Random vectors satisfying Khinchine-Kahane type inequalities for linear and quadratic forms

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We study the behaviour of moments of order p $(1 < p < \infty)$ of affine and quadratic forms with respect to non log-concave measures and we obtain an extension of Khinchine-Kahane inequality for new families of random vectors by using Pisier's inequalities for martingales. As a consequence, we get some estimates for the moments of affine and quadratic forms with respect to a tail volume of the unit ball of ℓ_q^n $(0 < q < 1)$.

1 Introduction

It is well known the exponential decay of many systems of independent and identically distributed random variables and a similar situation occurs in other frameworks. For instance, M. Gromov and V. Milman (see [8]) proved such exponential decay for the uniform distribution on convex bodies in $\mathbb{R}^n$ and this fact is known as an extension of Khinchine-Kahane inequalities for convex bodies. More exactly, what they proved was that for every convex body in $\mathbb{R}^n$, and for every f, linear form defined on $\mathbb{R}^n$, the following inequality is true

$$\left(\frac{1}{|K|}\int_K |f(x)|^p\,dx\right)^{1/p} \leq Cp\frac{1}{|K|}\int_K |f(x)|\,dx, \tag{1.1}$$

for all $p \geq 1$ and for some absolute constant $C > 0$ independent of K and of the dimension. From this fact it is clear that

$$\mu\left\{x \in K; |f(x)| > t\,\|f\|_2\right\} \leq C\exp(-Ct)$$

where μ is the uniform distribution on K and

$$\|f\|_2 = \left(\frac{1}{|K|}\int_K |f(x)|^2\,dx\right)^{1/2}.$$

Moreover,

$$\|f\|_{L_{\psi_1}(K,d\mu)} \leq C\|f\|_{L_1(K,d\mu)} \tag{1.2}$$

for some absolute constant $C > 0$, where $L_{\psi_1}(K, dx)$ is the Orlicz space generated by the Orlicz function $\psi_1(t) = e^t - 1$ with respect to the Lebesgue measure normalized on K.

By using C. Borel inequality (see [3], [4]) a simple proof of this fact can be given in a more general framework, that of log-concave measures on $\mathbb{R}^n$ (note that the uniform distribution on a convex body is a log-concave probability on $\mathbb{R}^n$ as a consequence of the Brunn-Minkowski inequality, see [13]). R. Latała

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([11]) and O. Guédon ([9]) extended the inequality 1.1 to the range $-1 < r < p < \infty$ for log-concave probabilities, by proving that

$$\|f\|_{L^p(d\mu)} \le C \max\left\{p, \frac{1}{1+r}\right\} \|f\|_{L^r(d\mu)}.$$

J. Bourgain (cf. [5]) extended M. Gromov and V. Milman inequality to the class of polynomials, answering a question raised by V. Milman. S.G. Bobkov (see [2]), by using localization lemma, extended Bourgain's result to any log-concave probability on $\mathbb{R}^n$ given the right estimate, i.e.

$$\|f\|_{L_{\psi_{1/d}}(d\mu)} \le C^d \|f\|_1,$$

for any polynomial f of degree d and some absolute constant C, where $\psi(t) = \exp\left(t^{1/d}\right) - 1$ and the same method prove that

$$\|f\|_{L^1(d\mu)} \le \left(\frac{C}{1+r}\right)^d \|f\|_{L^{r/d}(d\mu)}$$

for $-1 < r < 1$. More recently A. Brudnyi studied the corresponding result for analytic functions in terms of their Chebyshev degree (see [6]) and F. Nazarov, M. Sodin and A. Volberg give another approach to this kind of result by using a geometric Kannan-Lóvasz-Simonovits localization lemma (see [14]).

The extension of Khinchine inequalities for quadratic forms appears, for instance, in [7] and [10].

The main goal of this note is to exhibit families of random vectors in $\mathbb{R}^n$ verifying similar inequalities to the ones given above for affine and quadratic forms in the range $1 \le p \le \infty$. The uniform distribution on the q-balls B_q^n, $(0 < q < 1)$ are particular examples.

