A COIFMAN-ROCHBERG TYPE CHARACTERIZATION OF QUASI-POWER WEIGHTS

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Abstract. We give a characterization of quasipower weights in terms of Calderón transform of measures on $(0,\infty)$, similar to the one given by Coifman and Rochberg for the Muckenhoupt class $\mathcal{A}_1$.

In the last few years (cf. [BMR 1,2,3,4,5] and the references therein) we have been studying the connections between weighted norm inequalities and interpolation theory. For example, in [Mi] and [BMR3] we have shown that certain basic self improving inequalities in the theory of weights can be reinterpreted as inverse reiteration theorems. In this fashion the classical self improving results in the theory of weights follow as a consequence of the properties of solutions of certain elementary differential inequalities associated, via reiteration, to the K-functionals of the weights in question (cf. [Mi], [BMR 2, 3] and the references therein.) Our approach leads to new methods to attack the classical problems while at the same time producing new results in interpolation theory.

In this note we illustrate once again the interplay between the theory of weights and interpolation theory focussing our analysis on the Coifman-Rochberg theorem [CR]. This celebrated result gives a very simple algorithm to construct all the weights in the Muckenhoupt class $\mathcal{A}_1$, and therefore, by the Jones factorization theorem, provides a constructive characterization of all weights in the $\mathcal{A}_p$ classes, $p > 1$.

To be more precise, the Coifman-Rochberg theorem gives a characterization of $\mathcal{A}_1$ weights based on the properties of the Hardy-Littlewood maximal operator, our analysis here will lead us to an analogous characterization of quasipower weights in terms of the Calderón operator.

It is instructive to see the route we take to arrive to the formulation of the results.

Recall that $w \in \mathcal{A}_1$ iff there exists $C > 0$ such that

$$Mw \leq Cw, \tag{1}$$

where M denotes the Hardy-Littlewood maximal operator.

1991 *Mathematics Subject Classification.* Primary 46E30, 46B70.

Key words and phrases. Calderón operator, weighted norm inequalities, interpolation of operators, commutators, BMO.

*Partially supported by DGES (Spain).

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Typeset by $\mathcal{AMS}$-TEX

The Coifman-Rochberg theorem provides the following characterization of $\mathcal{A}_1$: $w \in \mathcal{A}_1$ if and only if there exists $0 < h \in L^\infty$, $f \in L^1_{\text{loc}}, \alpha \in (0,1)$, such that $w = h(Mf)^\alpha$. This, of course, is based on the validity of the formula

$$M((Mf)^\alpha) \leq C\,(Mf)^\alpha . \tag{2}$$

If we let $p = 1/\alpha$, $g = f^\alpha$, then (2) takes the following form: for any $g \in L^p_{\text{loc}}$,

$$M(M_p g) \leq C M_p g. \tag{3}$$

Taking rearrangements and using well known estimates for the maximal operator we get

$$\frac{1}{t}\int_0^t \left(\frac{1}{s}\int_0^s g^*(u)^p du\right)^{1/p} ds \leq C\left(\frac{1}{t}\int_0^t g^*(u)^p du\right)^{1/p}. \tag{3'}$$

Let us now recall that given a compatible pair of Banach spaces $\overline{A} = (A_0, A_1)$, we let for $a \in A_0 + A_1$, $t > 0$, $K(t, a; \overline{A})$ be defined by

$$K(t, a; \overline{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\},$$

where the inf runs over all possible decompositions $a = a_0 + a_1$, with $a_i \in A_i, i = 0, 1$.

In terms of $K-$functionals, (3') can be rewritten as

$$K(t, \frac{K((\cdot)^{1/p}, g; L^p, L^\infty)}{(\cdot)^{1/p}}; L^1, L^\infty) \leq Ct\frac{K(t^{1/p}, g; L^p, L^\infty)}{t^{1/p}}. \tag{4}$$

This suggests to define the class $\mathcal{A}(L^1, L^\infty)$ as follows: $f \in \mathcal{A}(L^1, L^\infty)$ iff f is non increasing and there exists a constant $C > 0$ such that for all $t > 0$ we have,

$$\frac{K(t, f, L^1, L^\infty)}{t} \leq Cf(t). \tag{5}$$

Remark. Note that if we let $p = 1$ in (3)-(3') the resulting inequality can be reinterpreted as a limit case of Gehring's Lemma studied in detail in [BMR3].

