An extension of Milman’s reverse Brunn-Minkowski inequality

by

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0. Introduction

The classical Brunn-Minkowski inequality states that for $A_1, A_2 \subset \mathbb{R}^n$ compact,

$$|A_1 + A_2|^{1/n} \geq |A_1|^{1/n} + |A_2|^{1/n} \quad (1)$$

where $| \cdot |$ denotes the Lebesgue measure on $\mathbb{R}^n$. Brunn [Br] gave the first proof of this inequality for $A_1, A_2$ compact convex sets, followed by an analytical proof by Minkowski [Min]. The inequality (1) for compact sets, not necessarily convex, was first proved by Lusternik [Lu]. A very simple proof of it can be found in [Pi 1], Ch. 1.

It is easy to see that one cannot expect the reverse inequality to hold at all, even if it is perturbed by a fixed constant and we restrict ourselves to balls (i.e. convex symmetric compact sets with the origin as an interior point). Take for instance $A_1 = \{(x_1 \ldots x_n) \in \mathbb{R}^n \mid |x_1| \leq \varepsilon, |x_i| \leq 1, 2 \leq i \leq n\}$ and $A_2 = \{(x_1 \ldots x_n) \in \mathbb{R}^n \mid |x_n| \leq \varepsilon, |x_i| \leq 1, 1 \leq i \leq n - 1\}$.

In 1986 V. Milman [Mil 1] discovered that if $B_1$ and $B_2$ are balls there is always a relative position of $B_1$ and $B_2$ for which a perturbed inverse of (1) holds. More precisely: “There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and any balls $B_1, B_2 \subset \mathbb{R}^n$ we can find a linear transformation $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $|\det(u)| = 1$ and

$$|u(B_1) + B_2|^{1/n} \leq C(|B_1|^{1/n} + |B_2|^{1/n})$$

The nature of this reverse Brunn-Minkowski inequality is absolutely different from others (say reverse Blaschke-Santaló inequality, etc.). Brunn-Minkowski inequality is an isoperimetric inequality, (in $\mathbb{R}^n$ it is its first and most important consequence till now) and there is no inverse to isoperimetric inequalities. So, it was a new idea that in the class of affine images of convex bodies there is some kind of inverse.

The result proved by Milman used hard technical tools (see [Mi1 1]). Pisier in [Pi 2] gave a new proof by using interpolation and entropy estimates. Milman in [Mil 2] gave another proof by using the “convex surgery” and achieving also some entropy estimates.

The aim of this paper is to extend this Milman’s result to a larger class of sets. Note that simple examples show that some conditions on a class of sets are clearly necessary.

For $B \subset \mathbb{R}^n$ body (i.e. compact, with non empty interior), consider $B_1 = B - x_0$, where $x_0$ is an interior point. If we denote by $N(B_1) = \cap_{|a| \geq 1} aB_1$ the balanced kernel of $B_1$, it is clear that $N(B_1)$ is a balanced compact neighbourhood of the origin, so there exists $c > 0$ such that $B_1 + B_1 \subset cN(B_1)$.

The Aoki-Rolewicz theorem (see [Ro], [K-P-R]) implies that there is $0 < p \leq 1$, namely $p = \log_2^{1/2}(c)$, such that $B_1 \subset B \subset 2^{1/p}B_1$, where $B$ is the unit ball of some $p$-norm. This observation will allow us to work in a $p$-convex environment.

The above construction allows us to define the following parameter. For $B$ a body let $p(B)$, $0 < p(B) \leq 1$, be the supremum of the $p$ for which there exist a measure preserving affine transformation

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of $B$, $T(B)$, and a $p$-norm with unit ball $\bar{B}$ verifying $T(B) \subset \bar{B}$ and $|\bar{B}| \leq |8^{1/p}B|$, (by suitably adapting the results appearing in [Mil 2], it is clear that $p(B) \geq p$ for any $p$-convex body $B$).