We should note that q-balls, $(0 < q < 1)$, are q-convex sets in $\mathbb{R}^n$ and not convex ones. A. Litvak in a recent paper proved that we cannot obtain an inequality of the type Gromov-Milman (see (1.1)) for linear forms with constant independent of the dimension, when we consider the uniform distribution, μ_K, on a q-convex body K in $\mathbb{R}^n$; so, μ_K (the uniform distribution on K) is not log-concave (see [12] for the definition of q-convex sets and for this result).

The methods we use in the proofs are quite elementary and are based on a recent result by G. Pisier (see [16]), where he gives a new proof of the inequalities for martingales in commutative and non-commutative L^p-space using Moebius inversion formula.

Next we introduce some notation. Let μ be a random vector on $\mathbb{R}^n$. We can see μ as the joint distribution of n real random variables (no necessarily independent), $(X_i)_{i=1}^n$, defined in some probability space Ω, $\mu = \mu_{X_1,\dots,X_n}$. $\mathbb{E}$ denotes either the expectation in $\mathbb{R}^n$ with respect to μ or the expectation in the probability space, depending on the representation we choose.

We say that μ is *unconditional* if

$$\mu_{X_1,\dots,X_n} = \mu_{\varepsilon_1 X_1,\dots,\varepsilon_n X_n}$$

for any choice of signs $\varepsilon_i = \pm 1$. We say that μ is *orthogonal* if

$$\mathbb{E} x_{i_1} \dots x_{i_k} = 0,$$

for all choice of indexes $1 \le i_1 < \cdots < i_k \le n$, $1 \le k \le n$, We say that μ is *strongly orthogonal* if

$$\mathbb{E} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k} = 0,$$

for all choice of indexes $1 \le i_1 < \cdots < i_k \le n$, $1 \le k \le n$ and $\alpha_j \in \mathbb{N} \cup \{0\}$, whenever $\min\{\alpha_1, \dots, \alpha_k\} = 1$. It is easy to see that an unconditional probability having finite all the moments for all the variables is strongly orthogonal and that the three concepts are different.

We shall introduce the following notation for Borel probabilities μ in $\mathbb{R}^n$ whose moments are all finite for all variables:

$$\varphi(p, \mu) = \max_{1 \le i \le n} \frac{\left(\mathbb{E}|x_i|^p\right)^{1/p}}{\left(\mathbb{E}|x_i|^2\right)^{1/2}} \tag{1.3}$$

and

$$\gamma(p,\mu) = \max_{1\le i\neq j\le n} \frac{\left(\mathbb{E}|x_ix_j|^p\right)^{1/p}}{\left(\mathbb{E}|x_ix_j|^2\right)^{1/2}} \tag{1.4}$$

for all $2 \le p$.

If the measure μ is the Lebesgue measure normalized in a compact $K \subset \mathbb{R}^n$ with $|K| > 0$, we will denote $\varphi(p,K) = \varphi(p,\mu)$ and $\gamma(p,K) = \gamma(p,\mu)$.

As usual we denote by

$$\|x\|_q = \left(\sum_{1=1}^n |x_i|^q\right)^{1/q}$$

for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $0 < q \le \infty$. B_q^n will denote the corresponding unit ball for $\|\cdot\|_q$. $|\cdot|$ will denote so Lebesgue measure as the absolute value depending on the context. It is clear that the Lebesgue measure normalized on B_q^n, i.e.

$$\mu(A) = \frac{|A\cap B_q^n|}{|B_q^n|}$$

is unconditional and so strongly orthogonal. The letter C or C_q will denote an absolute constant or a constant depending only on q which can vary from line to line.

2 Inequalities for linear and quadratic forms

Let μ be a random vector in $\mathbb{R}^n$. In the sequel we shall assume that all the moments of μ with respect to all the variables are finite. Next result gives an inequality of Khinchine type for affine forms in terms of the parameter $\varphi(p,\mu)$.