From our previous discussion, we see that the Coifman-Rochberg theorem in this context implies that for any $p > 1$, and $g \in L^p$, we have

$$\frac{K((\cdot)^{1/p}, g, L^p, L^\infty)}{(\cdot)^{1/p}} \in \mathcal{A}(L^1, L^\infty). \tag{6}$$

Of course once we know the validity of (6) a direct elementary proof of it can be established without relying on the Coifman-Rochberg theorem and the rearrangement inequalities for the maximal operator. Indeed, more generally the following reiteration formula is valid for pairs of Banach spaces

$$K(t, \frac{K((\cdot)^{1-\theta}, g; A_{\theta,p}, A_1)}{(\cdot)^{1-\theta}}; L^1, L^\infty) \leq ct\frac{K(t^{1-\theta}, g; A_{\theta,p}, A_1)}{t^{1-\theta}}. \tag{6'}$$

We only consider the elementary proof of the (6) in detail, the proof of (6') is analogous, if we use Holmstedt's formula. Now, we want to write

$$\frac{K(s^{1/p},g;L^p,L^\infty)}{s^{1/p}}\sim\left(\frac{1}{s}\int_0^s g^*(u)^p du\right)^{1/p}=f(s,t)+h(s,t),$$

in such a way that

$$\|f(\cdot,t)\|_1+t\,\|h(\cdot,t)\|_\infty\sim K(t,\frac{K((\cdot)^{1/p},g,L^p,L^\infty)}{(\cdot)^{1/p}};L^1,L^\infty).$$

Note that since $\left(\frac{1}{t}\int_0^t g^*(u)^p du\right)^{1/p}$ decreases, the optimal way to do this is apparently to write

$$\begin{aligned}\left(\frac{1}{s}\int_0^s g^*(u)^p du\right)^{1/p} &= \underbrace{\left(\frac{1}{s}\int_0^s f^*(u)^p du\right)^{1/p}\chi_{(0,t)}(s)}_{g(s,t)}\\ &\quad+\underbrace{\left(\frac{1}{s}\int_0^s f^*(u)^p du\right)^{1/p}\chi_{(t,\infty)}(s)}_{h(s,t)}.\end{aligned}$$

Then,

$$\begin{aligned}\|f(\cdot,t)\|_1 &= \int_0^t\left(\frac{1}{s}\int_0^s g^*(u)^p du\right)^{1/p}ds\\ &= \int_0^t s^{-1/p}\left(\int_0^s g^*(u)^p du\right)^{1/p}ds\\ &\le \left(\int_0^t g^*(u)^p du\right)^{1/p}\int_0^t s^{-1/p}ds\\ &= c_p t\left(\frac{1}{t}\int_0^t g^*(u)^p du\right)^{1/p}\end{aligned}$$

and similarly

$$t\,\|h(\cdot,t)\|_\infty\le t\left(\frac{1}{t}\int_0^t g^*(u)^p du\right)^{1/p}.$$

Remark. In this note we do not consider the most general results that can be obtained by our methods. For example, a more general version of (6) (resp.(6')) is closely related to the iteration results for maximal operators in [Ne], (cf. our related article [BMR5] for rearrangement estimates of variants of the Hardy-Littlewood maximal operator.)

In order to continue our discussion we need to introduce some classes of weights and develop notation.

Let us denote by $\overline{L^1}$ the compatible pair of Banach spaces

$$\overline{L^1}=(L^1(dx),L^1(dx/x))$$

on the interval $(0,\infty)$. It is well known (cf. [BK]) that

$$K(t,f;\overline{L^1}) = \int_0^\infty \min\left\{\frac{t}{x},1\right\} f(x)dx.$$

In the sequel w will denote a weight, i.e., a non negative Lebesgue measurable function defined on the interval $(0,\infty)$. We say that w satisfies the M_1 condition if there exists a constant $C>0$ such that for almost all $t>0$

$$\int_t^\infty \frac{w(x)}{x}dx \leq Cw(t). \tag{7}$$

We shall say that a weight satisfies an M^1 condition if there exists a constant $C>0$ such that for almost all $t>0$

$$\frac{1}{t}\int_0^t w(x)dx \leq Cw(t). \tag{8}$$

It is well known (see [Mu], [Ma]) that a weight w satisfies the M_1 condition if and only if $Pf \in L^1(w)$ for all $f \in L^1(w)$, where P is the Hardy operator defined by

$$Pf(t) = \frac{1}{t}\int_0^t f(x)dx$$

and $L^1(w)$ is the class of Lebesgue measurable functions f defined on the interval $(0,\infty)$ such that $\int_0^\infty |f(t)|w(t)dt < +\infty$. Similarly the class M^1 controls the boundedness of the operator Q, the adjoint of P, defined by

