

INTERPOLATION OF OPERATORS WHEN THE EXTREME SPACES ARE L^∞

by

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ABSTRACT. In this paper, equivalence between interpolation properties of linear operators and monotonicity conditions are studied, for a pair (Y, Z) of rearrangement invariant quasi Banach spaces, when the extreme spaces of the interpolation are L^∞ and a pair (A_0, B_0) under some assumptions. Weak and restricted weak intermediate spaces fall in our context. Applications to classical Lorentz and Lorentz-Orlicz spaces are given.

§ 0. Introduction.

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be two compatible couples of rearrangement invariant Banach function spaces (in short r.i.) over the interval I ($I = [0, 1]$ or $[0, \infty)$) (see definitions below). We denote by $\mathcal{A}(\bar{A}, \bar{B})$ the class of linear (or quasilinear or Lipschitz) operators which are bounded from A_0 into B_0 and from A_1 into B_1 . If Y, Z are intermediate r.i. spaces with respect to \bar{A}, \bar{B} respectively, we say that the pair (Y, Z) has the *linear (or quasilinear or Lipschitz) interpolation property with respect to the class \mathcal{A}* if every member of \mathcal{A} can be extended to a bounded operator from Y into Z .

This interpolation property has been extensively studied in its connection with many aspects concerning r.i. spaces, for instance, Boyd or Zippin's indexes, monotonicity conditions, boundedness of some suitable "maximal" operators and so on. Here we are concerned with the case $A_1 = B_1 = L^\infty$ and particularly in connection with the monotonicity property (\mathcal{M}) given in § 1 and the boundedness of only one operator.

In this direction the former result is contained in Calderon's paper [5] where it is showed that both properties, say linear interpolation and monotonicity, are equivalent in the case of $A_0 = B_0 = L^1$. Later on, Lorentz and Shimogaki [11] extended this result to the case $A_0 = B_0 = L^p$ with $p > 1$. The technique used by them consists on a linearization process of the L^p case.

Sharpley, Maligranda and other authors (see [12] and references quoted there) studied the case $A_0 = \Lambda(X)$, $B_0 = M(X)$ (see definitions in § 2) and $A_1 = B_1 = L^\infty$ or $A_1 = \Lambda(Y)$, $B_1 = M(Y)$ relating the interpolation properties with the boundedness of only one "maximal" operator ([19, theorem 4.7], [12, theorem 4.5]). On the other hand, Maligranda [12] obtained equivalence between the

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interpolation property for Lipschitz operators and monotonicity condition in the case $A_0 = \Lambda(X)$, $B_0 = M(X)$ and $A_1 = B_1 = L^\infty$. When $X = L^p$, $p > 1$, then $\Lambda(X) = L^{p,1}$ and $M(X) = L^{p,\infty}$. So we can see Maligranda's result is close to Lorentz-Shimogaki's one. The spaces with the interpolation property, when the extreme spaces are $\Lambda(X)$ and $M(X)$, are generally known in the literature as *weak type intermediate spaces*.

These papers leave out the more "natural" case where $A_0 = L^p$, $B_0 = L^{p,\infty}$ or, more generally, $A_0 = X$, $B_0 = M(X)$. In fact, following the usual terminology in Fourier Analysis, it should be reserved the term *weak type intermediate spaces* to spaces having the interpolation property in this last setting, while the spaces with the interpolation property in the setting before stated should be named *restricted weak type intermediate spaces*.

Our final purpose is to study this "intermediate" case between Lorentz-Shimogaki's and Maligranda's one. In order to do that, our main tool consists of obtaining, in a very general context, equivalence between interpolation properties of linear, quasilinear or Lipschitz type and monotonicity condition (\mathcal{M}) , which ensure that we are working out with Calderon couples. When this result is established it is an easy consequence to reduce the linear interpolation property to the boundedness of only one quasilinear operator.

This general result can be applied in the both cases stated before, namely, weak and restricted weak intermediate. So, on one hand we obtain some generalizations of Maligranda's results and on the other one we obtain several results in the case of $A_0 = X$, $B_0 = M(X)$. When $A_0 = L^p$, the quasilinear operator can be iterated and, as a consequence, we obtain that the weak type intermediate spaces are exactly the restricted weak intermediate spaces.

Moreover, by using a characterization about the boundedness of the Hardy operator in Lorentz spaces due to Ariño and Muckenhoupt, we can characterize the Lorentz spaces which are intermediate in terms of handy conditions on the weights. Finally, the last part of the paper is devoted to extend some of the previous results to the more general case of Lorentz-Orlicz spaces.

The paper is organized in two sections: the first one contains the notations and the general results and the second one the applications.

§ 1. General results

A Banach space $(X, \|\cdot\|)$ of real-valued, locally integrable, Lebesgue measurable functions on I ($I = [0, 1]$ or $[0, \infty)$) is said to be a rearrangement invariant Banach function space over I (in short r.i. space) if it satisfies the following conditions:

- i) If $|g| \leq |f|$ a.e. and $f \in X$, then $g \in X$ and $\|g\| \leq \|f\|$.
- ii) $0 \leq f_n \uparrow$, $\sup_{n \in \mathbb{N}} \|f_n\| \leq M$, imply that $f = \sup f_n \in X$ and $\|f\| = \sup_{n \in \mathbb{N}} \|f_n\|$.
- iii) X contains the simple integrable functions.
- iv) $f \in X \iff f^* \in X$ and $\|f\| = \|f^*\|$, where f^* denotes the nonincreasing rearrangement of the function f .

Fact ii) is known in the literature as Fatou property (cf. [10]). It is quite clear that if X is r.i. then $L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty$ (where the symbol \hookrightarrow signifies continuously embedded).

A classical result by Lorentz and Luxemburg ensures that for these spaces

- v) $\|f\| = \sup_{\|g\|_{X'} \leq 1} \left| \int_I fg \right|$, where X' is the associated space of X which is also r.i. space. In particular $X = X''$ isometrically.

The fundamental function ϕ_X of an r.i. space is defined by

$$\phi_X(t) = \|\chi_{[0,t]}\|, t \in I.$$

There is no loss of generality if we assume for ϕ_X to be positive, nondecreasing, absolutely continuous far from the origin, concave and to verify (see [19], [22]):

- vi) $\phi_X(t)\phi_{X'}(t) = t$, for all $t \in I$;
- vii) $\frac{d\phi_X(t)}{dt} \leq \frac{\phi_X(t)}{t}$, a.e. on I .