Proposition 2.1 *Let μ be as before. Let a be a vector in $\mathbb{R}^n$, $m \in \mathbb{R}$ and let $p \ge 2$ any even integer, then, for some absolute constant $C > 0$,we have*

$$\left(\mathbb{E}|m + \langle a,x\rangle|^p\right)^{1/p} \le Cp\varphi(p,\mu)\left(\mathbb{E}|m+\langle a,x\rangle|^2\right)^{1/2}, \tag{2.1}$$

whenever μ is orthogonal and

$$\left(\mathbb{E}|m + \langle a,x\rangle|^p\right)^{1/p} \le C\sqrt{p}\varphi(p,\mu)\left(\mathbb{E}|m+\langle a,x\rangle|^2\right)^{1/2} \tag{2.2}$$

whenever μ is unconditional.

P r o o f. First of all we suppose that μ is orthogonal. We can use the following Pisier's result quoted below (see [16], Theorem 2.1),

Let $(d_i)_{i\in I}$ be a finite sequence in $L^p(\Omega, d\mu)$ a measure space. Let p be an even integer. If we assume that

$$\int_\Omega d_{i_1}\dots d_{i_p} d\mu = 0 \tag{2.3}$$

whenever $i_j \neq i_k$, $(1 \le j,k \le p)$ then

$$\left\|\sum_{i\in I} d_i\right\|_p \le 2p \left\|\left(\sum_{i\in I}|d_i|^2\right)^{1/2}\right\|_p.$$

Let $\{d_i\}_{i=0}^{\infty}$ be the sequence of random variables given by

$$d_i = \begin{cases} m, & \text{if } i = 0, \\ a_i x_i, & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n \end{cases}$$

Since μ is orthogonal, by using Minkowski inequality, we have

$$\begin{aligned} \mathbb{E}|m + \langle a, x\rangle|^p = \left\| \sum d_i \right\|_p^p &\leq 2^p p^p \mathbb{E} \left(m^2 + \sum_1^n a_i^2 x_i^2 \right)^{p/2} \\ &\leq 2^p p^p \left(m^2 + \sum_1^n a_i^2 \left(\mathbb{E}|x_i|^p\right)^{2/p} \right)^{p/2} \\ &\leq 2^p p^p \varphi(p,\mu)^p \left(m^2 + \sum_1^n a_i^2 \mathbb{E}|x_i|^2 \right)^{p/2} \\ &= 2^p p^p \varphi(p,\mu)^p \left(\mathbb{E}|m + \langle a, x\rangle|^2 \right)^{p/2}. \end{aligned}$$

Next we consider that μ is unconditional. Let now $\{\varepsilon_i\}_{i=1}^n$ a sequence of independent, independent of μ and identically distributed random variables taking values ± 1 with probability $1/2$. It is clear that

$$\mathbb{E}|\langle a, x\rangle|^p = \mathbb{E} \left| \sum_1^n a_i \varepsilon_i x_i \right|^p$$

for all choice of signs ε_i, $1 \leq i \leq n$. Hence averaging and by Khinchine and Minkowski inequalities we have

$$\begin{aligned} \mathbb{E}|\langle a, x\rangle|^p = \mathbb{E}_x \mathbb{E}_\varepsilon \left| \sum_1^n \varepsilon_i a_i x_i \right|^p &\leq C^p p^{p/2} \mathbb{E} \left(\sum_1^n a_i^2 x_i^2 \right)^{p/2} \\ &\leq C^p p^{p/2} \left(\sum_1^n a_i^2 \left(\mathbb{E}|x_i|^p\right)^{2/p} \right)^{p/2} \\ &\leq C^p p^{p/2} \varphi(p,\mu)^p \left(\sum_1^n a_i^2 \mathbb{E}|x_i|^2 \right)^{p/2} \\ &= C^p p^{p/2} \varphi(p,\mu)^p \left(\mathbb{E}|\langle a, x\rangle|^2 \right)^{p/2}. \end{aligned}$$