$$Qf(t) = \int_t^\infty \frac{f(x)}{x}dx.$$

Actually, $Qf \in L^1(w)$ for all $f \in L^1(w)$ if and only w satisfies M^1. We define the Calderón operator S by $S = P + Q = P\circ Q = Q\circ P$, so that

$$Sf(t) = \int_0^\infty \min\left\{\frac{1}{x},\frac{1}{t}\right\} f(x)dx.$$

We shall say that a function defined on $(0,\infty)$ is S-locally integrable ($f \in S_{loc}$) if $Sf(t)$ exists everywhere $t>0$. In a similar way a non negative Borel measure μ, defined on the borelians $(0,\infty)$, is S-locally finite if

$$S\mu(t) = \int_0^\infty \min\left\{\frac{1}{x},\frac{1}{t}\right\} d\mu(x) < \infty,$$

for all $t>0$.

It will be also convenient to use the notation $f\sim g$ whenever for some constant $C>0$ and for all x we have

$$\frac{1}{C}f(x) \leq g(x) \leq Cf(x).$$

We say that a weight w on $(0,\infty)$ is a quasipower, i.e. $w \in \mathcal{Q}$, if

$$Pw \sim w, \;\; Qw \sim w.$$

In particular, quasipower weights satisfy $w \sim Sw$. These weights are frequently used in interpolation theory to extend the classical interpolation spaces of Lions and Peetre (see [BL], [BK], [G] and [K]).

Remark. In [BMR1] we considered real interpolation spaces constructed using the more general class of $\mathcal{C}_p$-weights. These weights control the weighted norm inequalities of S in $L^p(w)$. Since these classes change with the parameter "p", real interpolation spaces of Lions-Peetre type based on these weights have reiteration properties that effectively depend on the "second" parameter. In this context extrapolation theorems of Rubio de Francia type come up as substitutes for reiteration theorems.

In reference to the previous remark recall that, in particular, $\mathcal{C}_1$ are the weights that satisfy both M^1 and M_1 conditions.

Since $w \sim Qw$, every quasipower weight is equivalent to a non increasing quasipower weight. Remark also that a quasipower weight is an $\mathcal{A}_1$-weight.

At this point we should note that the classes $\mathcal{Q}$, $\mathcal{C}_1$ and $\mathcal{A}_1$ are all different from each other.

The class $\mathcal{C}_1$ is strictly larger than $\mathcal{Q}$ as it is shown in [BMR1]. Indeed, let

$$w(t) = \begin{cases} 1/\sqrt{t} & \text{if } 0 < t \leq 1 \\ 1/\sqrt{t-1} & \text{if } 1 < t. \end{cases}$$

It is easy to compute

$$Sw(t) = \begin{cases} \pi - 2 + \frac{4}{\sqrt{t}} & \text{if } 0 < t \leq 1 \\ \frac{2}{t} + \frac{2\sqrt{t-1}}{t} + 2\arctan\frac{1}{\sqrt{t-1}}, & \text{if } 1 < t \end{cases}$$

and therefore we see that $w \in \mathcal{C}_1$ but $w \notin \mathcal{Q}$. It is also easy to see that w is not in $\mathcal{A}_1$. Indeed, if $w \in \mathcal{A}_1$ then for some constant $C > 0$ we would have

$$\frac{1}{b-a}\int_a^b w(x)dx = \frac{2}{b-a}(1-\sqrt{a}+\sqrt{b-1}) < C,$$

for almost all $a \in (1/2, 1)$ and for all $b > 1$, which is false.

On the other hand, the weight $w = 1$ is in $\mathcal{A}_1$ but is not in $\mathcal{C}_1$.

The class $\mathcal{A}_1$ is different from $\mathcal{Q}$. In fact, consider the weight

$$w(t) = \begin{cases} \dfrac{1}{t(\log t)^2} & \text{if } 0 < t < \frac{1}{e} \\ \dfrac{1}{(\log t)^2} & \text{if } t > e \\ C, & \text{otherwise,} \end{cases} \tag{9}$$

it is readily seen that $w \in \mathcal{A}_1$, and moreover,

$$Sw(t) \sim \begin{cases} \dfrac{1}{t|\log t|} & \text{when } t \to 0 \,; \\ \dfrac{1}{\log t} & \text{when } t \to \infty. \end{cases} \tag{10}$$

Therefore, $Sw \notin M^1$ and consequently $w \notin \mathcal{Q}$.

