\lambda} \leq |(1-\lambda)A + \lambda B| \tag{1}$$

or equivalently

$$|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} \leq |A + B|^{\frac{1}{n}} \tag{2}$$

whenever $A \neq \emptyset \neq B$.

It is clear that (1) and (2) are equivalent:

- (1) $\Longrightarrow$ (2) We take
$$A' = A/|A|^{\frac{1}{n}} \qquad B' = B/|B|^{\frac{1}{n}} \qquad \lambda = |B|^{\frac{1}{n}}/(|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.)$$
- (2) $\Longrightarrow$ (1)
$$|(1-\lambda)A + \lambda B|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}} + \lambda |B|^{\frac{1}{n}} \geq |A|^{\frac{1-\lambda}{n}} |B|^{\frac{\lambda}{n}}.$$
(by the arithmetic-geometric mean inequality)

2.2. **Log-concave measures.** A measure (or probability measure) $d\mu(x)$ in $\mathbb{R}^n$ is log-concave if

$$d\mu(x) = e^{-V(x)} dx$$

where $V : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function (the support of V is an n-dimensional convex subset in $\mathbb{R}^n$).

Examples.

- The Lebesgue measure in $\mathbb{R}^n$
- The uniform measure on K, convex body in $\mathbb{R}^n$ (compact, convex with non empty interior)
- The exponential measure, $d\mu(x) = e^{-|x|} dx$ in $\mathbb{R}^n$
- The classical Gaussian measure in $\mathbb{R}^n$, $d\mu(x) = \dfrac{1}{\left(\sqrt{2\pi}\right)^n} \exp\left(-\dfrac{|x|^2}{2}\right) dx$

Brunn-Minkowski inequality for log-concave probabilities

Theorem 2.2. *Any log-concave probability μ on $\mathbb{R}^n$ satisfies Brunn-Minkowski inequality i.e.,*

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}$$

for any $A, B \subseteq \mathbb{R}^n$ borelians and any $0 \leq \lambda \leq 1$.

Proof. We take $f(x) = \chi_A(x)e^{-V(x)}$, $g(y) = \chi_B(y)e^{-V(y)}$ and

$$h(z) = \chi_{(1-\lambda)A+\lambda B}(z)e^{-V(z)}.$$

Then we apply Prékopa-Leindler inequality. $\square$

2.3. **Isoperimetric inequality in $\mathbb{R}^n$.** The classical isoperimetric inequality says that among all the Borel sets having the same volume the corresponding Euclidean ball is the one with the smallest perimeter or, reciprocally, among all the Borel sets having the same perimeter the corresponding Euclidean ball is the one with the greatest volume. This fact can be expressed in the following way

Theorem 2.3. *Let A any bounded Borel set in $\mathbb{R}^n$, then*

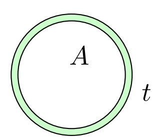
$$\frac{|\partial A|^{\frac{1}{n-1}}}{|A|^{\frac{1}{n}}} \geq \frac{|S^{n-1}|^{\frac{1}{n-1}}}{|B_2^n|^{\frac{1}{n}}},$$

where

$$|\partial A| = \liminf_{t\to 0} \frac{|A^t| - |A|}{t}$$

and A^t the t-dilation of A is

$$A^t = \{x \in \mathbb{R}^n; d(x, A) \leq t\} = A + tB_2^n.$$



Proof.

$$\begin{aligned}
|A^t| - |A| &= |A + tB_2^n| - |A| \\
&\geq \left(|A|^{\frac{1}{n}} + t\omega_n^{\frac{1}{n}}\right)^n - |A| \\
&= nt|A|^{\frac{n-1}{n}}\omega_n^{\frac{1}{n}} + o(t).
\end{aligned}$$

Hence

$$\begin{aligned}
|\partial A| &= \liminf_{t\to 0} \frac{|A^t| - |A|}{t} \\
&\geq n|A|^{\frac{n-1}{n}}\omega_n^{\frac{1}{n}}
\end{aligned}$$

and then

$$\frac{|\partial A|^{\frac{1}{n-1}}}{|A|^{\frac{1}{n}}} \geq \frac{|S^{n-1}|^{\frac{1}{n-1}}}{|B_2^n|^{\frac{1}{n}}}.$$

$\square$

2.4. **C. Borell's inequality.**

Theorem 2.4. *Let μ be a log-concave probability in $\mathbb{R}^n$. Then for any symmetric convex set $A \subseteq \mathbb{R}^n$ with $\mu(A) \geq \theta \geq \frac{1}{2}$ we have*

$$\mu(tA)^c \leq \theta \left(\frac{1-\theta}{\theta}\right)^{1+\frac{t}{2}} \qquad \forall\, t > 1$$

For instance, if $\mu(A) \geq 2/3$,

$$\mu(tA)^c \leq \frac{1}{2} \exp\left(-\frac{t \log 2}{2}\right) \qquad \forall\, t > 1$$

This inequality means that there is an exponential decay of the mass for $(t > 1)$-dilations of A symmetric, with absolute constants

Proof. It is a consequence of the fact that

$$A^c \supseteq \frac{2}{t+1}(tA)^c + \frac{t-1}{t+1}A$$

and Brunn-Minkowski inequality

$$1 - \theta \geq \mu(A^c) \geq \mu((tA)^c)^{\frac{2}{t+1}} \mu(A)^{\frac{t-1}{t+1}}.$$

