COMMUTATORS FOR THE MAXIMAL AND SHARP FUNCTIONS

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Abstract. The class of functions for which the commutator with the Hardy-Littlewood maximal function or the maximal sharp function are bounded on L^q are characterized and proved to be the same.

For the Hilbert transform H, and other classical singular integral operators, a well known and important result due to Coifman, Rochberg and Weiss (cf. [CRW]) states that a locally integrable function b in $\mathbb{R}^n$ is in BMO if and only if the commutator $[H, b]$, defined by

$$[H, b]f = H(bf) - bH(f),$$

is bounded in L^q, for some (and for all) $q \in (1, \infty)$. The cancellation implied by the commutator operation and the properties of singular integrals are crucial for the validity of the result. Later in [MS], using real interpolation techniques, Milman and Schonbeck proved a commutator result that applies to the Hardy-Littlewood maximal operator M as well as the sharp maximal operator. In fact the commutator result is valid for a large class of nonlinear operators which we now describe.

Let us say that T is a positive quasilinear operator if it is defined on a suitable class of locally integrable functions $D(T)$ and satisfies

i) $Tf \geq 0$, for $f \in D(T)$

ii) $T(\alpha f) = |\alpha| Tf$, for $\alpha \in \mathbb{R}$ and $f \in D(T)$

iii) $|Tf - Tg| \leq T(f - g)$ for $f, g \in D(T)$.

We have (cf. [MS])

Proposition 1. *Let b be a non negative BMO function and suppose that T is a positive quasilinear operator which is bounded on $L^q(w)$, for some $1 \leq q < \infty$ and for all w weights belonging to the Muckenhoupt class $\mathcal{A}_r$ for some $r \in [1, +\infty)$. Then $[T, b]$ is bounded on L^q.*

In particular the result applies to the maximal operator and the sharp maximal function. Note that since the Hardy-Littlewood maximal operator M is a positive

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1991 *Mathematics Subject Classification.* Primary 42B25, 46E30.

Key words and phrases. Maximal functions, sharp function, BMO, commutators..

*Partially supported by DGES

**Research partially done while Milman was visiting the University of Zaragoza suppported by the “Comisión de Doctorado” from this University.

***Partially supported by DGICYT and IER.

operator, the positivity condition on b seems crucial to effect the necessary cancellation. In fact, the closed graph theorem implies that if b is a negative locally integrable function then, $[M, b]$ is bounded on L^q for some $q \in (1, \infty)$ if and only if $b \in L^\infty$, since in this case we have

$$M(bf) - bM(f) \geq |b|M(f) \geq |bf|.$$

However, the question of obtaining a complete characterization of commutators with the Hardy-Littlewood maximal operator in the spirit of the Coifman-Rochberg-Weiss theorem has apparently remained open. In this note we show that a slightly extended form of positivity is a necessary and sufficient condition to characterize the boundedness of $[M, b]$. To see what this condition should be we observe that if M were a linear operator then, given that everything we do is modulo bounded operators, the correct requirement would appear to be that $b \in BMO$ with its negative part b^- bounded. Indeed, the sufficiency of the condition $b \in BMO$ with b^- bounded formally follows from previous Proposition, the fact that $b \in BMO \Rightarrow |b| \in BMO$ and the estimate

$$|[M, b]f - [M, |b|]f| \leq 2\left(b^- M(f)\right). \tag{1}$$

We summarize the previous discussion with the following

Proposition 2. *If b is a BMO function such that its negative part b^- is bounded then the commutator $[M, b]$ is bounded on L^q, for all $q \in (1, \infty)$.*

The purpose of this note is to prove the converse of the previous Proposition and to show that a similar characterization also holds for the sharp maximal operator.

Let us formally note that the proof of Proposition 1 given in [MS] implies the following more general result

Proposition 3. *Let b be a function in BMO, with $b^- \in L^\infty$ and let T be a positive quasilinear operator, which is bounded on $L^q(w)$, for some $1 \leq q < \infty$ and for all w weights belonging to the Muckenhoupt class $\mathcal{A}_r$ for some $r \in [1, +\infty)$. Then $[T, b]$ is bounded on L^q.*

Proof. Note that the following variant of (1) holds for any positive quasilinear operator T:

$$|[T, b]f - [T, |b|]f| \leq 2\left(b^- T(f) + T(b^- f)\right).$$

Therefore, since the right hand side is bounded, the result follows immediately from Proposition 1.