The main result of this note is the following

Theorem. *Let f be a non negative function in $f \in S_{loc}$. Then, for every $p > 1$ there exists $q > 1$ such that*

$$\left(\frac{K(t,f;\overline{L^1})}{t^q}\right)^{1/p} \sim \frac{1}{t} K\left(t, \left(\frac{K(\cdot,f;\overline{L^1})}{(\cdot)^q}\right)^{1/p}; \overline{L^1}\right).$$

Conversely, if F is a non negative function on $(0,\infty)$ such that

$$F(t) \sim \frac{1}{t} K(t,F;\overline{L^1}),$$

then there exist $p, q > 1$ such that

$$F(t) \sim \left(\frac{K(t,f;\overline{L^1})}{t^q}\right)^{1/p}.$$

The proof of the theorem can be easily deduced from the following auxiliary results. We begin with a very simple technical lemma.

Lemma 1. *Let f be a S-locally integrable function. Then,*

1. $Sf(x) = Sf(1) - \int_1^x \frac{Pf(y)}{y} dy.$
2. *Sf is a $\mathcal{C}^1$ function.*
3. *If μ is non negative S-locally finite measure then $xS\mu(x)$ is non decreasing, and $S\mu(x)$ is non increasing.*

Proof.

(1) By Fubini's theorem we have

$$\begin{aligned}
\int_1^x \frac{Pf(y)}{y} dy &= \int_1^x \frac{dy}{y^2} \int_0^y f(z)dz \\
&= \int_0^1 f(z)dz \int_1^x \frac{dy}{y^2} + \int_1^x f(z)dz \int_z^x \frac{dy}{y^2} \\
&= \left(1 - \frac{1}{x}\right) \int_0^1 f(z)dz + \int_1^x f(z)\left(\frac{1}{z} - \frac{1}{x}\right) dy \\
&= -Sf(x) + Sf(1).
\end{aligned}$$

(2) is an inmediate consequence of (1).

(3) If the measure μ is absolutely continuous with respect to Lebesgue measure, i.e. $d\mu = fdx$, $f \geq 0$, we note that $(xSf(x))' = Qf(x) \geq 0$ and $(Sf(x))' = -x^{-1}Pf(x) \leq 0$, for all $x > 0$. For a general measure, let $0 < x < x'$. Then we have

$$\begin{aligned}
xS\mu(x) &= \mu((0,x]) + \int_{(x,\infty)} \frac{x}{y} d\mu(y) \\
&= \mu((0,x']) - \mu((x,x']) + \int_{(x,x']} \frac{x}{y} d\mu(y) + \int_{(x',\infty)} \frac{x}{y} d\mu(y) \\
&\leq \mu((0,x']) + \int_{(x,x']} \left(\frac{x}{y} - 1\right) d\mu(y) + \int_{(x',\infty)} \frac{x}{y} d\mu(y) \\
&\leq x'S\mu(x')
\end{aligned}$$

and

$$S\mu(x) - S\mu(x') = \left(\frac{1}{x} - \frac{1}{x'}\right)\mu((0,x]) + \int_{(x,x']}\left(\frac{1}{y} - \frac{1}{x'}\right)d\mu(y)$$
$$\geq \left(\frac{1}{x} - \frac{1}{x'}\right)\mu((0,x]) \geq 0.$$

Remark. For μ absolutely continuous with respect to the Lebesgue measure, and using the expression

$$Sf(t) = \frac{1}{t}K(t,f;\overline{L^1}),$$

it is also easy to prove (3).

Lemma 2. *Let w be a weight on $(0,\infty)$, then*

(1) *If $xw(x)$ is a non decreasing, then w^α satisfies M^1, for all $0<\alpha<1$.*
(2) *If $w \sim Pw$, then $w \leq CQw$, for some constant $C>0$.*
(3) *If $w \sim Qw$, then $w \leq CPw$, for some constant $C>0$.*

Proof.
(1) Let $t>0$. Since $x<t$ implies that $x^\alpha w(x)^\alpha \leq t^\alpha w(t)^\alpha$ we have

$$\frac{1}{t}\int_0^t w(x)^\alpha dx \leq \frac{1}{1-\alpha}w(t)^\alpha.$$

(2) For some constants $C, C' > 0$ we have

$$w \leq CPw \leq CSw = CQ(Pw) \leq C'Qw.$$

(3) is similar to (2).