In which follows it may be convenient to let X be a quasi-Banach r.i.. The main difference occurs in the triangle inequality satisfied in X , i.e. $\|f + g\| \leq C(\|f\| + \|g\|)$, for some constant $C \geq 1$. In this case we suppose that a quasi-Banach space X satisfies properties i), ii), iii), iv) but, in general, no other conditions will be assumed. We say that a quasi-Banach function space is σ -order continuous if every order bounded nondecreasing sequence converges in the quasi-norm topology (cf. [10, Proposition 1.a.8]).

Let $(A_0, L_\infty), (B_0, L_\infty)$, be two compatible couples of r.i. quasi-Banach spaces on I . Let Y and Z be intermediate r.i. spaces on I with respect to (A_0, L_∞) and (B_0, L_∞) respectively. We say that the couple (Y, Z) belongs to:

- $\mathcal{LI}(A_0, B_0; L^\infty)$ if any linear operator $T : A_0 + L^\infty \rightarrow B_0 + L^\infty$ which is bounded from A_0 into B_0 and from L^∞ into L^∞ is also bounded from Y into Z . The closed graph theorem implies that there exists a constant $C \geq 1$ such that

$$\|T\|_{Y \rightarrow Z} \leq C \max\{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{L^\infty \rightarrow L^\infty}\}.$$

- $\mathcal{QLI}(A_0, B_0; L^\infty)$ if any quasilinear operator $T : A_0 + L^\infty \rightarrow B_0 + L^\infty$ which is bounded from A_0 into B_0 and from L^∞ into L^∞ is also bounded from Y into Z .

- $\mathcal{LPT}(A_0, B_0; L^\infty)$ if any operator $T : A_0 + L^\infty \rightarrow B_0 + L^\infty$ that is a Lipschitz operator from A_0 into B_0 and from L^∞ into L^∞ also maps Y into Z .

Recall that a map $T : A_0 \rightarrow B_0$ is bounded quasilinear if there are constants $K, C \geq 1$ such that $|T(\lambda f)| = |\lambda| |T(f)|$, $|T(f + g)| \leq K(|T(f)| + |T(g)|)$ and $\|T(f)\| \leq C\|f\|$. We define $\|T\|_{A_0 \rightarrow B_0} = \inf C$. In the same way, a map $T : A_0 \rightarrow B_0$ is a Lipschitz operator if there is a constant $C \geq 1$ such that $T(0) = 0$, $\|Tf - Tg\| \leq C\|f - g\|$; we define $\|T\|_{A_0 \rightarrow B_0} = \inf C$.

Next we shall introduce another class of spaces and for that we need the following

Lemma 1. *If X is a quasi-Banach r.i. space and $f \in X + L^\infty$ and $m(E) < \infty$ then $f\chi_E \in X$.*

Proof.- Let $f = g + h$ with $g \in X$ and $h \in L^\infty$. Since $g\chi_E \in X$ and $|h\chi_E| \leq \|h\|_\infty \chi_E \in X$ the result follows immediately. Q.E.D.

We say that the couple (Y, Z) belongs to:

- $\mathcal{M}(A_0, B_0)$ if there exists a constant $C \geq 1$ such that if $f \in Y$, $g \in B_0 + L^\infty$ and

$$\|f^* \chi_{[0,t]}\|_{A_0} \geq \|g^* \chi_{[0,t]}\|_{B_0} \quad \forall t > 0 \quad (\mathcal{M})$$

then $g \in Z$ and $\|g\|_Z \leq C\|f\|_Y$.

It is clear that

$$\mathcal{QLI}(A_0, B_0; L^\infty) \cup \mathcal{LPT}(A_0, B_0; L^\infty) \subseteq \mathcal{LI}(A_0, B_0; L^\infty).$$

Under some more restrictive assumptions the four classes of maps introduced before coincide.

Proposition 1. *Let (A_0, B_0) a couple of quasi-Banach r.i. spaces such that $\phi_{B_0}(t) \leq C\phi_{A_0}(t)$ for all $t > 0$, then*

$$(1.1) \quad \mathcal{M}(A_0, B_0) \subseteq \mathcal{QLI}(A_0, B_0; L^\infty)$$

(1.2) *If $I = [0, 1]$ and A_0 is σ -order continuous then*

$$\mathcal{M}(A_0, B_0) \subseteq \mathcal{LPT}(A_0, B_0; L^\infty).$$

Proof.- (1.1) Let (Y, Z) be a couple in $\mathcal{M}(A_0, B_0)$ and let T be a quasilinear map $T : A_0 + L^\infty \rightarrow B_0 + L^\infty$, bounded from A_0 into B_0 and from L^∞ into L^∞ . Suppose that $\|T\|_{A_0 \rightarrow B_0} \leq 1$ and $\|T\|_{L^\infty \rightarrow L^\infty} \leq 1$. We have to show that T is bounded from Y into Z . In order to do it let f be an element in Y . We only need to prove that

$$\|(Tf)^* \chi_{[0,t]}\|_{B_0} \leq \|(Cf)^* \chi_{[0,t]}\|_{A_0} \quad (*)$$

for all $t > 0$.

We know that

$$\|(Tf)^* \chi_{[0,t]}\|_{B_0} = \sup \|(Tf)\chi_E\|_{B_0}$$

where E runs over the borelians in I with $m(E) \leq t$. Set $s = f^*(t)$ and define

$$f_{(s)} = \begin{cases} s, & \text{if } f > s ; \\ -s, & \text{if } f < -s ; \\ f, & \text{otherwise} \end{cases}$$

and $f^{(s)} = f - f_{(s)}$. Since $f_{(s)} \in L^\infty$ and $f^{(s)} \in A_0$ we have

$$\begin{aligned} \|(Tf_{(s)})\chi_E\|_{B_0} &\leq \|Tf_{(s)}\|_\infty \|\chi_E\|_{B_0} \\ &\leq \|f_{(s)}\|_\infty \phi_{B_0}(t) \leq Cs\phi_{A_0}(t) \\ &\leq C\|f^*(t)\chi_{[0,t]}\|_{A_0} \leq C\|f^*\chi_{[0,t]}\|_{A_0}. \end{aligned}$$

Now

$$\begin{aligned} \|(Tf^{(s)})\chi_E\|_{B_0} &\leq \|Tf^{(s)}\|_{B_0} \\ &\leq C\|f^{(s)}\|_{A_0} \leq C\|f^*\chi_{[0,t]}\|_{A_0}. \end{aligned}$$

and hence we easily obtain the inequality (*).

