An extension of Milman's reverse Brunn-Minkowski inequality

by

Jesús Bastero *, Julio Bernués * and Ana Peña *

Departamento de Matemáticas. Facultad de Ciencias
Universidad de Zaragoza
50009-Zaragoza (Spain)

0. Introduction

The classical Brunn-Minkowski inequality states that for $A_1, A_2 \subset \mathbb{R}^n$ compact,

$$|A_1 + A_2|^{1/n} \geq |A_1|^{1/n} + |A_2|^{1/n} \tag{1}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$. Brunn **[Br]** gave the first proof of this inequality for A_1, A_2 compact convex sets, followed by an analytical proof by Minkowski **[Min]**. The inequality (1) for compact sets, not necessarily convex, was first proved by Lusternik **[Lu]**. A very simple proof of it can be found in **[Pi 1]**, Ch. 1.

It is easy to see that one cannot expect the reverse inequality to hold at all, even if it is perturbed by a fixed constant and we restrict ourselves to balls (i.e. convex symmetric compact sets with the origin as an interior point). Take for instance $A_1 = \{(x_1 \dots x_n) \in \mathbb{R}^n \mid |x_1| \leq \varepsilon, |x_i| \leq 1, 2 \leq i \leq n\}$ and $A_2 = \{(x_1 \dots x_n) \in \mathbb{R}^n \mid |x_n| \leq \varepsilon, |x_i| \leq 1, 1 \leq i \leq n-1\}$.

In 1986 V. Milman **[Mil 1]** discovered that if B_1 and B_2 are balls there is always a relative position of B_1 and B_2 for which a perturbed inverse of (1) holds. More precisely: "*There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and any balls $B_1, B_2 \subset \mathbb{R}^n$ we can find a linear transformation $u: \mathbb{R}^n \to \mathbb{R}^n$ with $|\det(u)| = 1$ and*

$$|u(B_1) + B_2|^{1/n} \leq C(|B_1|^{1/n} + |B_2|^{1/n})\text{"}$$

The nature of this reverse Brunn-Minkowski inequality is absolutely different from others (say reverse Blaschke-Santaló inequality, etc.). Brunn-Minkowski inequality is an isoperimetric inequality, (in $\mathbb{R}^n$ it is its first and most important consequence till now) and there is no inverse to isoperimetric inequalities. So, it was a new idea that in the class of affine images of convex bodies there is some kind of inverse.

The result proved by Milman used hard technical tools (see **[Mil 1]**). Pisier in **[Pi 2]** gave a new proof by using interpolation and entropy estimates. Milman in **[Mil 2]** gave another proof by using the "convex surgery" and achieving also some entropy estimates.

The aim of this paper is to extend this Milman's result to a larger class of sets. Note that simple examples show that some conditions on a class of sets are clearly necessary.

For $B \subset \mathbb{R}^n$ body (i.e. compact, with non empty interior), consider $B_1 = B - x_0$, where x_0 is an interior point. If we denote by $N(B_1) = \cap_{|a| \geq 1} aB_1$ the balanced kernel of B_1, it is clear that $N(B_1)$ is a balanced compact neighbourhood of the origin, so there exists $c > 0$ such that $B_1 + B_1 \subset cN(B_1)$. The Aoki-Rolewicz theorem (see **[Ro], [K-P-R]**) implies that there is $0 < p \leq 1$, namely $p = \log_2^{-1}(c)$, such that $B_1 \subset \bar{B} \subset 2^{1/p} B_1$, where $\bar{B}$ is the unit ball of some p-norm. This observation will allow us to work in a p-convex enviroment.

The above construction allows us to define the following parameter. For B a body let $p(B)$, $0 < p(B) \leq 1$, be the supremum of the p for which there exist a measure preserving affine transformation

* Partially supported by Grant DGICYT PS 90-0120
AMS Class.: 46B20, . Key words: p-convexity, entropy

of B, $T(B)$, and a p-norm with unit ball $\bar{B}$ verifying $T(B) \subset \bar{B}$ and $|\bar{B}| \leq |8^{1/p}B|$, (by suitably adapting the results appearing in [**Mil 2**], it is clear that $p(B) \geq p$ for any p-convex body B).