□

Corollary 2.2 (Reverse Hölder's inequality and exponential decay of semi-norms)**.** *There exist absolute constants $C_1, C_2 > 0$ such that for any log-concave probability on $\mathbb{R}^n$ and for any semi-norm $f : \mathbb{R}^n \to [0, \infty)$ we have*

i) $\left(\int_{\mathbb{R}^n} f^p d\mu\right)^{\frac{1}{p}} \leq C_1 p \int_{\mathbb{R}^n} f d\mu, \qquad \forall\, p > 1$

ii) $\mu\left\{x \in \mathbb{R}^n : f(x) \geq C_2 t \int_{\mathbb{R}^n} f d\mu\right\} \leq 2 \exp\left(-t \log 2\right), \qquad \forall\, t > 0.$

Proof. i) Since any semi-norm is integrable we can assume that $\int_{\mathbb{R}^n} f d\mu = 1$. Let $A = \{f < 3\}$. By Markov's inequality $\mu(A) \geq 2/3$. Then

$$\mu\left\{f \geq 3t\right\} = \mu(tA)^c \leq \frac{1}{2} \exp\left(-t\frac{\log 2}{2}\right), \qquad t > 1.$$

Let $p > 1$

$$\begin{aligned}\int_{\mathbb{R}^n} f^p d\mu &= \int_0^3 p\, t^{p-1} \mu\{f > t\} dt + \int_3^\infty p\, t^{p-1} \mu\{f > t\} dt \\ &\leq 3^p + 3^p \int_1^\infty p\, s^{p-1} e^{-2s} ds \leq (C_1 p)^p\end{aligned}$$

for some absolute constant $C_1 > 0$ and i) follows.

ii) Assume that $\int_{\mathbb{R}^n} f d\mu = 1$.

By Markov's inequality, taking $p = t \geq 1$

$$\mu\{f > 2C_1 t\} = \mu\left\{\frac{f}{2C_1 t} > 1\right\} \le \int \frac{f^t}{(2C_1 t)^t} d\mu$$
$$\le \frac{(C_1 t)^t}{(2C_1 t)^t} = e^{-t\log 2}$$

For $0 < t \le 1$, the trivial bound 1 does the job.

□

Remark. Inequalities for the moments:

Consider the case in which $f(x) = |x|$. We use the fact that $\mathbb{E}_\mu |x|^p = \int_{\mathbb{R}^n} |x|^p d\mu$

- By Borell's inequality

$$(\mathbb{E}_\mu |x|^p)^{\frac{1}{p}} \le C_1 p\, \mathbb{E}_\mu |x| \qquad \forall\, p > 1$$

and

$$\mu\left\{|x| \ge C_2 t \mathbb{E}_\mu |x|\right\} \le 2\exp(-t), \qquad \forall\, t > 0.$$

- Paouris (2006) improved the inequality to

$$(\mathbb{E}_\mu |x|^p)^{\frac{1}{p}} \le C \max\{\mathbb{E}_\mu |x|, p\lambda_\mu\}$$

where $\lambda_\mu = \sup_{\theta\in S^{n-1}} \left(\mathbb{E}_\mu |\langle x, \theta\rangle|^2\right)^{\frac{1}{2}}$. We also have

$$\mu\{|x| \ge Ct\mathbb{E}_\mu |x|\} \le \exp\left(-3\,\frac{t\mathbb{E}_\mu|x|}{\lambda_\mu}\right)$$

for $t \ge 1$, which is is stronger that Borell's inequality.

3. Isoperimetric inequalities

In this section we will study isoperimetric inequalities with respect to log-concave probabilities. We will consider their corresponding functional Poincaré's inequalities and will show E. Milman's result on the role on convexity.

3.1. **Isoperimetric versus functional inequalities.** As we said before the classical isoperimetric inequality on $\mathbb{R}^n$ says that

$$|\partial A| \ge C\,|A|^{1-\frac{1}{n}} \qquad \forall \text{ bounded borel } A \subseteq R^n,$$

where $C = \frac{|S^{n-1}|}{|B_2^n|^{1-\frac{1}{n}}}$ and $|\partial A|$ is the outer Minkowski content of A, defined by

$$|\partial A| = \liminf_{\varepsilon\to 0} \frac{|A^\varepsilon| - |A|}{\varepsilon},$$

being

$$A^\varepsilon = \{a + x; a \in A, |x| < \varepsilon\} = A + \varepsilon B$$

is the ε-dilation of A. The outer Minkowski content coincides with the $(n-1)$-dimensional Hausdorff measure of the boundary for bounded Borel sets with smooth enough boundary.

We know that another approach for proving the classical isoperimetric inequality is to establish the corresponding Sobolev inequality in the extreme

Theorem 3.1 (Ferderer-Fleming). *The following statements are equivalent with the same constant C:*

- *For any bounded Borel set $A \subset \mathbb{R}^n$*

$$|\partial A| \geq C\,|A|^{1-\frac{1}{n}}.$$

- *For any locally Lipschitz compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$*

$$\|\,|\nabla f|\,\|_1 \geq C\|f\|_{\frac{n}{n-1}}.$$

Here

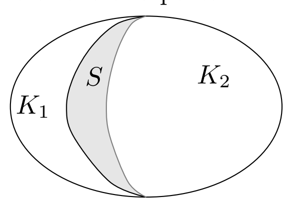
$$\|f\|_{\frac{n}{n-1}} = \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}$$

and

$$|\nabla f(x)| = \limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|},$$

which is defined for every $x \in \mathbb{R}^n$ and coincides almost everywhere with the classical modulus of the gradient due to Rademacher's theorem.