(1.2) In order to show now that the couple (Y, Z) belongs to $\mathcal{LPT}(A_0, B_0; L^\infty)$ we will follow the ideas of [12, lemma 4.4]. For the sake of completeness, we include the proof here. Suppose that T is a Lipschitz operator mapping $A_0 + L^\infty$ into $B_0 + L^\infty$ with $\|T\|_{A_0 \rightarrow B_0} \leq 1$ and $\|T\|_{L^\infty \rightarrow L^\infty} \leq 1$.

If $0 < t \leq 1$ and $f \in Y$, we set $f^*(t) = s$. Consider $f_{(s)}$ and $[(Tf)^*]_{(s)}$ defined in a similar way as before. Since $f_{(s)} \in L^\infty$ we have $\|T(f_{(s)})\|_\infty \leq s$ and so,

$$|Tf - (Tf)_{(s)}| \leq |Tf - T(f_{(s)})|.$$

If $0 < x \leq t$,

$$|(Tf)^*(x)| \leq |(Tf)^*(x) - [(Tf)^*]_{(s)}(x)| + [(Tf)^*]_{(s)}(x)$$

thus

$$\begin{aligned} \|(Tf)^*\chi_{[0,t]}\|_{B_0} &\leq C \left(\left\| |(Tf)^* - [(Tf)^*]_{(s)}| \chi_{[0,t]} \right\|_{B_0} + \|f^*(t)\chi_{[0,t]}\|_{B_0} \right) \\ &\leq C \left(\left\| [Tf - (Tf)_{(s)}] \chi_{[0,t]} \right\|_{B_0} + \|f^*\chi_{[0,t]}\|_{A_0} \right) \\ &\leq C \left(\left\| Tf - T(f_{(s)}) \right\|_{B_0} + \|f^*\chi_{[0,t]}\|_{A_0} \right) \\ &\leq C \left(\|Tf - T(f_{(s)})\|_{B_0} + \|f^*\chi_{[0,t]}\|_{A_0} \right) \\ &\leq C \left(\|(f - f_{(s)})\|_{A_0} + \|f^*\chi_{[0,t]}\|_{A_0} \right) \\ &\leq C\|f^*\chi_{[0,t]}\|_{A_0} \end{aligned}$$

This implies that T maps Y into Z . Next by using the fact that A_0 is σ -order continuous we can follow the proof of the theorem 4.5 from [11, theorem 4.5], and conclude the proof of this part. (The constants C appearing above may change from line to line). Q.E.D.

In order to obtain another implications we need to restrict our attention to Banach spaces.

Proposition 2. *Let A_0 be a Banach r.i. space and let B_0 be a quasi-Banach r.i. space. Suppose that $\phi_{A_0} \leq C\phi_{B_0}$ and $\frac{1}{\phi_{B_0}} \in B_0$. Then*

$$\mathcal{LI}(A_0, B_0; L^\infty) \subseteq \mathcal{M}(A_0, B_0).$$

Proof.- Let $f \in Y$, $g \in B_0 + L^\infty$ such that

$$\|f^* \chi_{[0,t]}\|_{A_0} \geq \|g^* \chi_{[0,t]}\|_{B_0}, \quad \forall t > 0$$

For every $k \in \mathbf{Z}$ we define $E_k = [2^k, 2^{k+1})$. We have

$$\begin{aligned} g^*(t) &= \sum_{-\infty}^{\infty} g^*(t) \chi_{E_k}(t) \\ &\leq \sum_{-\infty}^{\infty} g^*(2^k) \chi_{E_k}(t) \\ &\leq \sum_{-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \|f^* \chi_{[0,2^k]}\|_{A_0} \chi_{E_k}(t) \end{aligned}$$

For every $k \in \mathbf{Z}$ we can choose a function $h_k \in A'_0$ with $\|h_k\|_{A'_0} \leq 1$, such that

$$\|f^* \chi_{[0,2^k]}\|_{A_0} \leq 2 \int_I f^* \chi_{[0,2^k]} h_k.$$

Then

$$g^*(t) \leq 2 \sum_{-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \left(\int_0^{2^k} f^* h_k \right) \chi_{E_k}(t).$$

For any locally integrable function φ on I we define the “linear” operator T by

$$T\varphi(t) = 2 \sum_{-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \left(\int_0^{2^k} \varphi h_k \right) \chi_{E_k}(t).$$

It is clear that if $\varphi \in L^\infty$ and $t \in E_k$ we have

$$\begin{aligned} |T\varphi(t)| &\leq \frac{2}{\phi_{B_0}(2^k)} \|\varphi\|_\infty \|\chi_{[0,2^k]}\|_{A_0} \\ &\leq C \|\varphi\|_\infty. \end{aligned}$$

On the other hand, if $\varphi \in A_0$ then

$$|T\varphi(t)| \leq 2 \|\varphi\|_{A_0} \sum_{-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \chi_{E_k}(t).$$

For $t \in E_k$ the triangle inequality of the quasi-norm implies that

$$\phi_{B_0}(t) \leq \phi_{B_0}(2^{k+1}) \leq C\phi_{B_0}(2^k)$$

hence

$$\begin{aligned} |T\varphi(t)| &\leq C\|\varphi\|_{A_0} \sum_{-\infty}^{\infty} \frac{\chi_{E_k}(t)}{\phi_{B_0}(t)} \\ &= C\|\varphi\|_{A_0} \frac{1}{\phi_{B_0}(t)} \end{aligned}$$

Since $\frac{1}{\phi_{B_0}} \in B_0$ we obtain that T is also bounded from A_0 into B_0 . Eventually we have that $g \in B_0$ and $\|g\|_{B_0} \leq C\|f\|_{A_0}$ because $g^* \leq Tf^* \in B_0$. Q.E.D.

Among the results above we want to emphasize the following

Theorem 1. *Let A_0 be a Banach r.i. space and let B_0 be a quasi-Banach r.i. space. Suppose that $C^{-1}\phi_{B_0} \leq \phi_{A_0} \leq C\phi_{B_0}$ for some constant C and $\frac{1}{\phi_{B_0}} \in B_0$.*

Then

- (i) $\mathcal{LI}(A_0, B_0; L^\infty) = \mathcal{M}(A_0, B_0)$.
- (ii) *A couple of quasi Banach r.i. spaces (Y, Z) belongs to any of these classes if and only if the quasilinear operator*

$$Q\varphi(t) = \frac{1}{\phi_{B_0}(t)} \|\varphi\chi_{[0,t]}\|_{A_0}$$

is bounded from Y into Z for nonincreasing functions.