Our main theorem is,

Theorem 1. *Let $0 < p \leq 1$. There exists $C = C(p) \geq 1$ such that for all $n \in \mathbb{N}$ and all $A_1, A_2 \subset \mathbb{R}^n$ bodies such that $p(A_1), p(A_2) \geq p$, there exists an affine transformation $T(x) = u(x) + x_0$ with $x_0 \in \mathbb{R}^n$, $u: \mathbb{R}^n \to \mathbb{R}^n$ linear and $|\det(u)| = 1$ such that*

$$|T(A_1) + A_2|^{1/n} \leq C(|A_1|^{1/n} + |A_2|^{1/n})$$

In particular, for the class of p-balls the constant C is universal (depending only on p).

We prove this theorem in section 2. The key is to estimate certain entropy numbers. We will use the convexity of quasi-normed spaces of Rademacher type $r > 1$, as well as interpolation results and iteration procedures.

We want to thank Gilles Pisier for a useful conversation during the preparation of this paper.

1. Notation and background

Throughout the paper X, Y, Z will denote finite dimensional real vector spaces. A quasi-norm on a real vector space X is a map $\|\cdot\|: X \to \mathbb{R}^+$ such that

i) $\|x\| > 0 \ \forall \ x \neq 0$.

ii) $\|tx\| = |t| \ \|x\| \ \forall \ t \in \mathbb{R}, x \in X$.

iii) $\exists C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|) \ \forall \ x, y \in X$

If iii) is substituted by

iii') $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for $x, y \in X$ and some $0 < p \leq 1$,

$\|\cdot\|$ is called a p-norm on X. Denote by B_X the unit ball of a quasi-normed or a p-normed space.

The above observations concerning the p-convexification of our problem can be restated using p-norm and quasi-norm notation. Recall that any compact balanced set with 0 in its interior is the unit ball of a quasi-norm.

By the concavity of the function t^p, any p-norm is a quasi-norm with $C = 2^{1/p-1}$. Conversely, by the Aoki-Rolewicz theorem, for any quasi-norm with constant C there exists p, namely $p = \log_2^{-1}(2C)$, and a p-norm $|\cdot|$ such that $|x| \leq \|x\| \leq 4^{1/p}|x|, \ \forall \ x \in X$.

A set $K \subset X$ is called p-convex if $\lambda x + \mu y$, whenever $x, y \in K$, $\lambda, \mu \geq 0$, $\lambda^p + \mu^p = 1$. Given $K \subseteq X$, the p-convex hull (or p-convex envelope) of K is the intersection of all p-convex sets that contain K. It is denoted by p-conv (K). The closed unit ball of a p-normed space $(X, \|\cdot\|)$ will simply be called a p-ball. Any symmetric compact p-convex set in X with the origin as an interior point is the p-ball associated to some p-norm.

We say that a quasi-normed space $(X, \|\cdot\|)$ is of (Rademacher) type $q, 0 < q \leq 2$ if for some constant $T_q(X) > 0$ we have

$$\frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \|\sum_{i=1}^n \varepsilon_i x_i\| \leq T_q(X) (\sum_{i=1}^n \|x_i\|^q)^{1/q}, \quad \forall x_i \in X, 1 \leq i \leq n, \ \forall n \in \mathbb{N}$$

Kalton, [**Ka**], proved that any quasi-normed space $(X, \|\cdot\|)$ of type $q > 1$ is convex. That is, the quasi-norm $\|\cdot\|$ is equivalent to a norm and moreover, the equivalence constant depends *only* on $T_q(X)$, (for a more precise statement and proof of this fact see [**K-S**]).

Given $f, g: \mathbb{N} \to \mathbb{R}^+$ we write $f \sim g$ if there exists a constant $C > 0$ such that $C^{-1} f(n) \leq g(n) \leq C f(n), \ \forall \ n \in \mathbb{N}$. Numerical constants will always be denoted by C (or C_p if it depends only on p) although their value may change from line to line.