Hence

$$\begin{aligned} \left(\mathbb{E}|m + \langle x, a\rangle|^p\right)^{1/p} &\leq m + \left(\mathbb{E}|\langle a, x\rangle|^p\right)^{1/p} \\ &\leq C p^{1/2} \varphi(p,\mu) \left(\left(\mathbb{E}|\langle a, x\rangle|^2\right)^{1/2} + m \right) \\ &\leq C p^{1/2} \varphi(p,\mu) \left(\mathbb{E}|\langle a, x\rangle|^2 + m^2 \right)^{1/2} \\ &= C p^{1/2} \varphi(p,\mu) \left(\mathbb{E}|m + \langle a, x\rangle|^2 \right)^{1/2}. \end{aligned}$$

□

If we don't assume any cancellation at all, we also obtain an estimate similar to the one in part i), but only for $m = 0$ and for the values of $p > n$, for which (2.3) is obvious.

Next result offers an inequality for quadratic forms. The parameters $\varphi(p,\mu)$ and $\gamma(p,\mu)$ appear explicitly.

Proposition 2.2 *Let μ be as before. Let $C = (c_{ij})$ a real $n \times n$ symmetric matrix such that $c_{ij} = c_{ji}$ and $c_{ii} = 0$. Consider the quadratic form on $\mathbb{R}^n$ defined by $Q(x) = \sum_{i,j=1}^n c_{ij} x_i x_j$. Let $p \geq 2$ an even integer, then*

(i) Suppose μ is strongly orthogonal then we have

$$(\mathbb{E}\,|Q|^p)^{1/p} \leq Cp^2 \left[\sum_{1\leq i<j\leq n} c_{ij}^2 \left(\gamma^2(p,\mu)\mathbb{E}|x_i x_j|^2 + \varphi(p,\mu)^4 \mathbb{E}|x_i|^2 \mathbb{E}|x_j|^2 \right) \right]^{1/2},$$

(ii) If μ is unconditional we have

$$(\mathbb{E}|Q|^p)^{1/p} \leq Cp\gamma(p,\mu) \left(\mathbb{E}|Q|^2\right)^{1/2}$$

where $C > 0$ is an absolute constant.

Proof. *(i)* We are going to use the ideas appearing in [10] and [16].

$$\mathbb{E}\,|Q|^p = 2^p \mathbb{E} \left| \sum_{1\leq i<j\leq n} c_{ij} x_i x_j \right|^p.$$

By using the properties of μ and Jensen inequalities we get that

$$\begin{aligned} \mathbb{E}\,|Q|^p &= 2^p \mathbb{E}_x \left| \mathbb{E}_y \left(\sum_{1\leq i<j\leq n} c_{ij}(x_i x_j - y_i y_j) \right) \right|^p \\ &= 2^p \mathbb{E}_x \left| \mathbb{E}_y \left(\sum_{1\leq i<j\leq n} c_{ij}(x_i - y_i)(x_j + y_j) \right) \right|^p \\ &\leq 2^p \mathbb{E}_{x,y} \left| \sum_{1\leq i<j\leq n} c_{ij}(x_i - y_i)(x_j + y_j) \right|^p = 2^p \mathbb{E} \left| \sum_{i=i}^n d_i \right|^p \end{aligned}$$

where $\{d_i\}_{i=1}^\infty$, defined on the probability space $(\mathbb{R}^n \times \mathbb{R}^n, \mu \otimes \mu)$, is given by

$$d_i = (x_i - y_i) \sum_{j=i+1}^n c_{ij}(x_j + y_j),$$

if $1 \leq i \leq n-1$, and $d_i = 0$ if $i \geq n$.

The sequence $\{d_i\}_{i=1}^\infty$ is p-orthogonal in the sense of G. Pisier for any $p \geq 2$, with respect to the probability space $(\mathbb{R}^n \times \mathbb{R}^n, \mu \otimes \mu)$. Indeed, the condition imposed to μ implies that

$$\mathbb{E}_{x,y} d_{i_1} \ldots d_{i_p} = 0$$

whenever $i_1 < \cdots < i_p$, since the integrand is a polynomial in the variables x_k, y_k $(1 \leq k \leq n)$ and the corresponding exponents for x_{i_1}, y_{i_1} are equal to 1. So, by using Pisier and Minkowski inequalities we get

