

# THE VARIANCE CONJECTURE ON SOME POLYTOPES

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ABSTRACT. We show that any random vector uniformly distributed on any hyperplane projection of  $B_1^n$  or  $B_\infty^n$  verifies the variance conjecture

$$\text{Var } |X|^2 \leq C \sup_{\xi \in S^{n-1}} \mathbb{E} \langle X, \xi \rangle^2 \mathbb{E} |X|^2.$$

Furthermore, a random vector uniformly distributed on a hyperplane projection of  $B_\infty^n$  verifies a negative square correlation property and consequently any of its linear images verifies the variance conjecture.

## 1. INTRODUCTION AND NOTATION

Let  $X$  be a random vector in  $\mathbb{R}^n$  with a log-concave density, *i.e.*  $X$  is distributed on  $\mathbb{R}^n$  according to a probability measure,  $\mu_X$ , whose density with respect to the Lebesgue measure is  $\exp(-V)$  for some convex function  $V : \mathbb{R}^n \rightarrow (-\infty, \infty]$ . For instance, vectors uniformly distributed on convex bodies and Gaussian random vectors are log-concave.

A random vector  $X$  is said to be isotropic if:

- i) The barycenter is at the origin, *i.e.*,  $\mathbb{E}X = 0$ , and
- ii) The covariance matrix  $M_X$  is the identity  $I_n$ , *i.e.*  $\mathbb{E} \langle X, e_i \rangle \langle X, e_j \rangle = \delta_{i,j}$ ,  $1 \leq i, j \leq n$ ,

where  $\{e_i\}_{i=1}^n$  denotes the canonical basis in  $\mathbb{R}^n$  and  $\delta_{i,j}$  denotes the Kronecker delta. It is well known that every centered random vector with full dimensional support has a unique, up to orthogonal transformations, linear image which is isotropic.

Given a log-concave random vector  $X$ , we will denote by  $\lambda_X^2$  the highest eigenvalue of the covariance matrix  $M_X$  and by  $\sigma_X$  its “thin shell width” *i.e.*

$$\begin{aligned} \lambda_X^2 &= \|M_X\|_{\ell_2^n \rightarrow \ell_2^n} = \sup_{\xi \in S^{n-1}} \mathbb{E} \langle X, \xi \rangle^2, \\ \sigma_X &= \sqrt{\mathbb{E} \left| |X| - (\mathbb{E}|X|^2)^{\frac{1}{2}} \right|^2}. \end{aligned}$$

( $S^{n-1}$  represents the Euclidean unit sphere in  $\mathbb{R}^n$ ).

In Asymptotic Geometric Analysis, the *variance* conjecture states the following:

**Conjecture 1.1.** *There exists an absolute constant  $C$  such that for every isotropic log-concave vector  $X$ , if we denote by  $|X|$  its Euclidean norm,*

$$\text{Var } |X|^2 \leq C \mathbb{E} |X|^2 = Cn.$$

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*Date:* September 18, 2012.

The first named author is partially supported by Spanish grants MTM2010-16679 and “Programa de Ayudas a Grupos de Excelencia de la Región de Murcia”, Fundación Séneca, 04540/GERM/06.

The second author is partially supported by Spanish grant MTM2010-16679.

This conjecture was considered by Bobkov and Koldobsky in the context of the Central Limit Problem for isotropic convex bodies (see [BK]). It was conjectured before by Antilla, Ball and Perissinaki, (see [ABP]) that for an isotropic log-concave vector  $X$ ,  $|X|$  is highly concentrated in a “thin shell” more than the trivial bound  $\text{Var}|X| \leq \mathbb{E}|X|^2$  suggested. Actually, it is known that the variance conjecture is equivalent to the *thin shell width* conjecture:

**Conjecture 1.2.** *There exists an absolute constant  $C$  such that for every isotropic log-concave vector  $X$*

$$\sigma_X = \sqrt{\mathbb{E}||X| - \sqrt{n}|^2} \leq C.$$

It is also known (see [BN], [EK]) that these two equivalent conjectures are stronger than the hyperplane conjecture, which states that every convex body of volume 1 has a hyperplane section of volume greater than some absolute constant.