- (iii) *The couples (A_0, L^∞) and (B_0, L^∞) are relative Calderón couples.*

Proof.- The first part follows from propositions 1 and 2. For the second part, observe that Q is bounded from A_0 into B_0 and from L^∞ into L^∞ and so if $(Y, Z) \in \mathcal{LI}(A_0, B_0; L^\infty) (= \mathcal{QLI}(A_0, B_0; L^\infty))$, then Q is bounded from Y into Z . On the other hand, note that condition (\mathcal{M}) implies that $g^*(t) \leq Qf^*(t)$ and, therefore, the boundedness of Q for nonincreasing functions implies that $(Y, Z) \in \mathcal{M}(A_0, B_0; L^\infty)$

For the third part we recall (see [7]) that the couples (A_0, L^∞) , (B_0, L^∞) are termed relative Calderón couples if a couple $(Y, Z) \in \mathcal{LI}(A_0, B_0; L^\infty)$ if and only if for some constant $C > 0$ and for each $f \in Y$, whenever the inequality $K(t, g; B_0, L^\infty) \leq K(t, f; A_0, L^\infty)$ holds for all $t > 0$ and some $g \in B_0 + L^\infty$ then $g \in Z$ and $\|g\|_Z \leq C\|f\|_Y$ (K is the K -functional introduced by Peetre). It is easy to prove that the quasilinear operator Q satisfies

$$C^1 Q\varphi(t) \leq \frac{1}{\phi_{B_0}(t)} K(\phi_{B_0}(t), \varphi; A_0, L^\infty) \leq 2CQ\varphi(t)$$

for all $t > 0$ and for all nonnegative nonincreasing function $\varphi \in A_0 + L^\infty$. Therefore this part holds trivially. Q.E.D.

Remarks. i) Under the same hypotheses as in proposition 2, but supposing only that A_0 is quasi-Banach, we can prove in a simpler way that $\mathcal{QLI}(A_0, B_0; L^\infty) \subseteq \mathcal{M}(A_0, B_0)$. The operator we have to use instead of T is Q .

ii) Proposition 2 is not true when A_0 is quasi-Banach. For instance, let $I = [0, 1]$, $A_0 = B_0 = L^{p, \infty}$, $0 < p \leq 1$. In this case it is easy to see that the couple $(L^p, L^p) \notin \mathcal{M}(L^{p, \infty}, L^{p, \infty}; L^\infty) = \mathcal{QLI}(L^{p, \infty}, L^{p, \infty}; L^\infty)$, but by using a result by Kalton [8, **Theorem 1.1**], we can deduce that $(L^p, L^p) \in \mathcal{LI}(L^{p, \infty}, L^{p, \infty}; L^\infty)$, (this result was quoted to the authors by Oscar Blasco).

§ 2. Applications.

Not all Banach r.i. spaces B_0 satisfy the condition $\frac{1}{\phi_{B_0}} \in B_0$. In order to study this property we introduce the Lorentz spaces as they appear in [2], [12], [19], [22].

In what follows we assume that X is a Banach r.i. space. We denote by:

- $\Lambda(X)$ the space of all measurable functions with

$$\|f\|_{\Lambda(X)} = \int_I f^*(t) d\phi_X(t) < \infty.$$

Since ϕ_X is concave, the expression $\|f\|_{\Lambda(X)}$ is a norm and moreover $\Lambda(X)$ is a Banach r.i. space.

- $M(X)$ the space of all measurable functions f for which there exists f^{**} and

$$\|f\|_{M(X)} = \sup_{t \in I} \phi_X(t) f^{**}(t) < \infty.$$

Recall that f^{**} , the *Hardy transform* of f^* , is defined by

$$H(f^*)(t) = f^{**}(t) = \frac{1}{t} \int_0^t f^*.$$

$M(X)$ is again a Banach r.i. space.

- $M^*(X)$ the space of all measurable functions for which

$$\|f\|_{M^*(X)} = \sup_{t \in I} \phi_X(t) f^*(t) < \infty.$$

The function $\|\cdot\|_{M^*(X)}$ is a quasinorm on $M^*(X)$.

It is clear that for these spaces we have:

- i)** $X \subseteq M^*(X)$, $\Lambda(X) \hookrightarrow X \hookrightarrow M(X)$, $M(X) \subseteq M^*(X)$
- ii)** $\phi_{\Lambda(X)} = \phi_{M^*(X)} = \phi_X = \phi_{M(X)}$,
- iii)** $\frac{1}{\phi_X} \in M^*(X)$,
- iv)** $\frac{1}{\phi_X} \in X \iff X = M^*(X)$.

Lemma 2. Let X be a Banach r. i. space. The following conditions are equivalent:

- (a) The space $M^*(X)$ is convexifiable (i.e. there is a norm on $M^*(X)$ equivalent to $\|\cdot\|_{M^*(X)}$)
- (b) $M(X) = M^*(X)$
- (c) $\frac{1}{\phi_X} \in M(X)$
- (d) There exist a constant $C > 0$ such that

$$\frac{\phi_X(t)}{t} \int_0^t \frac{ds}{\phi_X(s)} \leq C, \quad \forall t \in I$$

- (e) $\|f\|_{M(X)} \sim \|f\|_{M^*(X)}, \quad \forall f \in M^*(X)$.

Proof.- We only will sketch (a) \Rightarrow (b). We may assume there is a r.i. norm $\|\cdot\|$ on $M^*(X)$ equivalent to $\|\cdot\|_{M^*(X)}$. If $f \in M^*(X)$ and $t > 0$

$$f^{**}(t)\phi_X(t) \leq C\|f^{**}(t)\chi_{[0,t]}\| \leq C\|f^*\chi_{[0,t]}\|$$

and therefore $\|f\|_{M(X)} \leq C\|f\|_{M^*(X)}$. Q.E.D.

Now we can state the results of preceding section in the framework of Lorentz spaces.

Proposition 3. If X is a r.i. Banach space then $\mathcal{LI}(\Lambda(X), M^*(X); L^\infty) = \mathcal{QLI}(\Lambda(X), M^*(X); L^\infty) = \mathcal{M}(\Lambda(X), M^*(X)) = \mathcal{LPI}(\Lambda(X), M^*(X); L^\infty)$ (the last fact if $I = [0, 1]$). Furthermore, a couple of spaces (Y, Z) belongs to any of the classes before stated if and only if the quasilinear operator $Q_{\Lambda(X)}$ defined by

$$Q_{\Lambda(X)}\varphi(t) = \frac{1}{\phi_X(t)} \int_0^t \varphi(x)d\phi_X(x) \quad (2.1)$$

is bounded from Y into Z for nonincreasing functions.