$$\begin{aligned}\mathbb{E}|Q|^p &\leq C^p p^p \mathbb{E}_{x,y} \left| \sum_{i=1}^n (x_i - y_i)^2 \left(\sum_{j=i+1}^n c_{ij}(x_j + y_j) \right)^2 \right|^{p/2} \\ &\leq C^p p^p \mathbb{E}_{x,y} \left(\sum_{i=1}^n (x_i^2 + y_i^2) \left(\sum_{j=i+1}^n c_{ij}(x_j + y_j) \right)^2 \right)^{p/2} \\ &\leq C^p p^p \left[\sum_{i=1}^n \left(\mathbb{E}_{x,y} \left| x_i \sum_{j=i+1}^n c_{i,j}(x_j + y_j) \right|^p \right)^{2/p} \right]^{p/2} .\end{aligned}$$

In order to compute

$$\mathbb{E}_{x,y} \left| x_i \sum_{j=i+1}^n c_{i,j}(x_j + y_j) \right|^p$$

we consider again $\{\widetilde{d}_j\}_{j=1}^\infty$ by

$$\widetilde{d}_j = \begin{cases} x_i c_{ij}(x_j + y_j), & \text{if } i+1 \leq j \leq n, \\ 0 & \text{otherwise} \end{cases}$$

we also have p-orthogonality and then

$$\begin{aligned}\mathbb{E}_{x,y} \left| x_i \sum_{j=i+1}^n c_{i,j}(x_j + y_j) \right|^p &\leq C^p p^p \mathbb{E}_{x,y} \left(\sum_{j=i+1}^n x_i^2 c_{ij}^2 (x_j + y_j)^2 \right)^{p/2} \\ &\leq C^p p^p \left[\sum_{j=i+1}^n \left(\mathbb{E}_{x,y} \left(|c_{ij} x_i (x_j + y_j)| \right)^p \right)^{2/p} \right]^{p/2} .\end{aligned}$$

Since

$$\begin{aligned}\left(\mathbb{E}_{x,y} \left(|x_i(x_j + y_j)| \right)^p \right)^{1/p} &\leq \left(\mathbb{E} |x_i x_j|^p \right)^{1/p} + \left(\mathbb{E}_{x,y} |x_i y_j|^p \right)^{1/p} \\ &\leq \gamma(p,\mu) \left(\mathbb{E} |x_i x_j|^2 \right)^{1/2} \\ &\qquad + \varphi(p,\mu)^2 \left(\mathbb{E} |x_i|^2 \right)^{1/2} \left(\mathbb{E} |x_j|^2 \right)^{1/2} ,\end{aligned}$$

we eventually we arrive at

$$\mathbb{E} |Q|^p \leq C^p p^{2p} \left[\sum_{1 \leq i < j \leq n} c_{ij}^2 \left(\gamma(p,\mu)^2 \mathbb{E}|x_i x_j|^2 + \varphi(p,\mu)^4 \mathbb{E}|x_i|^2 \mathbb{E}|x_j|^2 \right) \right]^{p/2} .$$

(ii) We follow the ideas of proposition 2.1. Let now $\{\varepsilon_i\}_{i=1}^n$ be a sequence of independent and identically distributed random variables taking values ± 1 with probability $1/2$. Then

$$\mathbb{E}|Q|^p = \mathbb{E} \left| \sum_{i,j=1}^n c_{ij} \varepsilon_i \varepsilon_j x_i x_j \right|^p$$

for all choice of signs ε_i, $1 \le i \le n$. Hence averaging and by Khinchine inequalities for quadratic forms (see [10], [7]) and Minkowski inequalities we have

$$\begin{aligned}
\mathbb{E}|Q|^p = \mathbb{E}_{x,\varepsilon} \left| \sum_{i,j=1}^n c_{ij}\varepsilon_i\varepsilon_j x_i x_j \right|^p &\le C^p p^p \mathbb{E} \left| \sum_{i,j=1}^n c_{ij}^2 x_i^2 x_j^2 \right|^{p/2} \\
&\le C^p p^p \left(\sum_{i,j=1}^n c_{ij}^2 \left(\mathbb{E}|x_i x_j|^p\right)^{2/p} \right)^{p/2} \\
&\le C^p p^p \gamma(p,\mu)^p \left(\sum_{i,j=1}^n c_{ij}^2 \mathbb{E}|x_i x_j|^2 \right)^{p/2} = C^p p^p \gamma(p,\mu)^p \left(\mathbb{E}|Q|^2\right)^{p/2}.
\end{aligned}$$