As a consequence we obtain the following

Corollary 3. *Let μ be a non negative S-locally finite measure and let $0<\alpha<1$. If we consider the weight $w=(S\mu)^\alpha$ then $w \sim Pw$ and $w \leq CQw$, for some constant $C>0$.*

Remark. Corollary 3 is sharp. In fact if we consider once again the function w defined by (9) above, by using the computation of Sw given in (10) we see that $Sw \notin M^1$ and $(Sw)^\alpha \notin M_1$ for any $0<\alpha<1$.

However, a suitable modification of $(Sf)^\alpha$ does produce quasipower weights:

Proposition 4. *Let μ be a non negative S-locally finite measure and let $0<\alpha<1$. Then there exists a positive ε such that the weight*

$$w(x) = \frac{(S\mu(x))^\alpha}{x^\varepsilon}$$

is a quasipower.

Proof. Since $(S\mu)^\alpha \in M^1$ by [BMR1] Proposition 2.3, there exist $\varepsilon > 0$ such that

$$w(x) = \frac{(S\mu(x))^\alpha}{x^\varepsilon} \in M^1,$$

consequently, for some constant $C > 0$ and for all $x > 0$ we have

$$\frac{1}{x}\int_0^x \frac{S\mu(y)^\alpha}{y^\varepsilon}dy \leq C\frac{S\mu(x)^\alpha}{x^\varepsilon}.$$

For any $x > 0$ we have

$$\begin{aligned} Qw(x) &= \int_x^\infty \frac{S\mu(y)^\alpha}{y^{\varepsilon+1}}dy \\ &\leq S\mu(x)^\alpha \int_x^\infty \frac{dy}{y^{\varepsilon+1}} = C\frac{S\mu(x)^\alpha}{x^\varepsilon} = Cw(x). \end{aligned}$$

The result follows.

The converse is also true

Proposition 5. *Let w be a quasipower weight, then there exist f S-locally integrable function, $\alpha \in (0,1)$ and $\varepsilon > 0$, such that*

$$w \sim \frac{(Sf(x))^\alpha}{x^\varepsilon}.$$

Proof. Since $w \sim Qw$ there is no loss of generality if we assume that w is a non increasing function. Moreover, the reverse Hölder inequality for $\mathcal{C}_1$-weights (cf. [BMR1], proposition 2.3) implies that for some $\varepsilon > 0$ the weight $w_1(x) = x^\varepsilon w(x) \in M_1$. We are going to prove that w_1 is also a quasipower weight. The condition M_1 says that $Qw_1 \leq Cw_1$. Also

$$\begin{aligned} \frac{1}{t}\int_0^t w_1(x)dx &= \frac{1}{t}\int_0^t x^\varepsilon w(x)dx \\ &\leq \frac{1}{t}\int_0^t t^\varepsilon w(x)dx \leq Ct^\varepsilon w(t), \end{aligned}$$

thus,

$$Pw_1 \leq Cw_1. \tag{11}$$

On the other hand,

$$\begin{aligned} w_1(t) = t^\varepsilon w(t) &\leq C\int_t^\infty \frac{w(x)}{x}t^\varepsilon dx \\ &\leq C\int_t^\infty \frac{x^\varepsilon w(x)}{x}dx = C\int_t^\infty \frac{w_1(x)}{x}dx, \end{aligned}$$

therefore

$$w_1 \leq CQ(w_1).$$

By Lemma 2, we see that $w_1 \in \mathcal{Q}$.

As before, since $w_1 \sim Qw_1$, we can assume without loss that w_1 is non increasing.

In particular, from (11) we see that w_1 satisfies the Gehring condition (3.5) in [BMR3], pag 16. It follows from Theorem 2.1 of [BMR3] that there exists $\delta > 0$ such that

$$\left(\frac{1}{t}\int_0^t w_1^{1+\delta}\right)^{1/1+\delta} \leq \frac{C}{t}\int_0^t w_1 \leq C'w_1.$$

Thus, if we $f(x) = w_1(x)^{1+\delta}$, we have $f \sim P(f)$, and moreover

$$Qf(t) = \int_t^\infty \frac{f(x)}{x} dx \le w_1(t)^\delta \int_t^\infty \frac{w_1(x)}{x} dx \le Cf(t).$$

Again by Lemma 2, we have that

$$f \sim Pf \sim Qf \sim Sf,$$

and therefore

$$w_1 \sim (Pf)^\alpha \sim Q(f)^\alpha \sim (Sf)^\alpha,$$

for $\alpha = 1/(1+\delta)$, and the desired result follows.

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