Let $u: X \to Y$ be a linear map between two quasi-normed spaces and $k \geq 1$. Recall the definition of the following numbers:

Kolmogorov numbers: $d_k(u) = \inf\{\|Q_S \circ u\| \mid S \subset Y \text{ subspace and } \dim(S) < k\}$ where $Q_S: Y \to Y/S$ is the quotient map.

Covering numbers: For $A_1, A_2 \subset X$, $N(A_1, A_2) = \inf\{N \in \mathbb{N} \mid \exists\, x_1 \ldots x_N \in X$ such that $A_1 \subset \bigcup_{1 \leq i \leq N}(x_i + A_2)\}$.

Entropy numbers: $e_k(u) = \inf\{\varepsilon > 0 \mid N(u(B_X), \varepsilon B_Y) \leq 2^{k-1}\}$

The sequences $\{d_k(u)\}, \{e_k(u)\}$ are non-increasing and satisfy $d_1(u) = e_1(u) = \|u\|$. If $\dim(X) = \dim(Y) = n$ then $d_k(u) = 0$ for all $k > n$. Denote s_k either d_k or e_k. For all linear operators $u: X \to Y$, $v: Y \to Z$ we have $s_k(v \circ u) = \|u\| s_k(v)$ and $s_k(v \circ u) = \|v\| s_k(u)$, $\forall k \in \mathbb{N}$ (called the ideal property of s_k) and

$$s_{k+n-1}(v \circ u) \leq s_k(v) s_n(u) \quad \forall\, k, n \in \mathbb{N}$$

The following two lemmas contain useful information about these numbers. The first one extends to the p-convex case its convex analogue due to Carl ([**Ca**]). Its proof mimics the ones of Theorem 5.1 and 5.2 in [**Pi 1**] (see also [**T**]) with minor changes. In particular we identify X as a quotient of $\ell_p(I)$, for some I, and apply the metric lifting property of $\ell_p(I)$ in the class of p-normed spaces (see Proposition C.3.6 in [**Pie**]). The second one contains easy facts about $N(A, B)$ and its proof is similar to the one of Lemma 7.5. in [**Pi 1**] .

Lemma 1. *For all $\alpha > 0$ and $0 < p < 1$ there exists a constant $C_{\alpha,p} > 0$ such that for all linear map $u: X \to Y$, X, Y p-normed spaces and for all $n \in \mathbb{N}$ we have*

$$\sup_{k \leq n} k^\alpha e_k(u) \leq C_{\alpha,p} \sup_{k \leq n} k^\alpha d_k(u)$$

Lemma 2.

i) For all $A_1, A_2, A_3 \subset X$, $N(A_1, A_3) \leq N(A_1, A_2) N(A_2, A_3)$

ii) For all $t > 0$ and $0 < p < 1$ there is $C_{p,t} > 0$ such that for all X p-normed space of dimension n, $N(B_X, tB_X) \leq C_{p,t}^n$.

iii) For any $A_1, A_2, K \subset \mathbb{R}^n$, $|A_1 + K| \leq N(A_1, A_2)|A_2 + K|$.

iv) Let B_1, B_2 be p-balls in $\mathbb{R}^n$ for some p and $B_2 \subset B_1$; then $\dfrac{|B_1|}{|B_2|} \sim N(B_1, B_2)$.

For any $B \subseteq \mathbb{R}^n$ p-ball the polar set of B is defined as

$$B^\circ := \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1, \ \forall\, y \in B\}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. Given B, D p-balls in $\mathbb{R}^n$ we define the following two numbers:

$$s(B) := (|B| \cdot |B^\circ|)^{1/n}$$

and

$$M(B, D) := \left(\frac{|B + D|}{|B \cap D|} \cdot \frac{|B^\circ + D^\circ|}{|B^\circ \cap D^\circ|} \right)^{1/n}$$

Observe that for any linear isomorphism $u: \mathbb{R}^n \to \mathbb{R}^n$ we have $s(u(B)) = s(B)$ and

$$M(u(B), u(D)) = M(B, D).$$

Recall that $s(B_{\ell_p^n}) \sim n^{-1/p} \sim s(B_{\ell_2^n})^{1/p}, 0 < p \leq 1$ ([**Pi 1**] pg. 11).