□

3 Inequalities for B_q^n, $(0 < q < 1)$

In the previous section the inequalities we obtained depend on the asymptotic behaviour of the constant $\varphi(p,\mu)$ and $\gamma(p,\mu)$. Now we consider the special case of probabilities μ defined by the Lebesgue measure normalized on a compact K, i.e.

$$\mu(A) = \frac{|A \cap K|}{|K|},$$

for A any Borel set in $\mathbb{R}^n$ and K a compact with $|A \cap K| > 0$. There are two families for which we can give the right estimate of the parameters $\varphi(p,K)$ and $\gamma(p,K)$.

Proposition 3.1 *Let $0 < q < 1$ and $2 \le p < \infty$. Then for every $1 \le i \le n$,*

$$\left(\frac{1}{|B_q^n|} \int_{B_q^n} |x_i|^p dx \right)^{1/p} \sim_q \left(\frac{p}{n+p} \right)^{1/q}.$$

and

$$\left(\frac{1}{|B_q^n|} \int_{B_q^n} |x_i x_j|^p dx \right)^{1/p} \sim_q \left(\frac{p}{n+p} \right)^{2/q} \left(\frac{n}{n+p} \right)^{1/pq},$$

if $i \ne j$, $(1 \le i, j \le n)$. Therefore

$$\varphi(p, B_q^n) \le C_q p^{1/q},$$

and

$$\gamma(p, B_q^n) \le C_q p^{2/q}.$$

Proof. It is easy to compute

$$\frac{1}{|B_q^n|} \int_{B_q^n} |x_i|^p dx = \frac{2|B_q^{n-1}|}{|B_q^n|} \int_0^1 x^p (1-x^q)^{(n-1)/q} dx.$$

and it is also well known (see, for instance [15]) that

$$|B_q^n| = \frac{\left(2\Gamma(1+\frac{1}{q})\right)^n}{\Gamma\left(1+\frac{n}{q}\right)}.$$

We use Stirling formula

$$\Gamma(1+z) = \sqrt{2\pi} z^{z+1/2} e^{-z} e^{\mu(z)}$$

for all $z > 0$, where $\mu(z)$ is non increasing function and non negative, for $z \geq 1$ (cf. [17]).

We therefore obtain

$$\frac{1}{|B_q^n|} \int_{B_q^n} |x_i|^p dx = \frac{\Gamma\left(\frac{p+1}{q}\right)\Gamma\left(1+\frac{n}{q}\right)}{q\Gamma\left(1+\frac{1}{q}\right)\Gamma\left(1+\frac{n+p}{q}\right)}$$

$$= \frac{e^{p/q}\Gamma\left(\frac{p+1}{q}\right)}{q\Gamma\left(1+\frac{1}{q}\right)} \left(\frac{n}{n+p}\right)^{\frac{n}{q}+\frac{1}{2}} \left(\frac{q}{n+p}\right)^{\frac{p}{q}} \exp\left(\mu(\frac{n}{q}) - \mu(\frac{n+p}{q})\right).$$

Since

$$e^{p/q}\left(\frac{n}{n+p}\right)^{\frac{n}{q}} \searrow 1$$

when $n \to \infty$ and besides $1 \leq \mu(\frac{n}{q})$, $\mu(\frac{n+p}{q}) \leq e$, we get

$$\left(\frac{1}{|B_q^n|} \int_{B_q^n} |x_i|^p dx\right)^{1/p} \sim \left(\frac{p}{n+p}\right)^{1/q}.$$