3.2. **Isoperimetric inequalities for log-concave probabilities.** Kannan, Lovász and Simonovits [KLS] posed the following question which originally arose in relation with some problems in theoretical computer science, i.e. an algorithmic question about the complexity of volume computation for convex bodies: Given a convex body in $K \subset \mathbb{R}^n$ find a surface which divide K into two parts whose measure is minimal relative to the volume of the two parts.



Namely, which is the greatest constant that makes the following formula true?

$$\mathrm{vol}_{n-1}(\partial_K K_1) \geq C\frac{\mathrm{vol}(K_1)\cdot\mathrm{vol}(K_2)}{\mathrm{vol}(K)}.$$

If we normalize $\mu(A) = \dfrac{|A|}{|K|}$ we have

$$\begin{array}{rcl} \mathrm{vol}_{n-1}(\partial_K K_1) & \geq & C\ \dfrac{\mathrm{vol}(K_1)\cdot\mathrm{vol}(K_2)}{\mathrm{vol}(K)} \\ & \Updownarrow & \\ \mu^+(A) & \geq & C\ \mu(A)\mu(A^c) \end{array}$$

which is known as Cheeger-type isoperimetric inequality. Hence the problem is, given μ (the uniform probability on a convex body K or more generally any log-concave probability), estimate the best constant $C > 0$ for which

$$\mu^+(A) \geq C\mu(A)\mu(A^c) \qquad \forall \text{ Borel set } A \subseteq \mathbb{R}^n$$

$$\Updownarrow$$

$$\mu^+(A) \geq C' \min\{\mu(A), \mu(A^c)\} \qquad \forall \text{ Borel set } A \subseteq \mathbb{R}^n$$

$$C \geq C' \geq \frac{C}{2}$$

where

$$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon}$$

and

$$A^\varepsilon = \{a + x; a \in A, |x| < \varepsilon\}$$

In a similar way to the classical one we have the following result

Theorem 3.2 (Maz'ja, Cheeger)**.** *Let μ be a Borel probability measure in $\mathbb{R}^n$. The following statements are equivalent:*

i) *For any Borel set $A \subseteq \mathbb{R}^n$*

$$\mu^+(A) \geq C_1 \min\{\mu(A), \mu(A^c)\}$$

ii) *For any integrable and locally Lipschitz function f*

$$C_2 \|f - \mathbb{E}_\mu f\|_1 \leq \| \, |\nabla f| \, \|_1.$$

Moreover $C_2 \leq C_1 \leq 2C_2$.

Here we have used the following notation:

$$\mathbb{E}_\mu f = \int_{\mathbb{R}^n} f \, d\mu \qquad \text{and} \qquad \|g\|_1 = \mathbb{E}_\mu |g| = \int_{\mathbb{R}^n} |g| d\mu$$

Proof: i) $\Longrightarrow$ ii)

We use the coarea formula

Lemma 3.1 (Federer)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ smooth, $g : \mathbb{R}^n \to [0, \infty)$ measurable. Then*

$$\int_{\mathbb{R}^n} g(x) \cdot |\nabla f(x)| dx = \int_{-\infty}^{\infty} \int_{\{f(x)=t\}} g(f^{-1}(t)) d\mathcal{H}_{n-1} \, dt$$

If $g = 1$ then

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f(x)| dx &= \int_{-\infty}^{\infty} \int_{\{f(x)=t\}} d\mathcal{H}_{n-1} \, dt \\ &= \int_{-\infty}^{\infty} \mathcal{H}_{n-1}\{x \in \mathbb{R}^n; f(x) = t\} dt \end{aligned}$$

For locally Lipschitz functions we have

Lemma 3.2 ([BH])**.** *Assume that $f > 0$ is locally Lipschitz and μ is a log-concave probability*

$$\int_{\mathbb{R}^n} |\nabla f(x)| d\mu(x) \geq \int_0^\infty \mu^+\{f > t\} dt$$

where $A_t = \{x \in \mathbb{R}^n : f(x) > t\}$.

Assume that $f > 0$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f(x)| d\mu(x) &\geq \int_0^\infty \mu^+\{f > t\} dt \\ &\geq \text{ (by i))} \\ &\geq C_1 \int_0^\infty \min\{\mu(A_t), \mu(A_t^c)\} dt \geq (*) \end{aligned}$$

where $A_t = \{x \in \mathbb{R}^n : f(x) > t\}$ and so,

$$\begin{aligned}(*) &\geq C_1 \int_0^\infty \mu(A_t)\mu(A_t^c)dt = \frac{C_1}{2}\int_0^\infty \|\chi_{A_t} - \mathbb{E}_\mu \chi_{A_t}\|_1 dt \\ &= \frac{C_1}{2}\int_0^\infty \sup_{\|g\|_\infty = 1}\int_{\mathbb{R}^n}(\chi_{A_t}(x) - \mathbb{E}_\mu\chi_{A_t})g(x)d\mu(x)dt \\ &\geq \frac{C_1}{2}\sup_{\|g\|_\infty=1}\int_0^\infty\int_{\mathbb{R}^n}(\chi_{A_t}(x) - \mathbb{E}_\mu\chi_{A_t})g(x)d\mu(x)dt = (**)\end{aligned}$$