Our main theorem is,

**Theorem 1.** Let $0 < p \leq 1$. There exists $C = C(p) \geq 1$ such that for all $n \in \mathbb{N}$ and all $A_1, A_2 \subset \mathbb{R}^n$ bodies such that $p(A_1), p(A_2) \geq p$, there exists an affine transformation $T(x) = u(x) + x_0$ with $x_0 \in \mathbb{R}^n$, $u: \mathbb{R}^n \to \mathbb{R}^n$ linear and $|\det(u)| = 1$ such that

$$|T(A_1) + A_2|^{1/n} \leq C(|A_1|^{1/n} + |A_2|^{1/n})$$

In particular, for the class of $p$-balls the constant $C$ is universal (depending only on $p$).

We prove this theorem in section 2. The key is to estimate certain entropy numbers. We will use the convexity of quasi-normed spaces of Rademacher type $r > 1$, as well as interpolation results and iteration procedures.

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1. Notation and background

Throughout the paper $X, Y, Z$ will denote finite dimensional real vector spaces. A quasi-norm on a real vector space $X$ is a map $\| \cdot \|: X \to \mathbb{R}^+$ such that

i) $\|x\| > 0 \forall x \neq 0$.

ii) $\|tx\| = |t| \|x\| \forall t \in \mathbb{R}, x \in X$.

iii) $\exists C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|) \forall x, y \in X$

If iii) is substituted by

iii’) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for $x, y \in X$ and some $0 < p \leq 1$,

$\| \cdot \|$ is called a $p$-norm on $X$. Denote by $B_X$ the unit ball of a quasi-normed or a $p$-normed space.

The above observations concerning the $p$-convexification of our problem can be restated using $p$-norm and quasi-norm notation. Recall that any compact balanced set with 0 in its interior is the unit ball of a quasi-norm.

By the concavity of the function $t^p$, any $p$-norm is a quasi-norm with $C = 2^{1/p-1}$. Conversely, by the Aoki-Rolewicz theorem, for any quasi-norm with constant $C$ there exists $p$, namely $p = \log_2(2C)$, and a $p$-norm $\| \cdot \|$ such that $|x| \leq \|x\| \leq 4^{1/p}|x|$, $\forall x \in X$.

A set $K \subset X$ is called $p$-convex if $\lambda x + \mu y$, whenever $x, y \in K$, $\lambda, \mu \geq 0$, $\lambda^p + \mu^p = 1$. Given $K \subset X$, the $p$-convex hull (or $p$-convex envelope) of $K$ is the intersection of all $p$-convex sets that contain $K$. It is denoted by $p\text{-conv~}(K)$. The closed unit ball of a $p$-normed space $(X, \| \cdot \|)$ will simply be called a $p$-ball. Any symmetric compact $p$-convex set in $X$ with the origin as an interior point is the $p$-ball associated to some $p$-norm.

We say that a quasi-normed space $(X, \| \cdot \|)$ is of (Rademacher) type $q, 0 < q \leq 2$ if for some constant $T_q(X) > 0$ we have

$$\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \| \sum_{i=1}^n \epsilon_i x_i \| \leq T_q(X) \left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q}, \forall x_i \in X, 1 \leq i \leq n, \forall n \in \mathbb{N}$$

Kalten, [Ka], proved that any quasi-normed space $(X, \| \cdot \|)$ of type $q > 1$ is convex. That is, the quasi-norm $\| \cdot \|$ is equivalent to a norm and moreover, the equivalence constant depends only on $T_q(X)$, (for a more precise statement and proof of this fact see [K-S]).

Given $f, g: \mathbb{N} \to \mathbb{R}^+$ we write $f \sim g$ if there exists a constant $C > 0$ such that $C^{-1}f(n) \leq g(n) \leq Cf(n), \forall n \in \mathbb{N}$. Numerical constants will always be denoted by $C$ (or $C_p$ if it depends only on $p$) although their value may change from line to line.
Let $u: X \to Y$ be a linear map between two quasi-normed spaces and $k \geq 1$. Recall the definition of the following numbers:

1. **Kolmogorov numbers**: $d_k(u) = \inf\{\|Q_S \circ u\| : S \subset Y \text{ subspace and } \dim(S) < k\}$ where $Q_S: Y \to Y/S$ is the quotient map.