Notes. (i) This result has been already obtained by Maligranda [12] in the case $Y = Z$.

(ii) The operator appearing in Proposition 3 is actually

$$Q_{\Lambda(X)}\varphi = H(\varphi \circ \phi_X^{-1}) \circ \phi_X.$$

The preceding proposition can be applied to the class of classical Lorentz spaces $\Lambda(w, q)$ with non monotone weights. Let w be an a.e. positive weight defined on $I = [0, \infty)$ such that $\int_0^t w < \infty, \forall t < \infty$ and $\int_0^\infty w = \infty$. We recall that the classical Lorentz space $\Lambda(w, q)$, $0 < q \leq \infty$ is the class of all real valued measurable functions on I such that

$$\|f\|_{\Lambda(w, q)} = \begin{cases} \left(\int_I f^*(t)^q w(t) dt \right)^{1/q} < \infty, & \text{if } 0 < q < \infty ; \\ \sup_{t>0} f^*(t)w(t) < \infty, & \text{if } q = \infty. \end{cases}$$

For $q = \infty$ we will only consider nondecreasing weights w . Ariño and Muckenhoupt [1] have showed that given $0 < q < \infty$, there exists a constant $C > 0$ such that the Hardy operator verifies

$$\|Hf\|_{\Lambda(w,q)} \leq C\|f\|_{\Lambda(w,q)}$$

for all nonnegative and nonincreasing functions f on \mathbb{R} if and only if the weight w satisfies

$$\int_t^\infty \frac{w(x)}{x^q} dx \leq \frac{B}{t^q} \int_0^t w(x) dx \quad (AM_q)$$

for some constant $B > 0$ and for all $t > 0$. Moreover, for $1 \leq q < \infty$, condition (AM_q) implies that $\Lambda(w, q)$ is a Banach space. Sawyer [18] proved that the converse is true for $1 < q < \infty$. Raynaud gave also another equivalent condition to this last fact by using quasi-concavity conditions for the function $W(t) = \int_0^t w(x) dx$ (see [17]).

In the case $q = \infty$ and w nondecreasing, the same reasons appearing in Lemma 2 show that $\Lambda(w, \infty)$ is a Banach space if and only if the weight w satisfies

$$\frac{w(t)}{t} \int_0^t \frac{dx}{w(x)} \leq C \quad (A_1)$$

for some constant $C > 0$ and for all $t > 0$.

If we suppose that the weights satisfy the conditions (AM_q) or (A_1) then

$$\|f^{**}\|_{\Lambda(w,q)} \sim \|f\|_{\Lambda(w,q)}$$

and reciprocally.

In next statements, when we say that $\Lambda(w, q)$ is a Banach space we will mean that conditions (AM_q) or (A_1) are satisfied.

By using Ariño-Muckenhoupt's result stated before we are going to be able to characterize the Lorentz spaces which are interpolated between $\Lambda(X)$, $M^*(X)$ and L^∞ .

Proposition 4. *Let X be a r.i. Banach space and suppose that $\Lambda(w, q)$ is a Banach space. Then the following assertions are true:*

(i) *For $1 \leq q < \infty$, $\Lambda(w, q) \in \mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ if and only if*

$$\int_t^\infty \frac{w(x)}{\phi_X(x)^q} dx \leq \frac{B}{\phi_X(t)^q} \int_0^t w(x) dx, \quad \forall t > 0 \quad (2.2)$$

(ii) *$\Lambda(w, \infty) \in \mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ if and only if*

$$\int_0^t \frac{d\phi_X(x)}{w(x)} \leq C \frac{\phi_X(t)}{w(t)}, \quad \forall t > 0. \quad (2.3)$$

Proof.- First of all we remark that condition (2.2) implies that w satisfies (AM_q) .

(i) We only have to prove that the operator $Q_{\Lambda(X)}$ defined by

$$Q_{\Lambda(X)}f = H(f \circ \phi_X^{-1}) \circ \phi_X$$

is bounded on $\Lambda(w, q)$, for nonnegative and nonincreasing functions.

$$\begin{aligned} \|Q_{\Lambda(X)}f\|_{\Lambda(w, q)}^q &= \int_0^\infty H(f \circ \phi_X^{-1})(\phi_X(t))^q w(t) dt \\ &= \int_0^\infty H(f \circ \phi_X^{-1})(y)^q w(\phi_X^{-1}(y)) (\phi_X^{-1})'(y) dy \\ &\leq C \int_0^\infty f(\phi_X^{-1}(y))^q w(\phi_X^{-1}(y)) (\phi_X^{-1})'(y) dy \\ &= C \int_0^\infty f(x)^q w(x) dx = C \|f\|_{\Lambda(w, q)}^q \end{aligned}$$

where, by using condition (AM_q) , the inequality is satisfied if and only if the weight

$$v(y) = w(\phi_X^{-1}(y)) (\phi_X^{-1})'(y)$$

satisfies

$$\int_t^\infty \frac{v(y)}{y^q} dy \leq \frac{B}{t^q} \int_0^t v(y) dy \quad \forall t > 0$$

for some constant $B > 0$ and this inequality is equivalent to (2.2).

(ii) This proof is simpler. Suppose that (2.3) holds. If $f \in \Lambda(w, \infty)$ we have

$$\begin{aligned} \sup_{t>0} \frac{w(t)}{\phi_X(t)} \int_0^t f^*(x) d\phi_X(x) &\leq \|f\|_{\Lambda(w, \infty)} \sup_{t>0} \frac{w(t)}{\phi_X(t)} \int_0^t \frac{d\phi_X(x)}{w(x)} \\ &\leq C \|f\|_{\Lambda(w, \infty)} \end{aligned}$$

For the converse implication, consider for each $t > 0$ the function

$f^*(x) = \frac{1}{w(x)} \chi_{[0, t]}(x) \in \Lambda(w, \infty)$. Then the inequality

$$\frac{w(t)}{\phi_X(t)} \int_0^t f^*(x) d\phi_X(x) \leq C \sup_{t>0} f^*(t) w(t), \quad \forall t > 0$$

implies (2.3). Q.E.D.