The following estimates on these numbers are known:

a) **[Sa]**. For every symmetric convex body $B \subset I\!R^n$, $s(B) \leq s(B_{\ell_2^n})$ with equality only if B is an ellipsoid. (*Blaschke-Santaló's inequality*).

b) **[B-M]**. There exists a numerical constant $C > 0$ such that for any $n \in I\!N$ and any symmetric convex body $B \subset I\!R^n$, $s(B) \geq Cs(B_{\ell_2^n})$.

c) **[Mil 1]**. There exists a numerical constant $C > 0$ such that for any $n \in I\!N$ and any symmetric convex body $B \subset I\!R^n$, there is an ellipsoid (called Milman ellipsoid) $D \subset I\!R^n$ such that $M(B, D) \leq C$,(*Milman ellipsoid theorem*).

2. Entropy estimates and reverse Brunn-Minkowski inequality

We first introduce some useful notation: Let $B_1, B_2 \subset I\!R^n$ be two p-balls and $u: I\!R^n \to I\!R^n$ a linear map. We denote $u: B_1 \to B_2$ the operator between p-normed spaces $u: (I\!R^n, \|\cdot\|_{B_1}) \to (I\!R^n, \|\cdot\|_{B_2})$ where $\|\cdot\|_{B_i}$ is the p-norm on $I\!R^n$ whose unit ball is B_i.

Proof of Theorem 1:

Let A_1, A_2 be two bodies in $I\!R^n$ such that $p(A_1)$, $p(A_2) \geq p$. It's clear from the definition that there exist two $\bar{p}$-balls, B_1, B_2, (for instance, $\bar{p} = p/2$) and two measure preserving affine transformations T_1, T_2, verifying

$$|T_2^{-1}T_1(A_1) + A_2| \leq |B_1 + B_2|$$

and

$$|B_1|^{1/n} + |B_2|^{1/n} \leq C_p \left(|A_1|^{1/n} + |A_2|^{1/n} \right).$$

So, we only have to prove the theorem for p-balls.

In the convex case a way to obtain the reverse Brunn-Minkowski inequality is to prove that, for any symmetric convex body B, there exists an ellipsoid D verifying $|B| = |D|$ and

$$|B + \Delta|^{1/n} \leq C|D + \Delta|^{1/n} \tag{2}$$

for any, say compact, subset $\Delta \subseteq I\!R^n$ (C is an universal constant independent of B and n).

Indeed, let B_1, B_2 be two balls in $I\!R^n$. Suppose w.l.o.g. that D_i, the ellipsoids associated to B_i satisfy $u_2 D_i = \alpha_i B_{\ell_2^n}$, where u_i are linear mappings with $|\det u_i| = 1$ and $|B_i|^{1/n} = \alpha_i |B_{\ell_2^n}|^{1/n}$.Then

$$\begin{aligned} |u_1 B_1 + u_2 B_2|^{1/n} &\leq C^2 |u_1 D_1 + u_2 D_2|^{1/n} \\ &= C^2(\alpha_1 + \alpha_2)|B_{\ell_2^n}|^{1/n} = C^2(|B_1|^{1/n} + |B_2|^{1/n}) \end{aligned}$$

In view of the preceding comments and of straightforward computations deduced from Lemma 2, in order to obtain (2) for p-balls it is sufficent to associate an ellipsoid D to each p-ball $B \subset I\!R^n$ in such a way that the corresponding covering numbers verify $N(B, D), N(D, B) \leq C^n$ for some constant C depending only on p.

It is important to remark now the fact that, what we deduce from covering numbers estimate is that the ellipsoid D associated to B actually verifies the stronger assertion

$$C^{-1}|B + \Delta|^{1/n} \leq |D + \Delta|^{1/n} \leq C|B + \Delta|^{1/n}$$

for any compact set Δ in $I\!R^n$, with constant depending only on p. Furthermore, the role of the ellipsoid can be played by any fixed p-ball in a "spetial position".