2. **Covering numbers**: For $A_1, A_2 \subset X$, $N(A_1, A_2) = \inf\{N \in \mathbb{N} : \exists x_1, \ldots, x_N \in X \text{ such that } A_1 \subset \bigcup_{1 \leq i \leq N} \langle x_i + A_2 \rangle\}$.

3. **Entropy numbers**: $e_k(u) = \inf\{\varepsilon > 0 : N(u(B_X), B_Y) \leq 2^{k-1}\}$

The following two lemmas contain useful information about these numbers. The first one extends the $p$-convex case its convex analogue due to Carl ([Ca]). Its proof mimics the ones of Theorem 5.1 and 5.2 in [Pi 1] (see also [T]) with minor changes. In particular we identify $X$ as a quotient of $\ell_p(I)$, for some $I$, and apply the metric lifting property of $\ell_p(I)$ in the class of $p$-normed spaces (see Proposition C.3.6 in [Pie]). The second one contains easy facts about $N(A, B)$ and its proof is similar to the one of Lemma 7.5. in [Pi 1].

**Lemma 1.** For all $\alpha > 0$ and $0 < p < 1$ there exists a constant $C_{\alpha, p} > 0$ such that for all linear map $u: X \to Y$, $X, Y$ $p$-normed spaces and for all $n \in \mathbb{N}$ we have

$$\sup_{k \leq n} k^\alpha e_k(u) \leq C_{\alpha, p} \sup_{k \leq n} k^\alpha d_k(u)$$

**Lemma 2.**

i) For all $A_1, A_2, A_3 \subset X$, $N(A_1, A_3) \leq N(A_1, A_2) N(A_2, A_3)$

ii) For all $t > 0$ and $0 < p < 1$ there is $C_{p,t} > 0$ such that for all $X$ $p$-normed space of dimension $n$, $N(B_X, tB_X) \leq C_{p,t}^n$.

iii) For any $A_1, A_2, K \subset \mathbb{R}^n$, $|A_1 + K| \leq N(A_1, A_2)|A_2 + K|$.

iv) Let $B_1, B_2$ be $p$-balls in $\mathbb{R}^n$ for some $p$ and $B_2 \subset B_1$; then $\frac{|B_1|}{|B_2|} \sim N(B_1, B_2)$.

For any $B \subset \mathbb{R}^n$ $p$-ball the polar set of $B$ is defined as

$$B^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \forall y \in B\}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. Given $B, D$ $p$-balls in $\mathbb{R}^n$ we define the following two numbers:

$$s(B) := (|B| \cdot |B^\circ|)^{1/n}$$

and

$$M(B, D) := \left(\frac{|B + D|}{|B \cap D|}, \frac{|B^\circ + D^\circ|}{|B^\circ \cap D^\circ|}\right)^{1/n}$$

Observe that for any linear isomorphism $u: \mathbb{R}^n \to \mathbb{R}^n$ we have $s(u(B)) = s(B)$ and $M(u(B), u(D)) = M(B, D)$.

Recall that $s(B_{\ell_p^n}) \sim n^{-1/p} \sim s(B_{\ell_2^n})^{1/p}, 0 < p \leq 1$ ([Pi 1] pg. 11).

The following estimates on these numbers are known:
a) [Sa]. For every symmetric convex body $B \subset \mathbb{R}^n$, $s(B) \leq s(B_{1/2})$ with equality only if $B$ is an ellipsoid. (Blaschke-Santaló’s inequality).

b) [B-M]. There exists a numerical constant $C > 0$ such that for any $n \in \mathbb{N}$ and any symmetric convex body $B \subset \mathbb{R}^n$, $s(B) \geq Cs(B_{1/2})$.

c) [Mil 1]. There exists a numerical constant $C > 0$ such that for any $n \in \mathbb{N}$ and any symmetric convex body $B \subset \mathbb{R}^n$, there is an ellipsoid (called Milman ellipsoid) $D \subset \mathbb{R}^n$ such that $M(B, D) \leq C$. (Milman ellipsoid theorem).