Hence

$$\begin{aligned}(**) &= \frac{C_1}{2}\sup_{\|g\|_\infty=1}\int_0^\infty\int_{\mathbb{R}^n}\chi_{A_t}(x)(g(x) - \mathbb{E}_\mu g)d\mu(x)dt \\ &= \frac{C_1}{2}\sup_{\|g\|_\infty=1}\int_{\mathbb{R}^n}(g(x) - \mathbb{E}_\mu g)f(x)d\mu = \frac{C_1}{2}\|f - \mathbb{E}_\mu f\|_1.\end{aligned}$$

In the general case we proceed for bounded below functions and then by an approximation argument.

Proof: ii) $\Longrightarrow$ i) Let A be a Borel set in $\mathbb{R}^n$. Given $0 < \varepsilon < 1$, we define

$$f_\varepsilon(x) = \max\left\{0, 1 - \frac{d(x, A^{\varepsilon^2})}{\varepsilon - \varepsilon^2}\right\}.$$

- $0 \leq f_\varepsilon(x) \leq 1$
- $f_\varepsilon(x) = 1$ if $x \in A^{\varepsilon^2}\big(\supseteq A\big)$
- $f(x) = 0$, whenever $d(x, A) > \varepsilon$
- $\lim_{\varepsilon\to 0} f_\varepsilon = \chi_{\overline{A}}$
-
$$|f_\varepsilon(x) - f_\varepsilon(y)| \leq \frac{1}{\varepsilon(1-\varepsilon)}\left|d(x, A^{\varepsilon^2}) - d(y, A^{\varepsilon^2})\right| \leq \frac{|x-y|}{\varepsilon(1-\varepsilon)}$$
- f_ε is locally Lipschtiz and
$$|\nabla f_\varepsilon(x)| \leq \frac{1}{\varepsilon - \varepsilon^2} \qquad x \in \mathbb{R}^n.$$

$|\nabla f_\varepsilon(x)| = 0$ whenever $x \in \{x \in \mathbb{R}^n; d(x, A) > \varepsilon\} \cup A^{\varepsilon^2} \supseteq \{x \in \mathbb{R}^n; d(x, A) > \varepsilon\} \cup A$.

Thus,

$$\int_{\mathbb{R}^n}|\nabla f_\varepsilon(x)|d\mu(x) \leq \frac{\mu(A^{\varepsilon+\varepsilon^2}) - \mu(A)}{\varepsilon - \varepsilon^2}.$$

By *ii)*,

$$C_2\|f_\varepsilon - \mathbb{E}_\mu f_\varepsilon\|_1 \leq \frac{\mu(A^{\varepsilon+\varepsilon^2}) - \mu(A)}{\varepsilon - \varepsilon^2}$$

and letting $\varepsilon \to 0^+$ we obtain

$$2C_2\mu(A)\mu(A^c) \leq \mu^+(A).$$

this gives

$$C_2\min\{\mu(A), \mu(A^c)\} \leq \mu^+(A).$$

3.3. **Poincaré inequalities associated to a log-concave μ.** Given $1 \le p \le q \le \infty$, we introduce

Definition 3.1. *$D_{p,q}(\mu)$ is the greatest constant that makes the following inequality true*

$$D_{p,q}\|f - \mathbb{E}_\mu f\|_p \le \| \, |\nabla f| \, \|_q$$

for any locally Lipschitz integrable functions $f \in L^p(d\mu)$.

- Case $p = q = 1$: $D_{1,1}$ is equivalent to the isoperimetric Cheeger constant for μ
- Case $p = q = 2$: is the Poincaré inequality for $d\mu = e^{V(x)}dx$.

$$\Updownarrow$$

$$D_{2,2}^2 \underbrace{\int_{\mathbb{R}^n} \left|f - \int_{\mathbb{R}^n} f \, d\mu\right|^2 d\mu}_{\mathrm{Var}_\mu(f)} \le \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu$$

- $D_{2,2}^2 = \lambda_2$ is known as the spectral gap of μ, which is the first eigenvalue of the Laplace-Beltrami operator

  $$L = \Delta - \langle \nabla V, \nabla \rangle$$

  associated to μ.