Remark. Since it could be useful for other purposes we note that an alternative proof of Proposition 3 can be given by following the proof given in [MS] verbatim and observing that, under the assumption that b^- is bounded, the pair $(L^q(e^b), L^q(e^{-b}))$ is ordered.

Recall that for a locally integrable function and for $1 \leq p < \infty$, the Hardy-Littlewood maximal function is defined by

$$M_p(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f|^p \right)^{1/p}$$

for all $x \in \mathbb{R}^n$, where Q denotes a cube with sides parallel to the coordinate axes and $|Q|$ the Lebesgue measure of Q. Note that for $p = 1$, $M_p = M$ is the classical Hardy-Littlewood maximal operator. The sharp function is given by

$$f^\sharp(x) = M^\sharp f(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f - f_Q|, \quad x \in \mathbb{R}^n$$

where as usual we let $f_Q = \frac{1}{|Q|}\int_Q f$.

If Q_0 is a fixed cube, then the Hardy-Littlewood maximal function relative to Q_0 is given by

$$M_{p,Q_0}(f)(x) = \sup_{x\in Q\subseteq Q_0} \left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p}$$

defined for all $x \in Q_0$.

Our main result for M_p can be now stated as follows

Proposition 4. *Let b be a real valued, locally integrable function in $\mathbb{R}^n$. The following assertions are equivalent:*

(4.1) *The commutator $[M_p, b]$ is bounded in L^q, for all q, $p < q < \infty$.*
(4.2) *The commutator $[M_p, b]$ is bounded in L^q, for some q, $p < q < \infty$.*
(4.3) *b is in BMO and $b^-$is in L^∞.*
(4.4) *There exists $q \in [1, \infty)$ such that*

$$\sup_Q \frac{1}{|Q|}\int_Q |b - M_{p,Q}b|^q < \infty.$$

(4.5) *For all $q \in [1, \infty)$ we have*

$$\sup_Q \frac{1}{|Q|}\int_Q |b - M_{p,Q}b|^q < \infty.$$

Proof. Since the implications (4.1) $\Rightarrow$ (4.2) and (4.5) $\Rightarrow$ (4.4) follow readily, we only have to prove (4.3) $\Rightarrow$ (4.1), (4.2) $\Rightarrow$ (4.4) and (4.4) $\Rightarrow$ (4.3) (the implication (4.1) $\Rightarrow$ (4.5) is similar to (4.2) $\Rightarrow$ (4.4))

(4.3) $\Rightarrow$ (4.1). The conclusion follows from the Proposition 3 and the fact that M_p is bounded in $L^q(w)$ for all w in the class $\mathcal{A}_{q/p}$.

(4.2) $\Rightarrow$ (4.4). We consider $f = \chi_Q \in L^q$. Then

$$\left(\int_Q |M_p(bf) - bM_p(f)|^q\right)^{1/q} \leq \|[M_p, b]f\|_q \leq C\|f\|_q = |Q|^{1/q}$$

implies the result, since $M_{p,Q}(\chi_Q) = \chi_Q$ and $M_p(b\chi_Q)(x) = M_{p,Q}(b)(x)$ for all $x \in Q$.

(4.4) $\Rightarrow$ (4.3). Let Q be a fixed cube. By hypothesis and Hölder's inequality we have

$$\frac{1}{|Q|}\int_Q |b - M_{p,Q}b| \leq \left(\frac{1}{|Q|}\int_Q |b - M_{p,Q}b|^q\right)^{1/q} \leq C.$$

Let $E = \{x \in Q; b(x) \leq b_Q\}$ and $F = \{x \in Q; b(x) > b_Q\}$. The following equality is trivially true

$$\int_E |b - b_Q| = \int_F |b - b_Q|.$$

Then

$$\begin{aligned} \frac{1}{|Q|}\int_Q |b - b_Q| &= \frac{2}{|Q|}\int_E |b - b_Q| \\ &\leq \frac{2}{|Q|}\int_E |b - M_{p,Q}(b)| \\ &\leq \frac{2}{|Q|}\int \end{aligned}$$

Proposition 6. *Let b be a real valued, locally integrable function in $\mathbb{R}^n$. The following assertions are equivalent:*

(6.1) *The commutator $[M^\sharp, b]$ is bounded in L^q for all q, $1 < q < \infty$.*
(6.2) *The commutator $[M^\sharp, b]$ is bounded in L^q for some q, $1 < q < \infty$*
(6.3) *b is in BMO and b^- is in L^∞*
(6.4) *There exists $q \in [1, \infty)$ such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - 2(b\chi_Q)^\sharp(x)|^q dx < \infty$$

(6.5) *For all $q \in [1, \infty)$ we have*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - 2(b\chi_Q)^\sharp(x)|^q dx < \infty.$$

Proof.