2. Entropy estimates and reverse Brunn-Minkowski inequality

We first introduce some useful notation: Let $B_1, B_2 \subset \mathbb{R}^n$ be two $p$-balls and $u: \mathbb{R}^n \to \mathbb{R}^n$ a linear map. We denote $u: B_1 \to B_2$ the operator between $p$-normed spaces $u: (\mathbb{R}^n, \| \cdot \|_{B_1}) \to (\mathbb{R}^n, \| \cdot \|_{B_2})$ where $\| \cdot \|_{B_i}$ is the $p$-norm on $\mathbb{R}^n$ whose unit ball is $B_i$.

Proof of Theorem 1:

Let $A_1, A_2$ be two bodies in $\mathbb{R}^n$ such that $p(A_1), p(A_2) \geq p$. It’s clear from the definition that there exist two $\bar{p}$-balls, $B_1, B_2$, (for instance, $\bar{p} = p/2$) and two measure preserving affine transformations $T_1, T_2$, verifying

$$|T_2^{-1}T_1(A_1) + A_2| \leq |B_1 + B_2|$$

and

$$|B_1|^{1/n} + |B_2|^{1/n} \leq C_p \left( |A_1|^{1/n} + |A_2|^{1/n} \right).$$

So, we only have to prove the theorem for $p$-balls.

In the convex case a way to obtain the reverse Brunn-Minkowski inequality is to prove that, for any symmetric convex body $B$, there exists an ellipsoid $D$ verifying $|B| = |D|$ and

$$|B + \Delta|^{1/n} \leq C|D + \Delta|^{1/n} \tag{2}$$

for any, say compact, subset $\Delta \subset \mathbb{R}^n$ ($C$ is an universal constant independent of $B$ and $n$).

Indeed, let $B_1, B_2$ be two balls in $\mathbb{R}^n$. Suppose w.l.o.g. that $D_i$, the ellipsoids associated to $B_i$ satisfy $u_2D_i = \alpha_iB_{1/2}$, where $u_i$ are linear mappings with $|\det u_i| = 1$ and $|B_i|^{1/n} = \alpha_i|B_{1/2}|^{1/n}$. Then

$$|u_1B_1 + u_2B_2|^{1/n} \leq C^2|u_1D_1 + u_2D_2|^{1/n} = C^2(\alpha_1 + \alpha_2)|B_{1/2}|^{1/n} = C^2(|B_1|^{1/n} + |B_2|^{1/n})$$

In view of the preceding comments and of straightforward computations deduced from Lemma 2, in order to obtain (2) for $p$-balls it is sufficient to associate an ellipsoid $D$ to each $p$-ball $B \subset \mathbb{R}^n$ in such a way that the corresponding covering numbers verify $N(B, D), N(D, B) \leq C^n$ for some constant $C$ depending only on $p$.

It is important to remark now the fact that, what we deduce from covering numbers estimate is that the ellipsoid $D$ associated to $B$ actually verifies the stronger assertion

$$C^{-1}|B + \Delta|^{1/n} \leq |D + \Delta|^{1/n} \leq C|B + \Delta|^{1/n}$$

for any compact set $\Delta$ in $\mathbb{R}^n$, with constant depending only on $p$. Furthermore, the role of the ellipsoid can be played by any fixed $p$-ball in a “special position”.

Denote by $\hat{B}$ the convex hull of $B$.

By definition of $e_n$, if $e_n(id: B \to D) \leq \lambda$ then $N(B, 2\lambda D) \leq 2^{n-1}$ and by Lemma 2-ii), $N(B, D) \leq \lambda^n$. (Of course, the same can be done with $N(D, B)$). Therefore our problem reduces to estimating entropy numbers. What we are going to prove is really a stronger result than we need, in the line of Theorem 7.13 of [Pi 2].
Lemma 3. Given \( \alpha > 1/p - 1/2 \), there exists a constant \( C = C(\alpha, p) \) such that, for any \( n \in \mathbb{N} \) and for any \( p \)-ball \( B \in \mathbb{R}^n \) we can find an ellipsoid \( D \in \mathbb{R}^n \) such that

\[
d_k(D \to B) + e_k(B \to D) \leq C \left( \frac{n}{k} \right)^\alpha
\]

for every \( 1 \leq k \leq n \).