Proposition 3.1. *The following relations hold:*

- *By Hölder's inequality, if $1 \le p \le q \le \infty$*

  $$D_{p,q} \le D_{p,\infty} \le D_{1,\infty}$$

  $$D_{p,q} \le D_{1,q} \le D_{1,\infty}$$

- *Maz'ja and Cheeger (1960)*

  $$D_{1,1} \le C D_{2,2},$$

  where C is an absolute constant
- *Easy modification of Hölder's inequality*

  $$D_{p,p} \le C p' D_{p',p'} \qquad \forall 1 \le p \le p' \le \infty,$$

  where C is an absolute constant

Much more important and difficult are

Theorem 3.3. *If μ is a log-concave probability we have*

i) *Ledoux (1994):*

$$D_{2,2} \le C D_{1,1}$$

ii) *E. Milman (2010):*

$$D_{1,\infty} \le C D_{1,1}$$

In consequence

$$D_{p,q} \le D_{1,\infty} \le C D_{1,1} \le C \, p \, D_{p,q}$$

C absolute (independent of μ and even of the dimension)

The part ii) of the theorem 3.3 is consequence of the following result

Theorem 3.4 (E. Milman [EM1])**.** *Let μ be a log-concave probability in $\mathbb{R}^n$. Assume that*

$$D_{1,\infty}\mathbb{E}_\mu|f-\mathbb{E}_\mu f| \le \| \, |\nabla f| \, \|_\infty \qquad \forall f \text{ locally Lipschitz}$$

Then

$$\mu^+(A) \ge CD_{1,\infty}\mu(A)^2 \qquad \forall \text{ borelian } \mu(A) \le \frac{1}{2},$$

where $C>0$ is an absolute constant. Moreover

$$CD_{1,\infty}\mathbb{E}_\mu|f-\mathbb{E}_\mu f| \le \mathbb{E}_\mu|\nabla f|$$

and

$$CD_{1,\infty} \le D_{1,1}.$$

Proof. The moreover part: $\mu^+(A) \ge CD_{1,\infty}\mu(A)^2$ is enough. Indeed,

$$\mu(A) = \frac{1}{2} \Longrightarrow \mu^+(A) \ge C\frac{D_{1,\infty}}{4}$$

The isoperimetric profile defned by

$$I_\mu(t) := \inf\{\mu^+(A) \, : \, \mu(A) = t\}, \qquad 0 \le t \le \frac{1}{2}$$

is a concave function (this result is true both in Riemannian geometry and for log-concave probabilities due to the work of a lot of people: Bavard-Pansu, Bérard-Besson-Gallot, Gallot, Morgan-Johnson, Sternberg-Zumbrun, Kuwert, Bayle-Rosales, Bayle, Morgan, Bobkov).

Given $0 \le t \le \frac{1}{2}$

$$I_\mu(t) \ge 2tI_\mu\left(\frac{1}{2}\right) \ge 2tCD_{1,\infty}\frac{1}{4} \ge CD_{1,\infty}\frac{t}{2}.$$

Then

$$\mu^+(A) \ge CD_{1,\infty}\mu(A) \qquad \text{whenever } \mu(A) \le \frac{1}{2}.$$

Hence, Cheeger's theorem implies

$$\mathbb{E}_\mu|\nabla f| \ge CD_{1,\infty}\mathbb{E}_\mu|f-\mathbb{E}_f|.$$

Next we will use the semigroups technique introduced by Ledoux: Given $d\mu = e^{-V(x)}dx$, V convex and smooth, let

$$L = \Delta - \langle \nabla V, \nabla \rangle$$

the associated Laplace-Beltrami operator. Let $(P_t)_{t\ge 0}$ be the semigroup generated by L It is characterized by the heat diffussion given by the following system of differential equations of second order

$$\frac{d}{dt}P_t(f) = L(P_t(f))$$
$$P_0(f) = f$$

for every bounded smooth function f.

The main properties are:

1) $P_t(1) = 1$
2) $f \ge 0 \Longrightarrow P_t(f) \ge 0$
3) $\mathbb{E}_\mu P_t(f) = \mathbb{E}_\mu f$
4) $\mathbb{E}_\mu|P_t(f)|^p \le \mathbb{E}_\mu|f|^p$, $\forall p \ge 1$

5) (Bakry-Ledoux). If $2 \le q \le \infty$ and f bounded and smooth
$$\| \, |\nabla P_t(f)| \, \|_{L^q(\mu)} \le \frac{1}{\sqrt{2t}} \|f\|_{L^q(\mu)}.$$
6) (Ledoux) If f bounded smooth
$$\|f - P_t(f)\|_{L^1(\mu)} \le \sqrt{2t} \| \, |\nabla f| \, \|_{L^1(\mu)}.$$

Assume now that A is closed, $\mu(A) \le \dfrac{1}{2}$. Given $\varepsilon > 0$, let $A^\varepsilon = \{x \in \mathbb{R}^n : d(x,A) < \varepsilon\}$. The function $\chi_{A,\varepsilon}(x) = \max\{1 - \frac{1}{\varepsilon} d(x,A), 0\}$ is Lipschitz and

- $\lim_{\varepsilon \to 0} \chi_{A,\varepsilon}(x) = 1 \Longleftrightarrow x \in A$
- $$|\nabla \chi_{A,\varepsilon}(x)| = \begin{cases} = 0, & \text{if} \quad x \in \mathrm{int} A \\ = 0, & \text{if} \quad d(x,A) > \varepsilon \\ \le \frac{1}{\varepsilon}, & \text{if} \quad x \notin \mathrm{int} A, 0 \le d(x,A) \le \varepsilon. \end{cases}$$

Then
$$\begin{aligned} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon} &\ge \int_{\mathbb{R}^n} |\nabla \chi_{A,\varepsilon}(x)| d\mu(x) \\ &\ge (\text{by } 6) \ge \frac{1}{\sqrt{2t}} \, \mathbb{E}_\mu |\chi_{A,\varepsilon} - P_t(\chi_{A,\varepsilon})|. \end{aligned}$$