These results should be compared with those appearing in [19]. In his paper Sharpley deals with the case of interpolation between $(\Lambda(X_1), M(X_1))$ and $(\Lambda(X_2), M(X_2))$ and characterizes when $\Lambda^\alpha(Y)$ or $M(Y)$ are interpolated spaces. Observe that $\Lambda^\alpha(Y)$ and $M(Y)$ are particular cases of $\Lambda(w, q)$.

Note that the operator $Q_{\Lambda(X)}$ given by (2.1) is actually “linear”. Hence, proposition 3 is more or less immediate in the sense that proposition 2 is not needed in order to prove proposition 3 since the monotonicity condition supplies

the linear operator $Q_{\Lambda(X)}$. That is, the linearization made in proposition 2 is not needed.

However, our general result also allows us to treat the less evident case of $(X, M(X))$. First of all, we will translate theorem 1 to this context:

Proposition 5. *If X is a r.i. Banach space then $\mathcal{LI}(X, M^*(X); L^\infty) = \mathcal{QLI}(X, M^*(X); L^\infty) = \mathcal{M}(X, M^*(X)) = \mathcal{LPI}(X, M^*(X); L^\infty)$ (the last fact if $I = [0, 1]$). Furthermore, a couple of spaces (Y, Z) belongs to any of the classes stated before if and only if the quasilinear operator Q_X defined by*

$$Q_X \varphi(t) = \frac{1}{\phi_X(t)} \|\varphi \chi_{[0,t]}\|_X$$

is bounded from Y into Z for nonnegative and nonincreasing functions.

When $X = L^p$, $1 < p < \infty$, it is not difficult to prove that for a p -convex space Y : $(Y, Y) \in \mathcal{LI}(L^p, L^{p,\infty}; L^\infty) \iff (Y, Y) \in \mathcal{LI}(L^{p,1}, L^{p,\infty}; L^\infty)$, (see [10] for a definition of p -convex spaces).

In the next result we characterize the spaces $\Lambda(w, q)$ which are interpolated in this context.

Proposition 6. *Let X be a r.i. Banach space and suppose that $\Lambda(w, q)$ is a Banach space, $1 \leq q \leq \infty$. Then the following is true:*

- (i) *If $1 \leq q < \infty$, $\Lambda(w, q) \in \mathcal{LI}(X, M^*(X); L^\infty) \iff$ condition (2.2) holds.*
- (ii) *$\Lambda(w, \infty) \in \mathcal{LI}(X, M^*(X); L^\infty)$ if and only if the following condition is satisfied*

$$\frac{w(t)}{\phi_X(t)} \left\| \frac{\chi_{[0,t]}}{w} \right\|_X \leq C \quad \forall t > 0. \quad (2.4)$$

Proof.- (i) We know that $\Lambda(w, q) \in \mathcal{LI}(X, M^*(X); L^\infty) \iff$ the operator

$$Q_X(f)(t) = \frac{1}{\phi_X(t)} \|f \chi_{[0,t]}\|_X$$

is bounded in $\Lambda(w, q)$ for nonincreasing and nonnegative functions. Since $Q_X(f) \leq Q_{\Lambda(X)}(f)$, we obtain that condition (2.2) implies the interpolation property. On the other hand, since

$$Q_X(\chi_{[0,s]})(t) = \chi_{[0,s]}(t) + \frac{\phi_X(s)}{\phi_X(t)} \chi_{[s,\infty)}$$

we have that if Q_X is bounded on $\Lambda(w, q)$, then

$$\int_0^s w(x) dx + \phi_X(s)^q \int_s^\infty \frac{w(x)}{\phi_X(x)^q} dx \leq C \int_0^s w(x) dx$$

and hence (2.2) holds.

(ii) Suppose $\Lambda(w, \infty) \in \mathcal{LI}(X, M^*(X); L^\infty)$. By observing that $\frac{1}{w} \in \Lambda(w, \infty)$ and that

$$Q_X \left(\frac{1}{w} \right) (t) = \frac{1}{\phi_X(t)} \left\| \frac{\chi_{[0,t]}}{w} \right\|_X$$

we get condition (2.4).

For the reverse part we know that $w(x)f^*(x) \leq \|f\|_{\Lambda(w, \infty)}$, $\forall x > 0$ and $\forall f \in \Lambda(w, \infty)$. Furthermore

$$\begin{aligned} Q_X(f^*)(t) &= \frac{1}{\phi_X(t)} \|f^* \chi_{[0,t]}\|_X \\ &\leq \frac{C}{\phi_X(t)} \|f\|_{\Lambda(w, \infty)} \left\| \frac{\chi_{[0,t]}}{w} \right\|_X \\ &\leq \frac{C}{w(t)} \|f\|_{\Lambda(w, \infty)} \end{aligned}$$

and since w is nondecreasing we obtain that the operator Q_X is bounded on $\Lambda(w, \infty)$. Q.E.D.

Remarks. (i) We observe from Propositions 4 and 6 that the spaces $\Lambda(w, q)$, $1 \leq q < \infty$, belong to $\mathcal{LI}(X, M^*(X); L^\infty)$ and $\mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ simultaneously. Furthermore, if $X = L^p$, $1 < p < q$, the weight w satisfies condition $(AM_{\frac{q}{p}})$ and then the space $\Lambda(w, q)$ is p -convex. The converse is also true, i.e., let $q > p \geq 1$, if the space $\Lambda(w, q)$ is p -convex then $\Lambda(w, q) \in \mathcal{LI}(L^p, L^{p, \infty}; L^\infty)$. This result should be compared with those appearing in [17].

(ii) If $\Lambda(X) \neq X = M^*(X)$ (for instance, $X = L^{p, \infty}$, $p > 1$) the space $X = \Lambda(\phi_X, \infty) \in \mathcal{LI}(X, M^*(X); L^\infty)$ but $X \notin \mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ as it is easy to prove.

(iii) $\Lambda(w, \infty) \in \mathcal{LI}(L^{p, 1}, L^{p, \infty}; L^\infty) \iff \Lambda(w, \infty)$ is p -convex.

In the case $X = L^p$, $1 < p < \infty$, the situation is clearer as the following result shows

Proposition 7. Let $1 \leq p < \infty$, a real number and let Y be a r.i. space. The following assertions are equivalent:

- (i) $Y \in \mathcal{LI}(L^{p, 1}, L^{p, \infty}; L^\infty)$
- (ii) $Y \in \mathcal{LI}(L^{p, r}, L^{p, \infty}; L^\infty)$, for some $1 < r < \infty$.
- (iii) The upper Boyd index $\alpha(Y) < \frac{1}{p}$. (See [4] for the definition of Boyd indices).