Thus

$$\varphi(p, B_q^n) \leq C_q p^{1/q}.$$

Let now $i \neq j$, $(1 \leq i, j \leq n)$,

$$\frac{1}{|B_q^n|} \int_{B_q^n} |x_i x_j|^p \, dx = \frac{2}{|B_q^n|} \int_0^1 x_1^p \left(\int_{(1-x_1^q)^{1/q} B_q^{n-1}} |x_2|^p dx_2 \dots dx_n\right) dx_1$$

$$= \frac{2\left|B_q^{n-1}\right|}{|B_q^n|} \left(\int_0^1 x_1^p (1-x_1^q)^{\frac{n-1+p}{q}} dx_1\right) \left(\frac{1}{|B_q^{n-1}|} \int_{B_q^{n-1}} |x_2|^p \, dx_2 \cdots dx_n\right)$$

$$\sim_q \left(\frac{p}{n-1+p}\right)^{p/q} n^{1/q} \frac{\Gamma\left(\frac{p+1}{q}\right)\Gamma\left(1+\frac{n-1+p}{q}\right)}{\Gamma\left(1+\frac{n+2p}{q}\right)}.$$

By using again Stirling formula we have

$$\left(\frac{1}{|B_q^n|} \int_{B_q^n} |x_i x_j|^p \, dx\right)^{1/p} \sim_q \left(\frac{p}{n-1+p}\right)^{1/q} n^{\frac{1}{pq}} \left(\frac{p+1}{n+2p}\right)^{\frac{1}{q}+\frac{1}{pq}}$$

$$\sim_q \left(\frac{p}{n+p}\right)^{2/q} \left(\frac{n}{n+p}\right)^{1/pq}.$$

□

Remark 3.2 It is easy to see that for fixed $n \in \mathbb{N}$, then

$$\varphi(p, B_q^n) \leq C_q n^{1/q}$$

for all $p \geq 2$. In consequence we get a better estimate that the corresponding to Gromov-Milman for general convex bodies. The same remark can be done for quadratic forms, since we would have

$$\gamma(p, B_q^n) \leq C_q n^{2/q}$$

for all $p \geq 2$.

Using now these estimate we can give the corresponding inequalities for affine forms and quadratic forms for B_q^n, $(0 < q < 1)$

Corollary 3.3 *Let $f, Q : \mathbb{R}^n \longrightarrow \mathbb{R}$ respectively an affine or a quadratic form and $K = B_q^n$, $(0 < q < 1)$. There exists a constant $C_q > 0$ such that*

$$\left(\mathbb{E}|f|^p\right)^{1/p} \leq C_q p^{\frac{1}{q}+\frac{1}{2}} \left(\mathbb{E}|f|^2\right)^{1/2},$$

or respectively

$$\left(\mathbb{E}|Q|^p\right)^{1/p} \leq C_q p^{1+\frac{4}{q}} \left(\mathbb{E}|Q|^2\right)^{1/2}$$

for $2 \leq p < +\infty$. Moreover

$$\mathbb{E}\exp\left(\left|\frac{f}{C_q' \left(\mathbb{E}|f|^2\right)^{1/2}}\right|^{\frac{2q}{2+q}}\right) \leq 2,$$

or respectively

$$\mathbb{E}\exp\left(\left|\frac{Q}{C_q' \left(\mathbb{E}|Q|^2\right)^{1/2}}\right|^{\frac{q}{4+q}}\right) \leq 2,$$

where the expectation is with respect to the normalized Lebesgue measure on K denoted μ_K.

Remark 3.4 Notice that proposition 3.1 and the last corollary also holds for every $0 < q \leq \infty$. In particular, we get that for every $0 < p \leq \infty$ and every affine form $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\|f\|_{L_{2q/(2+q)}(d\mu_K)} \leq C_q \|f\|_{L_1(d\mu_K)}, \tag{3.1}$$

i.e. the unit ball of ℓ_q^n $(0 < q \leq \infty)$ is a $\psi_{2q/(2+q)}$-body. Recently, F. Barthe and A. Koldobsky (see [1]) have proved that if $2 < q \leq \infty$, the unit ball of ℓ_q^n is a ψ_2-body.

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