The variance conjecture is a particular case of a stronger conjecture, due to Kannan, Lovász and Simonovits (see [KLS]) concerning the spectral gap of log-concave probabilities. This conjecture can be stated in the following way due to the work of Cheeger, Maz’ya and Ledoux:

**Conjecture 1.3.** *There exists an absolute constant  $C$  such that for any locally Lipschitz function,  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , and any centered log-concave random vector  $X$  in  $\mathbb{R}^n$*

$$\text{Var } g(X) \leq C\lambda_X^2 \mathbb{E}|\nabla g(X)|^2.$$

Note that Conjecture 1.1 is the particular case of Conjecture 1.3, when we consider only isotropic vectors and  $g(X) = |X|^2$ . Our purpose in this paper is to study the particular case of Conjecture 1.3 in which  $g(X) = |X|^2$  but  $X$  is not necessarily isotropic. Thus, we will study the following *general variance* conjecture:

**Conjecture 1.4.** *There exists an absolute constant  $C$  such that for every centered log-concave vector  $X$*

$$\text{Var } |X|^2 \leq C\lambda_X^2 \mathbb{E}|X|^2.$$

In the same way that Conjecture 1.1 is equivalent to Conjecture 1.2, Conjecture 1.4 can be shown (see Section 2) to be equivalent to the following *general thin shell width* conjecture:

**Conjecture 1.5.** *There exists an absolute constant  $C$  such that for every centered log-concave vector  $X$*

$$\sigma_X \leq C\lambda_X$$

Notice that since Conjecture 1.4 and Conjecture 1.5 are not invariant under linear maps, these two conjectures cannot easily be reduced to Conjecture 1.1 and Conjecture 1.2. We will study how these conjectures behave under linear transformations and we will also prove that random vectors uniformly distributed on a certain family of polytopes verify Conjecture 1.4 but, before stating our results, let us recall the results known, up to now, concerning the aforementioned conjectures.

Besides the Gaussian vectors only a few examples are known to satisfy Conjecture 1.3. For instance, the vectors uniformly distributed on  $\ell_p^n$ -balls,  $1 \leq p \leq \infty$ , the simplex and some revolution convex bodies ([S], [LW], [BW], [Hu]). In [K4], Klartag proved Conjecture 1.3 with an extra  $\log n$  factor for vectors uniformly distributed on unconditional convex bodies and recently Barthe and Cordero extended this

result for log-concave vectors with many symmetries (see [BC]). Kannan, Lovász and Simonovits proved Conjecture 1.3 with the factor  $(\mathbb{E}|X|)^2$  instead of  $\lambda_X^2$  (see [KLS]), improved by Bobkov to  $(\text{Var}|X|^2)^{1/2} \simeq \sigma_X \mathbb{E}|X|$  (see [Bo]).

In [K3], Klartag proved Conjecture 1.1 for random vectors uniformly distributed on isotropic unconditional convex bodies. The best known (dimension dependent) bound for general log-concave isotropic random vectors in Conjecture 1.2 was proved by Guédon and Milman with a factor  $n^{1/3}$  instead of  $C$ , improving down to  $n^{1/4}$  when  $X$  is  $\psi_2$  (see, [GM]). This results give better estimates than previous ones by Klartag (see [K2]) and Fleury (see [F]). Given the relation between Conjecture 1.1 and Conjecture 1.2 we have that Conjecture 1.1 is known to be true with an extra factor  $n^{2/3}$ .

Very recently Eldan, ([E]) obtained a breakthrough showing that Conjecture 1.2 implies Conjecture 1.3 with an extra logarithmic factor. By using the result of Guédon-Milman, Conjecture 1.3 is obtained with an extra factor  $n^{2/3}(\log n)^2$ .

Since the variance conjecture is not linearly invariant, in Section 2 we will study its behavior under linear transformations *i.e.*, given a centered log-concave random vector  $X$ , we will study the variance conjecture of the random vector  $TX$ ,  $T \in GL(n)$ . We will prove that if  $X$  is an isotropic random vector verifying Conjecture 1.1, then the non-isotropic  $T \circ U(X)$  verifies the variance conjecture (1.4) for a typical  $U \in O(n)$ . We will also show the equivalence between Conjecture 1.4 and Conjecture 1.5. As a consequence of Guédon and Milman's result we obtain that every centered log-concave random vector verifies the variance conjecture with constant  $Cn^{\frac{2}{3}}$  rather than the  $Cn^{\frac{2}{3}}(\log n)^2$  obtained from the best general known result in Conjecture 1.3.