Proof.- We only have to prove (ii) \implies (i). We restrict ourselves to the case $r = p$ because the proof is similar. By proposition 6, we know that the operator

$$Q_p f(t) = \frac{1}{t^{1/p}} \left(\int_0^t f^p \right)^{1/p}$$

is bounded in Y for nonincreasing nonnegative functions, i.e. $\|Q_p f\|_Y \leq C \|f\|_Y$, for some constant $C > 0$. Let $f = f^* \in Y$. It is quite easy to compute that

$$Q_p^{(n+1)} f(t) = \left(\int_0^1 f^p(tx) \frac{[\log(1/x)]^n}{n!} dx \right)^{1/p}$$

for any natural number $n \in \mathbf{N}$. If $\epsilon C < 1$ we define

$$Sf(t) = \left(\sum_{n=0}^{\infty} [\epsilon^n Q_p^{(n+1)} f(t)]^p \right)^{1/p}.$$

Since $S_N f(t) = \left(\sum_{n=0}^N [\epsilon^n Q_p^{(n+1)} f(t)]^p \right)^{1/p} \in Y$, $S_N f(t) \uparrow Sf(t)$ when $N \rightarrow \infty$, and

$$\begin{aligned} \|S_N f\|_Y &\leq \left\| \sum_{n=0}^N \epsilon^n Q_p^{(n+1)} f \right\|_Y \\ &\leq \left(\sum_{n=0}^{\infty} \epsilon^n C^{n+1} \right) \|f\|_Y \end{aligned}$$

we obtain that $Sf \in Y$ and $\|Sf\|_Y \leq C' \|f\|_Y$. But

$$\begin{aligned} Sf(t) &= \left(\sum_{n=0}^{\infty} \int_0^1 f(tx)^p \frac{[\epsilon \log(1/x)]^n}{n!} dx \right)^{1/p} \\ &= \left(\int_0^1 f(tx)^p \sum_{n=0}^{\infty} \frac{[\epsilon \log(1/x)]^n}{n!} dx \right)^{1/p} \\ &= \left(\int_0^1 f(tx)^p \frac{dx}{x^\epsilon} \right)^{\frac{1}{p}} = \frac{1}{t^{\frac{1-\epsilon}{p}}} \left(\int_0^t f(x)^p d(x^{1-\epsilon}) \right)^{1/p} \\ &= \frac{1}{t^{\frac{1-\epsilon}{p}}} \|f \chi_{[0,t]}\|_{L^{\frac{p}{1-\epsilon}, p}} \end{aligned}$$

Hence $(Y, Y) \in \mathcal{LI}(L^{\frac{p}{1-\epsilon}, p}, L^{\frac{p}{1-\epsilon}, \infty}; L^\infty)$ and so (i) is true. Q.E.D.

In the last part of this paper we consider a similar situation to the previous one in the framework of Lorentz-Orlicz spaces. Different versions of this class of spaces appear in [13], [20] and they have been also studied in [9], [14], [15] and [17]. Here we consider the Lorentz-Orlicz spaces as they appear in [13].

In the sequel φ will denote an Orlicz function, i.e. a convex, non-decreasing function on $[0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. We also suppose that φ satisfies the Δ_2 condition: there exists a constant $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$, for all $t > 0$, or equivalently, there exists $1 \leq q < \infty$, so that,

$$\varphi(at) \leq a^q \varphi(t), \quad \forall a \geq 1, t > 0.$$

The weight w is supposed to satisfy the same conditions appearing in the definition of Lorentz spaces, namely, w is an a.e. positive weight defined on $[0, \infty)$ such that $\int_0^t w < \infty, \forall t < \infty$ and $\int_0^\infty w = \infty$.

The space $\Lambda(w, \varphi)$ is the class of real valued measurable functions on I so that $\int_I \varphi(f^*(t))w(t)dt < \infty$.

Next results are an extension of Ariño-Muckenhoupt's inequality.

Lemma 3. *Let φ be an Orlicz function. Suppose that there exists a constant $B > 0$ such that*

$$\int_t^\infty \varphi\left(\frac{at}{x}\right) w(x) dx \leq B\varphi(a) \int_0^t w(x) dx \quad \forall t > 0, \forall a > 0 \quad (A_\varphi)$$

then, for some $\alpha < 1$ and $D > 0$, we have

$$\int_t^\infty \psi\left(\frac{at}{x}\right) w(x) dx \leq D\psi(a) \int_0^t w(x) dx \quad \forall t > 0, \forall a > 0$$

where $\psi(t) = \varphi(t^\alpha)$.

Proof.- We can adapt the arguments in [1] to our more general situation. Only a few changes are necessary. Using the notation in [1] the number $\alpha < 1$ is chosen in such a way that

$$2S^{1/q} = 2 \left(\frac{2B+1}{2B+2} \right)^{1/q} < 2^\alpha < 2.$$

Q.E.D.

As a consequence of this lemma and with the same reasons appearing in [1], we obtain the following

Proposition 8. *Let φ be an Orlicz function. A weight w satisfies condition (A_φ) if and only if there exists a constant $B' > 0$ such that for every nonnegative nonincreasing function f on $(0, \infty)$ we have*

$$\int_0^\infty \varphi(H(f))w \leq B' \int_0^\infty \varphi(f)w$$

where $H(f)$ is the Hardy transform of f .

Remark. If the weight satisfies (A_φ) , the expression

$$\|f\|_{\Lambda(w, \varphi)} = \inf \left\{ \rho; \int_0^\infty \varphi\left(\frac{f^*}{\rho}\right) w \leq 1 \right\}$$

defines a quasi-norm on $\Lambda(w, \varphi)$ which is equivalent to the norm

$$\| \|f\| \| = \|Hf^*\|_{\Lambda(w, \varphi)}$$

and therefore $\Lambda(w, \varphi)$ is a Banach space.

By introducing the Simonenko indices (see [13]) we can give necessary or sufficient conditions in order to verify condition (A_φ) for a weight. Given an Orlicz convex function φ and a number $T > 0$ we define

$$p_T = \inf_{t \geq T} \frac{t\varphi'(t)}{\varphi(t)}$$

$$q_T = \sup_{t \geq T} \frac{t\varphi'(t)}{\varphi(t)}$$

where $\varphi'(t)$ is supposed to be the right derivative of the Orlicz function φ . We also introduce $p_0 = \inf p_T$ and $q_0 = \sup q_T$. It is clear that $1 \leq p_0 \leq p_T \leq q_T \leq q_0 < \infty$ and

$$\alpha^{q_T} \varphi(t) \leq \varphi(\alpha t) \leq \alpha^{p_T} \varphi(t)$$

whenever $\alpha \leq 1$ and $T \leq \alpha t$.