Denote by $\hat{B}$ the convex hull of B.

By definition of e_n, if $e_n(id: B \to D) \leq \lambda$ then $N(B, 2\lambda D) \leq 2^{n-1}$ and by Lemma 2-ii), $N(B, D) \leq \lambda^n$. (Of course, the same can be done with $N(D, B)$). Therefore our problem reduces to estimating entropy numbers. What we are going to prove is really a stronger result than we need, in the line of Theorem 7.13 of [**Pi 2**].

Lemma 3. *Given $\alpha > 1/p - 1/2$, there exists a constant $C = C(\alpha, p)$ such that, for any $n \in I\!N$ and for any p-ball $B \in I\!R^n$ we can find an ellipsoid $D \in I\!R^n$ such that*

$$d_k(D \to B) + e_k(B \to D) \leq C \left(\frac{n}{k}\right)^\alpha$$

for every $1 \leq k \leq n$.

Proof of the Lemma. From Theorem 7.13 of **[Pi 2]** we can easily deduce the following fact: There exists a constant $C(\alpha) > 0$ such that for any $1 \leq k \leq n$, $n \in I\!N$ and any ball $\hat{B} \subset I\!R^n$, there is ellipsoid $D_0 \subset I\!R^n$ such that the identity operator $id\colon I\!R^n \to I\!R^n$ verifies

$$d_k(id\colon D_0 \to \hat{B}) \leq C(\alpha)\Big(\frac{n}{k}\Big)^\alpha \qquad \text{and} \qquad e_k(id\colon \hat{B} \to D_0) \leq C(\alpha)\Big(\frac{n}{k}\Big)^\alpha \tag{3}$$

For simplicity, since we are always going to deal with the identity operator, we will denote $id\colon B_1 \to B_2$ by $B_1 \to B_2$.

Let D_0 be the ellipsoid associated to $\hat{B}$ in (3). It is well known **[Pe]**, **[G-K]** that $B \subseteq \hat{B} \subseteq n^{1/p-1}B$. This means $\|B \to \hat{B}\| \leq 1$ and $\|\hat{B} \to B\| \leq n^{1/p-1}$. Now, (3) and the ideal property of d_k and e_k imply

$$d_k(D_0 \to B) \leq C(\alpha) n^{1/p-1} \left(\frac{n}{k}\right)^\alpha \qquad \text{and} \qquad e_k(B \to D_0) \leq C(\alpha) \left(\frac{n}{k}\right)^\alpha \qquad \forall\, k \leq n$$

This let us to introduce the constant C_n as the infimum of the constants $C > 0$ for which the conclusion of lemma 3 is true for all p-ball in $I\!R^n$. Trivially $C_n \leq C(\alpha)\left(1 + n^{1/p-1}\right)$. Let D_1 be an almost optimal ellipsoid such that

$$\begin{aligned} d_k(D_1 \to B) &\leq 2C_n \left(\frac{n}{k}\right)^\alpha \\ e_k(B \to D_1) &\leq 2C_n \left(\frac{n}{k}\right)^\alpha \end{aligned} \tag{4}$$

for every $1 \leq k \leq n$.

Use the real interpolation method with parameters $\theta, 2$ to interpolate the couple $id\colon B \to B$ and $id\colon D_1 \to B$. It is straightforward from its definition that for $B_\theta \colon= (B, D_1)_{\theta,2}$, we have

$$d_k(B_\theta \to B) \leq \|B \to B\|^{1-\theta}(d_k(D_1 \to B))^\theta \qquad \forall\, k \leq n$$

and therefore,

$$d_k(B_\theta \to B) \leq \left(2C_n \left(\frac{n}{k}\right)^\alpha\right)^\theta \qquad \forall\, k \leq n$$

Write $\lambda = 4C_n \left(\frac{n}{k}\right)^\alpha$. By definition of the entropy numbers, there exist $x_i \in I\!R^n$ such that $B \subset \bigcup_{i=1}^{2^{k-1}} x_i + 2\lambda D_1$. But by perturbing λ with an absolute constant we can suppose w.l.o.g. that $x_i \in B$. For all $z \in B$, there exists $x_i \in B$ such that $\|z - x_i\|_{D_0} \leq 2\lambda$. Also by p-convexity, $\|z - x_i\|_B \leq 2^{1/p}$.