Proof of the Lemma. From Theorem 7.13 of [Pi 2] we can easily deduce the following fact: There exists a constant \( C(\alpha) > 0 \) such that for any \( 1 \leq k \leq n, n \in \mathbb{N} \) and any ball \( \hat{B} \subset \mathbb{R}^n \), there is ellipsoid \( D_0 \subset \mathbb{R}^n \) such that the identity operator \( \text{id} : \mathbb{R}^n \to \mathbb{R}^n \) verifies

\[
d_k(\text{id} : D_0 \to \hat{B}) \leq C(\alpha) \left( \frac{n}{k} \right)^\alpha \quad \text{and} \quad e_k(\text{id} : \hat{B} \to D_0) \leq C(\alpha) \left( \frac{n}{k} \right)^\alpha
\]  

(3)

For simplicity, since we are always going to deal with the identity operator, we will denote \( \text{id} : B_1 \to B_2 \) by \( B_1 \to B_2 \).

Let \( D_0 \) be the ellipsoid associated to \( \hat{B} \) in (3). It is well known [Pe], [G-K] that \( B \subseteq \hat{B} \subseteq n^{1/p-1}B \). This means \( \|B \to \hat{B}\| \leq 1 \) and \( \|B \to B\| \leq n^{1/p-1} \). Now, (3) and the ideal property of \( d_k \) and \( e_k \) imply

\[
d_k(D_0 \to B) \leq C(\alpha)n^{1/p-1} \left( \frac{n}{k} \right)^\alpha \quad \text{and} \quad e_k(B \to D_0) \leq C(\alpha) \left( \frac{n}{k} \right)^\alpha \quad \forall \ k \leq n
\]

This let us to introduce the constant \( C_n \) as the infimum of the constants \( C > 0 \) for which the conclusion of lemma 3 is true for all \( p \)-ball in \( \mathbb{R}^n \). Trivially \( C_n \leq C(\alpha)(1 + n^{1/p-1}) \). Let \( D_1 \) be an almost optimal ellipsoid such that

\[
d_k(D_1 \to B) \leq 2C_n \left( \frac{n}{k} \right)^\alpha
\]

\[
e_k(B \to D_1) \leq 2C_n \left( \frac{n}{k} \right)^\alpha
\]  

(4)

for every \( 1 \leq k \leq n \).

Use the real interpolation method with parameters \( \theta, 2 \) to interpolate the couple \( \text{id} : B \to B \) and \( \text{id} : D_1 \to B \). It is straightforward from its definition that for \( B_\theta = (B, D_1)_{\theta, 2}, \) we have

\[
d_k(B_\theta \to B) \leq \|B \to B\|^{1-\theta}(d_k(D_1 \to B))^\theta \quad \forall \ k \leq n
\]

and therefore,

\[
d_k(B_\theta \to B) \leq \left( 2C_n \left( \frac{n}{k} \right)^\alpha \right)^\theta \quad \forall \ k \leq n
\]

Write \( \lambda = 4C_n \left( \frac{n}{k} \right)^\alpha \). By definition of the entropy numbers, there exist \( x_i \in \mathbb{R}^n \) such that \( B \subseteq \bigcup_{i=1}^{2^k-1} x_i + 2\lambda D_1 \). But by perturbing \( \lambda \) with an absolute constant we can suppose w.l.o.g. that \( x_i \in B \). For all \( z \in B \), there exists \( x_i \in B \) such that \( \|z - x_i\|_{\nu_0} \leq 2\lambda \). Also by \( p \)-convexity, \( \|z - x_i\|_B \leq 2^{1/p} \).