When $\varepsilon \to 0$ we have
$$\begin{aligned} \sqrt{2t} \, \mu^+(A) &\ge \mathbb{E}_\mu |\chi_A - P_t(\chi_A)| = 2 \left(\mu(A) - \int_A P_t(\chi_A)(x) d\mu(x) \right) \\ &= 2\big(\mu(A)\mu(A^c) - \mathbb{E}_\mu(\chi_A - \mu(A))(P_t(\chi_A) - \mu(A))\big) \\ &\le (\text{by Hölder}) \\ &\ge 2\big(\mu(A)\mu(A^c) - \|\chi_A - \mu(A)\|_\infty \mathbb{E}_\mu |P_t(\chi_A) - \mu(A)|\big). \end{aligned}$$

We use the hypothesis and we have
$$\mathbb{E}_\mu |P_t(\chi_A) - \mu(A)| \le \frac{1}{D_{1,\infty}} \mathbb{E}_\mu |\nabla P_t(\chi_A)|,$$
since $\mathbb{E}_\mu P_t(\chi_A) = \mu(A)$. Also $\nabla P_t(\chi_A - \mu(A)) = \nabla P_t(\chi_A)$, so

$$\begin{aligned} \mathbb{E}_\mu |\nabla P_t(\chi_A)| &\le \| |\nabla P_t(\chi_A)| \|_2 \\ &\le \ (\text{by } 5) \le \frac{1}{\sqrt{2t}} \|P_t \chi_A - \mu(A)\|_2 \\ &\le (\text{ by } 4) \le \frac{1}{\sqrt{2t}} \|\chi_A - \mu(A)\|_2 \le \frac{1}{\sqrt{2t}} \|\chi_A - \mu(A)\|_\infty \\ &\le \frac{1}{\sqrt{2t}}. \end{aligned}$$

Since $\mu(A) \le \dfrac{1}{2}$,
$$\sqrt{2t} \mu^+(A) \ge \mu(A) - \frac{2}{\sqrt{2t} D_{1,\infty}}$$

Choose $\sqrt{2t}D_{1,\infty} = 4/\mu(A)$ and we get

$$\mu^+(A) \geq 8D_{1,\infty}\mu(A)^2.$$

□

4. K-L-S spectral gap conjecture

Given μ a log-concave probability on $\mathbb{R}^n$, the Kannan-Lovász-Simonovits problem is to estimate the greatest constant in the inequality

$$\mu^+(A) \geq C\mu(A) \qquad \forall \text{ Borel set, } \mu(A) \leq \frac{1}{2}.$$

We know that this problem is equivalent, up to absolute constants, to estimate the best constant in Poincaré 's inequality

$$\lambda_2 \mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 \leq \mathbb{E}_\mu|\nabla f|^2 \qquad \forall \text{ locally Lipschitz } f$$

and also equivalent to estimate the best constant in

$$C'\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 \leq 1 \qquad \forall \text{ 1-Lipschitz } f$$

Kannan-Lovász-Simonovits conjectured that

KLS Conjecture *The greatest constant is attained for affine functions, up to an absolute constant.*

Let f be an affine function on $\mathbb{R}^n$, $f(x) = t + \langle a, x\rangle$, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$. Its Lipschitz constant is $|a|$. If we assume that f is affine and 1-Lipschitz, then $f(x) = t+\langle\theta, x\rangle$, $t \in \mathbb{R}$, $\theta \in S^{n-1}$. Thus, $\mathbb{E}_\mu f = t + \langle\theta, \mathbb{E}_\mu x\rangle$ and

$$\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 = \mathbb{E}_\mu\langle\theta, x - \mathbb{E}_\mu x\rangle^2$$

Let

$$\lambda_\mu^2 := \sup_{\theta\in S^{n-1}} \mathbb{E}_\mu\langle\theta, x - \mathbb{E}_\mu x\rangle^2$$

(λ_μ is greatest eigenvalue of the covariance matrix of μ) Hence

KLS Conjecture (equivalent formulation) *Let μ be a log-concave probability on $\mathbb{R}^n$ with barycenter $\mathbb{E}_\mu x$. Let $\lambda_\mu^2 = \sup_{\theta\in S^{n-1}} \mathbb{E}_\mu\langle x - \mathbb{E}_\mu x, \theta\rangle^2$. Then*

$$\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 \leq C\lambda_\mu^2 E_\mu|\nabla f|^2$$

for some absolute constant $C > 0$.

4.1. **Known results.**

Theorem 4.1 (KLS estimate, [KLS]). *Given μ (log-concave) and f Lipschitz*

$$\mathbb{E}_\mu|f - \mathbb{E}_\mu f|^2 \leq C\mathbb{E}_\mu|x - \mathbb{E}_\mu x|^2 \cdot \mathbb{E}_\mu|\nabla f|^2,$$

where $C > 0$ is an absolute constant.