(6.3) ⇒ (6.1). It follows from Proposition 3, since for all locally integrable functions f we have $M^\sharp f \le 2M(f)$ and the weighted estimates are guaranteed.

(6.2) ⇒ (6.4). Let Q be a fixed cube as before. Let Q_1 be another cube, it is easy to compute that

$$\frac{1}{|Q_1|} \int_{Q_1} |\chi_Q - (\chi_Q)_{Q_1}| = \frac{2|Q_1 \setminus Q||Q_1 \cap Q|}{|Q_1|^2} \le \frac{1}{2}$$

(recall that for non negative numbers: $4rs \le (r+s)^2$). On the other hand, given $x \in Q$ there always exists a cube Q_1 containing Q and such that $|Q_1| = 2|Q|$. This shows that $(\chi_Q)^\sharp(x) = 1/2$, for all $x \in Q$. Hence the conclusion follows readily as in Proposition 4.

(6.4) ⇒ (6.3). We proceed as in the corresponding portion of the proof of Proposition 4, but some extra difficulties appear.

First, our claim is to prove that

$$|b_Q| \le 2(b\chi_Q)^\sharp(x), \;\; x \in Q \tag{2}$$

Let $x \in Q$ and pick a cube Q_1 containing Q and with volume $|Q_1| = 2|Q|$. Then

$$\begin{aligned} (b\chi_Q)^\sharp(x) &\ge \frac{1}{|Q_1|} \int_{Q_1} |b\chi_Q(y) - (b\chi_Q)_{Q_1}|\, dy \\ &= \frac{1}{2|Q|} \left(\int_Q \left| b(y) - \frac{1}{2} b_Q \right| dy + \frac{1}{2}|Q_1 \setminus Q||b_Q| \right) \\ &= \frac{1}{2|Q|} \int_Q |b(y) - \frac{1}{2} b_Q| dy + \frac{1}{4}|b_Q| \end{aligned} \tag{3}$$

On the other hand

$$\begin{aligned}|b_Q| &\le \frac{1}{|Q|}\int_Q |b(y)-\frac{1}{2}b_Q|dy + \frac{1}{|Q|}\int_Q |\frac{1}{2}b_Q|dy\\ &= \frac{1}{|Q|}\int_Q |b(y)-\frac{1}{2}b_Q|dy + \frac{1}{2}|b_Q|,\end{aligned}$$

and so

$$\frac{1}{2}|b_Q| \le \frac{1}{|Q|}\int_Q |b(y)-\frac{1}{2}b_Q|dy \tag{4}$$

Finally, (3) and (4) lead us to (2).

We can now achieve that $b \in BMO$. Indeed, let $E = \{x \in Q; b(x) \le b_Q\}$. Then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx &= \frac{2}{|Q|}\int_E (b_Q - b(x))dx\\ &\le \frac{2}{|Q|}\int_E \left(2(b\chi_Q)^\sharp(x) - b(x)\right)dx\\ &\le \frac{2}{|Q|}\int_E \left|2(b\chi_Q)^\sharp(x) - b(x)\right|dx\\ &\le \frac{2}{|Q|}\int_Q \left|2(b\chi_Q)^\sharp(x) - b(x)\right|dx \le C,\end{aligned}$$

where we have applied (2) and the hypothesis.

In order to prove that $b^- \in L^\infty$ we also use (2). We start from the inequality

$$2(b\chi_Q)^\sharp(x) - b(x) \ge |b_Q| - b^+(x) + b^-(x), \ x \in Q.$$

Averaging on Q, we have

$$\begin{aligned}C &\ge \frac{1}{|Q|}\int_Q |2(b\chi_Q)^\sharp(x) - b(x)|dx\\ &\ge \frac{1}{|Q|}\int_Q (2(b\chi_Q)^\sharp(x) - b(x))dx\\ &\ge \frac{1}{|Q|}\int_Q (|b_Q| - b^+(x) + b^-(x))dx\\ &= |b_Q| - \frac{1}{|Q|}\int_Q b^+(x)dx + \frac{1}{|Q|}\int_Q b^-(x)dx\end{aligned}$$

Leting $|Q| \to 0$, with $x \in Q$, Lebesgue differentiation theorem assures that

$$C \ge |b(x)| - b^+(x) + b^-(x) = 2b^-(x)$$

and the desired result follows.

The remaining proofs are similar to the ones in Proposition 4 and we leave the details to the interested reader.

References

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