Proposition 9. *Let φ be an Orlicz function and let w be a weight. The following assertions are true:*

i) *If a weight w satisfies the condition (AM_{p_0}) then it also verifies (A_φ) . Furthermore the Hardy transform is bounded on $\Lambda(w, \varphi)$ and $\Lambda(w, \varphi)$ is a Banach space.*

ii) *If the Hardy transform is bounded on $\Lambda(w, \varphi)$ then w has to satisfy condition (AM_{q_0}) .*

Proof.- i) First of all suppose that w satisfies the condition (AM_{p_0}) , then

$$\int_t^\infty \varphi\left(\frac{at}{x}\right) w(x) dx \leq \varphi(a) \int_t^\infty \left(\frac{t}{x}\right)^{p_0} w(x) dx$$

and hence w verifies condition (A_φ) and so the Hardy transform is bounded on $\Lambda(w, \varphi)$. The other statements are clear.

ii) Now we assume that there exists a constant $C \geq 1$ such that

$$\|Hf\|_{\Lambda(w, \varphi)} \leq C \|f\|_{\Lambda(w, \varphi)}$$

for all non increasing function $f \in \Lambda(w, \varphi)$. In particular, given $t > 0$ if $s = \|\chi_{[0,t]}\|_{\Lambda(w, \varphi)}^{-1}$, we obtain that

$$\|H(\chi_{[0,t]})\|_{\Lambda(w, \varphi)} \leq \frac{C}{s}.$$

Therefore

$$\int_0^t \varphi\left(\frac{s}{C}\right) w(x) dx + \int_t^\infty \varphi\left(\frac{st}{Cx}\right) w(x) dx \leq 1.$$

Since $\frac{\varphi(s)}{C^{q_0}} \leq \varphi\left(\frac{s}{C}\right)$ and

$$\varphi\left(\frac{st}{Cx}\right) \geq \varphi\left(\frac{s}{C}\right) \left(\frac{t}{x}\right)^{q_0} \geq \varphi(s) \left(\frac{t}{Cx}\right)^{q_0}$$

we have that

$$\int_0^t w(x) dx + \int_t^\infty \left(\frac{t}{x}\right)^{q_0} w(x) dx \leq C^{q_0} \int_0^t w(x) dx.$$

and this completes the proof. Q.E.D.

In the case of considering the space $\Lambda(w, \varphi)$ on the unit interval $I = [0, 1]$ (or more generally $I = [0, l]$ for $l < \infty$) we can give a more precise result.

Proposition 10. *Suppose $I = [0, l]$, ($l < \infty$) and let φ an Orlicz function. Let $p = \liminf_{t \rightarrow \infty} \frac{t\varphi'(t)}{\varphi(t)}$ and $q = \limsup_{t \rightarrow \infty} \frac{t\varphi'(t)}{\varphi(t)}$. The following assertions are true:*

i) *If w satisfies condition (AM_p) then the Hardy transform is bounded on $\Lambda(w, \varphi)$ and $\Lambda(w, \varphi)$ is a Banach space.*

ii) *If the Hardy transform is bounded on $\Lambda(w, \varphi)$ then w satisfies condition $(AM_{q+\epsilon})$ for all $\epsilon > 0$.*

Proof.- i) If w satisfies condition (AM_p) by using Lemma 2.1 in [1] w also verifies condition $(AM_{p-\epsilon})$ for some $\epsilon > 0$. Then there exists $T > 0$ such that $p - \epsilon < p_T$. Define now the function $\bar{\varphi}_T$ by

$$\bar{\varphi}_T(t) = \begin{cases} \varphi(t), & \text{if } t \geq T ; \\ \varphi(T) \left(\frac{t}{T}\right)^{p_T}, & \text{otherwise.} \end{cases}$$

Note that the function $\bar{\varphi}_T$ is an Orlicz function for which

$$p_T = \inf_{t > 0} \frac{t\bar{\varphi}'(t)}{\bar{\varphi}(t)}.$$

Hence by using preceding proposition we obtain that Hardy transform is bounded on $\Lambda(w, \bar{\varphi}_T)$. Thus the Hardy transform is also bounded on $\Lambda(w, \varphi)$, as $m(I) < \infty$ and the Orlicz functions are equivalent at infinity.

ii) Let ϵ be a positive number. There exists $T > 0$ such that

$$\frac{t\varphi'(t)}{\varphi(t)} \leq q_T < q + \epsilon \quad \forall t \geq T$$

If we define the Orlicz function $\bar{\varphi}_T$ as before it is clear that $\sup_{t > 0} \frac{t\bar{\varphi}'(t)}{\bar{\varphi}(t)} \leq q_T$. Since $\bar{\varphi}_T$ is equivalent to φ at infinity and the Hardy transform is bounded on $\Lambda(w, \bar{\varphi}_T)$, by using again Proposition 9, we deduce that w satisfies the condition (AM_{q_T}) and so $(AM_{q+\epsilon})$.

This completes the proof. Q.E.D.

Now we can state the corresponding interpolation results whose proves follow the same lines as in Propositions 4 and 6.

Proposition 11. *Suppose that φ is an Orlicz function and w satisfies the condition (A_φ) . Let X be a r.i. function space. Then the following assertions are equivalent:*

i) *Any linear or cuasilinear operator T which is bounded from $\Lambda(X)$ into $M^*(X)$ and from L^∞ into L^∞ verifies*

$$\int_0^\infty \varphi((Tf)^*(x))w(x)dx \leq C' \int_0^\infty \varphi(f^*(x))w(x)dx$$

for some constant $C' > 0$ and for any function $f \in \Lambda(w, \varphi)$.

ii) The same as in i) but with operators mapping X into $M^*(X)$, instead of $\Lambda(X)$ into $M^*(X)$.

iii) There exist a constant $D > 0$ such that

$$\int_t^\infty \varphi \left(\frac{\phi_X(t)}{\phi_X(x)} \right) w(x) dx \leq D \int_0^t w(x) dx$$

Moreover, if one of these conditions is satisfied for $X = L^p$, $1 \leq p < \infty$, the space $\Lambda(w, \varphi)$ is p -convex.

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