A general result (see **[B-L]** Ch. 3.) assures the existence of a constant $C_p > 0$ such that

$$\|x\|_{B_\theta} \leq C_p \|x\|_B^{1-\theta} \|x\|_{D_1}^\theta.$$

Therefore, for all $z \in B$, there exists $x_i \in B$ such that $\|z - x_i\|_{B_\theta} \leq C_p \lambda^\theta$ which means

$$e_k(B \to B_\theta) \leq C_p\left(2C_n \left(\frac{n}{k}\right)^\alpha\right)^\theta.$$

Since $\alpha > 1/p - 1/2$, then we can pick $\theta \in (0,1)$ such that $\dfrac{2(1-p)}{2-p} < \theta < \min\{1, 1 - 1/2\alpha\}$. Then B_θ has Rademacher type strictly bigger than 1 because $\frac{1-\theta}{p} + \frac{\theta}{2} < 1$.

By Kalton's result quoted before, we can suppose that B_θ is a ball and therefore we can apply to it (3) for $\gamma = \alpha(1-\theta) > 1/2$ and assure the existence of another ellipsoid D_2 such that

$$d_k(D_2 \to B_\theta) \leq C(\gamma)\left(\frac{n}{k}\right)^\gamma \qquad \text{and} \qquad e_k(B_\theta \to D_2) \leq C(\gamma)\left(\frac{n}{k}\right)^\gamma \qquad \text{and} \qquad \forall\ k \leq n$$

Recall that $d_{2k-1}(D_2 \to B) \leq d_k(D_2 \to B_\theta) d_k(B_\theta \to B)$ and the same for the e_k's. Thanks to the monotonicity of the numbers s_k we can use the what is known about s_{2k-1} for all s_k. Using the estimates obtained above we get $\forall\ k \leq n$,

$$d_k(D_2 \to B) \leq C(p,\alpha) 2^\theta C_n^\theta \left(\frac{n}{k}\right)^{\gamma+\alpha\theta} \qquad \text{and} \qquad e_k(B \to D_2) \leq C(p,\alpha) 2^\theta C_n^\theta \left(\frac{n}{k}\right)^{\gamma+\alpha\theta}.$$

Hence by the election of γ and by minimality we obtain $C_n^{1-\theta} \leq C(p,\alpha)2^\theta$, and the conclusion of the lemma holds.

///

The theorem follows now from the estimate we achieved in Lemma 3 and by Lemma 1. Indeed, given any $\alpha > 1/p - 1/2$, if D is the ellipsoid associated to B by Lemma 3, we have

$$\begin{aligned} n^\alpha e_n(D \to B) &\leq \sup_{k \leq n} k^\alpha e_k(D \to B) \leq C(\alpha,p) \sup_{k \leq n} k^\alpha d_k(D \to B) \leq C(\alpha,p) \sup_{k \leq n} k^\alpha \frac{n^\alpha}{k^\alpha} \\ &= C(\alpha,p) n^\alpha \end{aligned}$$

and so, $e_n(D \to B) \leq C(\alpha,p)$. On the other hand just take $k = n$ in Lemma 3 and so, $e_n(B \to D) \leq C(\alpha,p)$.

Finally observe that since the constant $C(\alpha,p)$ depends only on p and α and we can take any $\alpha > 1/p - 1/2$ the thesis of the theorem as stated inmediately follows.

///

3. Concluding remarks

We conclude this note by stating the corresponding versions of a) Blaschke-Santaló, b) reverse Blaschke-Santaló and c) Milman ellipsoid theorem, cited in section 1, in the context of p-normed spaces.