Proof.

$$\begin{aligned}
\mathbb{E}_\mu|f-\mathbb{E}_\mu f|^2 &\le \mathbb{E}_\mu\big(|f(x)-f(\mathbb{E}_\mu x)|+|f(\mathbb{E}_\mu x)-\mathbb{E}_\mu f|\big)^2\\
&= \mathbb{E}_\mu\big(|f(x)-f(\mathbb{E}_\mu x)|+|\mathbb{E}_\mu(f(\mathbb{E}_\mu x)-f)|\big)^2\\
&\le \text{ (by Minkowski's inequality)}\\
&\le 4\,\mathbb{E}_\mu|f(x)-f(\mathbb{E}_\mu x)|^2\\
&\le 4\mathbb{E}_\mu|x-\mathbb{E}_\mu x|^2\cdot \||\nabla f|\|_\infty^2\\
&\le C\mathbb{E}_\mu|x-\mathbb{E}_\mu x|^2\cdot \mathbb{E}_\mu|\nabla f|^2.
\end{aligned}$$

This inequality is worse than the one conjectured by the authors since $\mathbb{E}_\mu|x-\mathbb{E}_\mu x|^2=\sum_{i=1}^n\mathbb{E}_\mu(x_i-\mathbb{E}_\mu x_i)^2\le n\lambda_\mu^2$. □

Theorem 4.2 (Payne-Weinberger (1960))**.** *If μ is the normalized uniform measure on a convex body K*

$$\mathbb{E}_\mu|f-\mathbb{E}_\mu f|^2\le \frac{4}{\pi^2}\,diam(K)^2\cdot\mathbb{E}_\mu|\nabla f|^2$$

If B_2^n is the Euclidean ball and μ the normalized uniform measure on it the sharp estimate is

$$\mathbb{E}_\mu|f-\mathbb{E}_\mu f|^2\le \frac{C}{n}\cdot\mathbb{E}_\mu|\nabla f|^2.$$

The first estimate in Payne-Weinberger inequality is a trivial consequence of KLS estimate since

$$E_\mu|x-\mathbb{E}_\mu x|^2\le \text{ (by Jensen) }\le \mathbb{E}_{\mu\otimes\mu}|x-y|^2\le \text{ (diam }K)^2.$$

The second one is much more acurate.

Theorem 4.3. *Talagrand (1991) Let $d\mu(x)=\frac{1}{2^n}e^{-\sum_{i=1}^n|x_i|}dx$. Then*

$$\mathbb{E}_\mu|f-\mathbb{E}_\mu f|^2\le C\cdot\mathbb{E}_\mu|\nabla f|^2$$

(This is the classical Talagrand's inequality for the exponential probability)

Theorem 4.4 (Gaussian case)**.** *Let $d\mu(x)=(2\pi)^{-n/2}e^{-|x|^2/2}$. Then*

$$Var_\mu f=\mathbb{E}_\mu|f-\mathbb{E}_\mu f|^2\le\mathbb{E}_\mu|\nabla f|^2$$

Proof. It is easy to see that $\lambda_\mu=1$. Let $u\in\mathcal{D}(\mathbb{R}^n)$ be a test functions. Consider the associated Laplace-Beltrami operator L

$$Lu(x)=\Delta u(x)-\langle x,\nabla u(x)\rangle.$$

We know that $\{Lu;u\in\mathcal{D}\}$ is dense in $\{f\in L^2(\mathbb{R}^n,d\mu):\mathbb{E}_\mu f=0\}$. Then, $\inf_{u\in\mathcal{D}}\big\{\mathbb{E}_\mu(Lu-f)^2\big\}=0$ and integrating by parts

- $\mathbb{E}_\mu f\,Lu=-\mathbb{E}_\mu\langle\nabla f,\nabla u\rangle$ (Green's formula)
- $\mathbb{E}_\mu(Lu)^2=\mathbb{E}_\mu\langle\nabla u,\nabla u\rangle+\mathbb{E}_\mu\sum_{i,j}(\partial_{ij}u(x))^2\ge\mathbb{E}_\mu|u|^2$

Assume that $\mathbb{E}_\mu f=0$. Since

$$\mathrm{Var}_\mu f=\mathbb{E}_\mu\big|f-\mathbb{E}_\mu f\big|^2=\mathbb{E}_\mu f^2$$

we have

$$\begin{aligned}\mathbb{E}_\mu f^2 - \mathbb{E}_\mu (Lu - f)^2 &= 2\mathbb{E}_\mu f\, Lu - \mathbb{E}_\mu (Lu)^2 \\ &\le -2\mathbb{E}_\mu \langle \nabla f, \nabla u\rangle - \mathbb{E}_\mu |u|^2 \\ &\le \mathbb{E}_\mu |\nabla f|^2.\end{aligned}$$

Taking the infimum in u we obtain the result.

$$\mathrm{Var}_\mu f \le \lambda_\mu^2 \cdot \mathbb{E}_\mu |\nabla f|^2$$

□

Theorem 4.5. *The normalized measure on the classes before verify the KLS conjecture*

- *p-balls, $1 \le p \le \infty$ (Sodin 2008 , Latala&Wojtaszczyk, 2008)*
- *The simplex (Barthe and Wolff, 2009)*
- *Some revolution bodies (Bobkov 2003, Hue 2011)*
- *Unconditional bodies (Klartag, 2009)with a $\log n$ constant, i.e.*

  $$Var_\mu f = \mathbb{E}_\mu \left| f - \mathbb{E}_\mu f\right|^2 \le C \log n\, \lambda_\mu^2\, \mathbb{E}_\mu |\nabla f|^2$$

  (A convex body K is unconditional if $(x_1, \dots, x_n) \in K$ if and only if $(|x_1|, \dots, |x_n|) \in K$).