A general result (see [B-L] Ch. 3.) assures the existence of a constant \( C_p > 0 \) such that

\[
\|x\|_{B_\theta} \leq C_p \|x\|_B^{1-\theta}\|x\|_{D_1}^\theta.
\]

Therefore, for all \( z \in B \), there exists \( x_i \in B \) such that \( \|z - x_i\|_{B_\theta} \leq C_p \lambda^\theta \) which means

\[
e_k(B \to B_\theta) \leq C_p \left( 2C_n \left( \frac{n}{k} \right)^\alpha \right)^\theta.
\]
Since $\alpha > 1/p - 1/2$, then we can pick $\theta \in (0, 1)$ such that
\[
\frac{2(1-p)}{2-p} < \theta < \min\{1, 1 - 1/2\alpha\}.
\]
Then $B_{\theta}$ has Rademacher type strictly bigger than 1 because
\[
\frac{1-\theta}{p} + \frac{\theta}{2} < 1.
\]
By Kalton’s result quoted before, we can suppose that $B_{\theta}$ is a ball and therefore we can apply to it (3) for $\gamma = \alpha(1 - \theta) > 1/2$ and assure the existence of another ellipsoid $D_{2}$ such that
\[
d_{k}(D_{2} \rightarrow B_{\theta}) \leq C(\gamma) \left(\frac{n}{k}\right)^{\gamma} \quad \text{and} \quad e_{k}(B_{\theta} \rightarrow D_{2}) \leq C(\gamma) \left(\frac{n}{k}\right)^{\gamma} \quad \text{and} \quad \forall \ k \leq n.
\]
Recall that $d_{2k-1}(D_{2} \rightarrow B) \leq d_{k}(D_{2} \rightarrow B_{\theta})d_{k}(B_{\theta} \rightarrow B)$ and the same for the $e_{k}$’s. Thanks to the monotonicity of the numbers $s_{k}$ we can use the what is known about $s_{2k-1}$ for all $s_{k}$. Using the estimates obtained above we get $\forall \ k \leq n,$
\[
d_{k}(D_{2} \rightarrow B) \leq C(p, \alpha)2^{\theta}C_{n}^{d}(\alpha)^{\gamma + \alpha\theta} \quad \text{and} \quad e_{k}(B \rightarrow D_{2}) \leq C(p, \alpha)2^{\theta}C_{n}^{d}(\alpha)^{\gamma + \alpha\theta}.
\]
Hence by the election of $\gamma$ and by minimality we obtain $C_{n}^{d-\theta} \leq C(p, \alpha)2^{\theta}$, and the conclusion of the lemma holds.

The theorem follows now from the estimate we achieved in Lemma 3 and by Lemma 1. Indeed, given any $\alpha > 1/p - 1/2$, if $D$ is the ellipsoid associated to $B$ by Lemma 3, we have
\[
n^{\alpha}e_{n}(D \rightarrow B) \leq \sup_{k \leq n} k^{\alpha}e_{k}(D \rightarrow B) \leq C(\alpha, p) \sup_{k \leq n} k^{\alpha}d_{k}(D \rightarrow B) \leq C(\alpha, p) \sup_{k \leq n} k^{\alpha}n^{\alpha} \leq C(\alpha, p) n^{\alpha}.
\]
and so, $e_{n}(D \rightarrow B) \leq C(\alpha, p)$. On the other hand just take $k = n$ in Lemma 3 and so, $e_{n}(B \rightarrow D) \leq C(\alpha, p)$.

Finally observe that since the constant $C(\alpha, p)$ depends only on $p$ and $\alpha$ and we can take any $\alpha > 1/p - 1/2$ the thesis of the theorem as stated immediately follows.
3. Concluding remarks

We conclude this note by stating the corresponding versions of a) Blaschke-Santaló, b) reverse Blaschke-Santaló and c) Milman ellipsoid theorem, cited in section 1, in the context of \( p \)-normed spaces.

**Proposition 1.** Let \( 0 < p \leq 1 \). There exists a numerical constant \( C_p > 0 \) such that for every \( p \)-ball \( B \subseteq \mathbb{R}^n \),
\[
C_p (s(B_{B})^{1/p}) \leq s(B) \leq s(B_{B})
\]
and in the second inequality, equality holds if only if \( B \) is an ellipsoid.