Proposition 1. *Let $0 < p \leq 1$. There exists a numerical constant $C_p > 0$ such that for every p-ball $B \subseteq I\!R^n$,*

$$C_p\big(s(B_{\ell_2^n})\big)^{1/p} \leq s(B) \leq s(B_{\ell_2^n})$$

and in the second inequality, equality holds if only if B is an ellipsoid

Proof: Denote by $\hat{B}$ the convex envelope of B. Since $\hat{B}^\circ = B^\circ$ we have $s(B) \leq s(\hat{B}) \leq s(B_{\ell_2^n})$. If $s(B) = s(B_{\ell_2^n})$, then $\hat{B}$ is an ellipsoid. We will show that $B = \hat{B}$. Every x in the boundary of $\hat{B}$ can be written as $x = \sum \lambda_i x_i, x_i \in B$, $\sum \lambda_i = 1$; but since $\hat{B}$ is an ellipsoid, x is an extreme point of $\hat{B}$ and so $x = x_i$ for some i that is $x \in B$. This shows $B = \hat{B}$ and we are done.

For the first inequality, $B \subseteq \hat{B} \subseteq n^{1/p-1}B$ easily implies $\left(\frac{|\hat{B}|}{|B|}\right)^{1/n} \leq n^{1/p-1}$ and so,

$$\begin{aligned} s(B) = (|B| \cdot |B^\circ|)^{1/n} = (|B| \cdot |\hat{B}^\circ|)^{1/n} = \left(\frac{|B|}{|\hat{B}|}\right)^{1/n} (|\hat{B}| \cdot |B^\circ|)^{1/n} \geq \frac{s(\hat{B})}{n^{1/p-1}} \\ \geq \frac{Cs(B_{\ell_2^n})}{n^{1/p-1}} = Cn^{-1/p} = C_p\big(s(B_{\ell_2^n})\big)^{1/p} \end{aligned}$$

///

The left inequality above is sharp since $s(B_{\ell_p^n}) = C_p\big(s(B_{\ell_2^n})\big)^{1/p}$. The right inequality is also sharp since every ball is a p-ball for every $0 < p < 1$. And it is sharp even if we restrict ourselves to the class of p-balls which are not q-convex for any $q > p$, as it is showed by the following example: Let $\varepsilon > 0$ and C_ε be a relatively open cap in S^{n-1} centered in $x = (0, \ldots, 0, 1)$ of radius ε. Write $K = S^{n-1} \setminus \{C_\varepsilon \cup -C_\varepsilon\}$. The p-ball p-conv (K) is not q-convex for any $q > p$ and we can pick ε such that $\frac{s(p\text{-conv }(K))}{s(B_{\ell_2^n})} \sim 1$.

Observe that the left inequality is actually *equivalent* to the existence of a constant $C_p > 0$ such that for every p-ball B, $\left(\frac{|\hat{B}|}{|B|}\right)^{1/n} \leq C_p n^{1/p-1}$ and by Lemma 2 iv), this is also equivalent to the inequality $N(\hat{B}, B) \leq C_p n^{1/p-1}$.

With respect to the Milman ellipsoid theorem we obtain

Proposition 2. *Let $0 < p < 1$. There exists a numerical constant $C_p > 0$ such that for every p-ball B there is an ellipsoid D such that $M(B, D) \leq C_p n^{1/p-1}$.*

Proof: Given a p-ball B let D be the Milman ellipsoid of $\hat{B}$. Then,

$$\begin{aligned} M(B, D) &= \left(\frac{|B + D|}{|B \cap D|} \cdot \frac{|B^\circ + D|}{|B^\circ \cap D|}\right)^{1/n} \\ &= \left(\frac{|\hat{B} + D|}{|\hat{B} \cap D|} \cdot \frac{|\hat{B}^\circ + D|}{|\hat{B}^\circ \cap D|}\right)^{1/n} \left(\frac{|B + D|}{|\hat{B} + D|}\right)^{1/n} \left(\frac{|\hat{B} \cap D|}{|B \cap D|}\right)^{1/n} \\ &\leq M(\hat{B}, D) \left(\frac{|\hat{B} \cap D|}{|B \cap D|}\right)^{1/n} \leq C_p n^{1/p-1} \end{aligned}$$