A general upper bound which is the best known estimate, up to now, is

Theorem 4.6 (Guédon-Milman (2011) + Eldan (2013))**.** *For any log-concave probability in $\mathbb{R}^n$, Poincaré's inequality is true in the following way*

$$Var_\mu f = \mathbb{E}_\mu \big| f - \mathbb{E}_\mu f\big|^2 \le C n^{2/3} (\log n)^2\, \lambda_\mu^2\, \mathbb{E}_\mu |\nabla f|^2$$

for any locally Lipschitz integrable function f.

4.2. **Relations with other conjectures.** There are other well-known geometric and probabilistic conjectures related with the KLS conjecture. They are the variance or thin shell conjecture and the slicing problem.

The slicing problem, Bourgain (1986): *There exists an absolute constant $C > 0$ such that every convex body K in $\mathbb{R}^n$ with volume 1 has, at least, one $(n-1)$-dimensional section such that*

$$|K \cap H|_{n-1} \ge C.$$

The slicing problem or hyperplane conjecture was introduced by Bourgain when he was proving the boundedness in L^p of the Hardy-Littlewood maximal function on convex bodies. It is known to be true in the following families

- Unconditional convex bodies
- Zonoids
- Random polytopes
- Polytopes in which the number of vertices is proportional to de dimension, i.e., for instance, $N/n \le 2$
- The unit balls of finite dimensional Schatten classes,for $1 \le p \le \infty$
- $(n-1)$-orthogonal projection of the classes above
- and more

and a general estimate is

Theorem 4.7. *There exists an absolute constant $C > 0$ such that for every convex body K in $\mathbb{R}^n$ with volume 1 at least one $(n-1)$-dimensional section satisfies*

- *Bourgain (1986),* $|K \cap H|_{n-1} \geq \dfrac{C}{n^{1/4} \log n}$
- *Klartag (2006),* $|K \cap H|_{n-1} \geq \dfrac{C}{n^{1/4}}$

Thin shell width or variance conjecture (2003, Bobkov-Koldobsky, Anttila-Ball-Perissinaki): *There exists an absolute constant $C > 0$ such that for every log-concave probability μ in $\mathbb{R}^n$*

$$\sigma_\mu = \sqrt{\mathbb{E}_\mu \big| |x| - \mathbb{E}_\mu |x| \big|^2} \leq C \lambda_\mu.$$

It is not difficult to prove that the thin shell conjecture is just the KLS conjecture to be true for the function $|x|$ or for $|x|^2$. The name is due to the following fact

Theorem 4.8. *If the thin shell width conjecture were true, we would have a stronger concentration of the mass around the mean for log-concave probabilities*

$$\mu \left\{ \big| |x| - \mathbb{E}_\mu |x| \big| > t \mathbb{E}_\mu |x| \right\} \leq 2 \exp \left(-C' t^{\frac{1}{2}} \frac{(\mathbb{E}_\mu |x|)^{\frac{1}{2}}}{\lambda_\mu^{\frac{1}{2}}} \right), \qquad \forall \, t > 0$$

The thin shell width conjecture is true for the uniform probability on

- Finite dimensional p-balls, $1 \leq p \leq \infty$
- Finite dimensional Orlicz-balls
- Revolution bodies
- $(n-1)$-dimensional orthogonal projections of the crosspolytope (1-ball)
- $(n-1)$-dimensional orthogonal projections of the cube and even all their linear deformations
- Although this conjecture is not linear invariant, in a random sense, more than half of linear deformations of the classe above also satisfy this conjecture.

The best known estimate

Theorem 4.9 (Guedon-Milman, 2010)**.** *There exists an absolute constant $C > 0$ such that for every log-concave probability μ in $\mathbb{R}^n$*

$$\sigma_\mu \leq C n^{1/3} \, \lambda_\mu.$$

The relation among the three conjectures is the following:

- Eldan-Klartag (2010) proved that if the thin shell conjecture is true for all log-concave probabilities then the slicing problem is also true for any convex body.
- Eldan (2013) proved that if the thin shell width conjecture were true for any log-concave probability then the Kannan-Lovász-Simonovits spectral gap conjecture would be true, up to a $\log n$ factor.
- In a parallel way Ball-Nguyen (2013) proved that if the KLS conjecture were true for a family of convex bodies the slicing problem would be true for this family.

The contribution of the authors respect to the thin shell width conjecture, are (see [AB1],[AB2])

- $(n-1)$-dimensional orthogonal projections of the cross-polytope (1-ball) verify this conjecture
- $(n-1)$-dimensional orthogonal projections of the cube and even all their linear deformation verify this conjecture
- $(n-1)$-dimensional orthogonal projections of the p-balls verify this conjecture when the orthogonal vectors to the hyperplane on which the projection is taken are sparse.
- If μ verifies the thin shell width conjecture then $\nu = \mu \circ T$ also verifies the thin shell width conjecture at least for half of T's (T linear map) in a probabilistic meaning and 'at random' if tha Schatten norm of $\|T\|_{c_4}$ satisfies
$$\frac{\|T\|_{HS}}{\|T\|_{c_4}} = o(n^{\frac{1}{4}}).$$

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