**Proof:** Denote by \( \hat{B} \) the convex envelope of \( B \). Since \( \hat{B}^p = B^p \) we have \( s(B) \leq s(\hat{B}) \leq s(B_{B}) \). If \( s(B) = s(B_{B}) \), then \( \hat{B} \) is an ellipsoid. We will show that \( B = \hat{B} \). Every \( x \) in the boundary of \( \hat{B} \) can be written as \( x = \sum \lambda_i x_i, x_i \in B, \sum \lambda_i = 1; \) but since \( \hat{B} \) is an ellipsoid, \( x \) is an extreme point of \( \hat{B} \) and so \( x = x_i \) for some \( i \) that is \( x \in B \). This shows \( B = \hat{B} \) and we are done.

For the first inequality, \( B \subseteq \hat{B} \subseteq n^{1/p-1}B \) easily implies \( \left( \frac{\hat{B}}{|B|} \right)^{1/n} \leq n^{1/p-1} \) and so,
\[
s(B) = (|B| \cdot |B^p|)^{1/n} = (|B| \cdot |\hat{B}^p|)^{1/n} = \left( \frac{|B|}{|\hat{B}|} \right)^{1/n} (|\hat{B}| \cdot |B^p|)^{1/n} \geq \frac{Cs(B_{B})}{n^{1/p-1}} = Cn^{1/p-1} = C_p (s(B_{B})^{1/p})
\]

The left inequality above is sharp since \( s(B_{B}) = C_p (s(B_{B})^{1/p}) \). The right inequality is also sharp since every ball is a \( p \)-ball for every \( 0 < p < 1 \). And it is sharp even if we restrict ourselves to the class of \( p \)-balls which are not \( q \)-convex for any \( q > p \), as it is showed by the following example: Let \( \varepsilon > 0 \) and \( C_\varepsilon \) be a relatively open cap in \( S^{n-1} \) centered in \( x = (0, \ldots, 0, 1) \) of radius \( \varepsilon \). Write \( K = S^{n-1} \setminus \{C_\varepsilon \cup -C_\varepsilon \} \). The \( p \)-ball \( p \)-conv \( (K) \) is not \( q \)-convex for any \( q > p \) and we can pick \( \varepsilon \) such that \( \frac{s(\text{p-conv } (K))}{s(B_{B})} \sim 1 \).

Observe that the left inequality is actually equivalent to the existence of a constant \( C_p > 0 \) such that for every \( p \)-ball \( B \), \( \left( \frac{|\hat{B}|}{|B|} \right)^{1/n} \leq C_p n^{1/p-1} \) and by Lemma 2 iv), this is also equivalent to the inequality \( N(B, B) \leq C_p n^{1/p-1} \).

With respect to the Milman ellipsoid theorem we obtain

**Proposition 2.** Let \( 0 < p < 1 \). There exists a numerical constant \( C_p > 0 \) such that for every \( p \)-ball \( B \) there is an ellipsoid \( D \) such that \( M(B, D) \leq C_p n^{1/p-1} \).

**Proof:** Given a \( p \)-ball \( B \) let \( D \) be the Milman ellipsoid of \( \hat{B} \). Then,
\[
M(B, D) = \left( \frac{|B + D|}{|B \cap D|}, \frac{|B^p + D|}{|B^p \cap D|} \right)^{1/n} \leq \left( \frac{|\hat{B} + D|}{|\hat{B} \cap D|}, \frac{|\hat{B}^p + D|}{|\hat{B}^p \cap D|} \right)^{1/n} \leq M(\hat{B}, D) \left( \frac{|\hat{B} \cap D|}{|B \cap D|} \right)^{1/n} \leq C_p n^{1/p-1}
\]
The bound for $M(B, D)$ is sharp. Indeed, if there was a function $f(n) << n^{1/p-1}$ such that for every a $p$-ball $B$ there was an ellipsoid $D$ with $M(B, D) \leq f(n)$, then

$$\frac{s(B_2)}{f(n)} = \frac{s(D)}{f(n)} \leq (|B \cap D| \cdot |B^o \cap D|)^{1/n} \leq s(B)$$

and we would have, $s(B) \geq \frac{s(B_2)}{f(n)} >> n^{-1/p}$ which is not possible.

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References