///

The bound for $M(B,D)$ is sharp. Indeed, if there was a function $f(n) << n^{1/p-1}$ such that for every a p-ball B there was an ellipsoid D with $M(B,D) \leq f(n)$, then

$$\frac{s(B_{\ell_2^n})}{f(n)} = \frac{s(D)}{f(n)} \leq \left(|B \cap D| \cdot |B^\circ \cap D|\right)^{1/n} \leq s(B)$$

and we would have, $s(B) \geq \frac{s(B_{\ell_2^n})}{f(n)} >> n^{-1/p}$ which is not possible.

Acknowledgments. The authors are indebted to the referee for some fruitful comments which led them to improve the presentation of the paper.

References

[B-L] Bergh, J. Löfström, J.: *Interpolation Spaces. An Introduction*. Springer-Verlag, 223. Berlin Heildelberg (1976).

[B-M] Bourgain, J., Milman, V.D.: *On Mahler's conjecture on the volume of a convex symmetric body and its polar*. Preprint I.H.E.S., March 1985.

[Br] Brunn, H.: *Über Ovale und Eiflächen*. Innaugural dissertation. München 1887.

[Ca] Carl B.: *Entropy numbers, s-numbers, and eigenvalue problems*. J. Funct. Anal. **41**, 290-306 (1981).

[G-K] Gordon, Y., Kalton, N.J.: *Local structure for quasi-normed spaces*. Preprint.

[Ja] Jarchow, H.: *Locally convex spaces*. B.G. Teubner Stuttgart, (1981).

[Ka] Kalton, N.: *Convexity, Type and the Three Space Problem*. Studia Math. **69**, 247-287 (1980-81).

[K-S] Kalton, N., Sik-Chung Tam: *Factorization theorems for quasi-normed spaces*. Houston J. Math. **19** (1993), 301-317.

[K-P-R] Kalton, N., Peck, N.T., Roberts, J.W. : *An F-space sampler*. London Math. Soc. Lecture Notes 89. Cambridge Univ. Press. Cambridge (1985).

[Lu] Lusternik, L.A.: *Die Brunn-Minkowskische Ungleichung für beliebige messbare Mengen*. C.R. Acad.Sci.URSS **8**, 55-58 (1935).

[Mil 1] Milman, V.D.: *Inégalité de Brunn-Minkowsky inverse et applications á la théorie locale des espaces normés*. C.R. Acad.Sci.Paris **302**,Sér 1, 25-28 (1986).

[Mil 2] Milman, V.D.: *Isomorphic symmetrizations and geometric inequalities*. In "Geometric Aspects of Functional Analysis" Israel Sem. GAFA. Lecture Notes in Mathematics, Springer, **1317**, (1988), 107-131.

[Min] Minkowski, H.:*Geometrie der Zahlen*. Teubner, Leipzig (1910).

[Pe] Peck, T.: *Banach-Mazur distances and projections on p-convex spaces*. Math. Zeits. **177**, 132-141 (1981).

[Pie] Pietsch, A.: *Operator ideals*. North-Holland, Berlin 1979.

[Pi 1] Pisier, G.: *The volume of convex bodies and Banach Space Geometry*. Cambridge University Press, Cambridge (1989).

[Pi 2] Pisier, G.: *A new approach to several results of V. Milman*. J. reine angew. Math. **393**, 115-131 (1989).

[Ro] Rolewicz, S.: *Metric linear spaces.* MM 56. PWN, Warsaw (1972).

[Sa] Santaló, L.A.: *Un invariante afín para los cuerpos convexos de n dimensiones.* Portugal Math. **8**, 155-161 (1949).

[T] Triebel, H.: *Relations between approximation numbers and entropy numbers.* Journal of Approx. Theory **78**, (1994), 112-116.