The main results in this paper will be included in Section 3, where we will show that random vectors uniformly distributed on hyperplane projections of  $B_1^n$  or  $B_\infty^n$  (the unit balls of  $\ell_1^n$  and  $\ell_\infty^n$  respectively) verify Conjecture 1.4. Furthermore, in the case of the hyperplane projections of  $B_\infty^n$  we will see that they verify a negative square correlation property with respect to any orthonormal basis, which will allow us to deduce that also a random vector uniformly distributed on any linear image of any hyperplane projection of  $B_\infty^n$  will verify Conjecture 1.4.

In order to compute some quantities on the hyperplane projections of  $B_1^n$  and  $B_\infty^n$  we will use Cauchy's formula which, in the case of polytopes can be stated like this:

Let  $K_0$  be a polytope with facets  $\{F_i : i \in I\}$  and  $K = P_H K_0$  be the projection of  $K_0$  onto a hyperplane. If  $X$  is a random vector uniformly distributed on  $K$ , for any integrable function  $f : K \rightarrow \mathbb{R}$  we have

$$\mathbb{E}f(X) = \sum_{i \in I} \frac{\text{Vol}(P_H(F_i))}{\text{Vol}(K)} \mathbb{E}f(P_H Y^i),$$

where  $Y^i$  is a random vector uniformly distributed on the facet  $F_i$  and  $\text{Vol}$  denotes the volume or Lebesgue measure.

Let us now introduce some notation. Given a convex body  $K$ , we will denote by  $\tilde{K}$  its homothetic image of volume 1 ( $\text{Vol}(\tilde{K}) = 1$ ).  $\tilde{K} = \frac{K}{\text{Vol}^{\frac{1}{n}}(K)}$ . We recall that a convex body  $K \subset \mathbb{R}^n$  is *isotropic* if it has volume  $\text{Vol}(K) = 1$ , the barycenter of  $K$  is at the origin and its inertia matrix is a multiple of the identity. Equivalently, there exists a constant  $L_K > 0$  called isotropy constant of  $K$  such that

$L_K^2 = \int_K \langle x, \theta \rangle^2 dx, \forall \theta \in S^{n-1}$ . In this case if  $X$  denotes a random vector uniformly distributed on  $K$ ,  $\lambda_X = L_K$ . Thus,  $K$  is isotropic if the random vector  $X$ , distributed on  $L_K^{-1}K$  with density  $L_K^n \chi_{L_K^{-1}K}$  is isotropic.

When we write  $a \sim b$ , for  $a, b > 0$ , it means that the quotient of  $a$  and  $b$  is bounded from above and from below by absolute constants.  $O(n)$  will always denote the orthogonal group on  $\mathbb{R}^n$ .

## 2. GENERAL RESULTS

In this section we are going to consider the variance conjecture for linear transformations of a fixed centered log-concave random vector in  $\mathbb{R}^n$ . Our first result shows that if such random vector is not far from being isotropic and verifies the variance conjecture, then the average perturbation (in the sense we will state in the proposition) will also verify the variance conjecture.

**Proposition 2.1.** *Let  $X$  be a centered isotropic, log-concave random vector in  $\mathbb{R}^n$  verifying the variance conjecture with constant  $C_1$ . Let  $T \in GL(n)$  be any linear transformation. If  $U$  is a random map uniformly distributed in  $O(n)$  then*

$$\mathbb{E}_U \text{Var} |T \circ U(X)|^2 \leq CC_1 \|T\|_{op}^2 \|T\|_{HS}^2 = CC_1 \lambda_{T \circ u(X)}^2 \mathbb{E} |T \circ u(X)|^2$$

for any  $u \in O(n)$ , where  $C$  is an absolute constant.

*Proof.* The non singular linear map  $T$  can be expressed by  $T = V\Lambda U_1$  where  $V, U_1 \in O(n)$  and  $\Lambda = [\lambda_1, \dots, \lambda_n]$  ( $\lambda_i > 0$ ) a diagonal map.

Given  $\{e_i\}_{i=1}^n$  the canonical basis in  $\mathbb{R}^n$ , we will identify every  $U \in O(n)$  with the orthonormal basis  $\{\eta_i\}_{i=1}^n$  such that  $U_1 U \eta_i = e_i$  for all  $i$ . Thus, by uniqueness of the Haar measure invariant under the action of  $O(n)$  we have that, for any integrable function  $F$

$$\begin{aligned} \mathbb{E}_U F(U) &= \mathbb{E}_U F(\eta_1, \dots, \eta_n) \\ &= \int_{S^{n-1}} \int_{S^{n-1} \cap \eta_1^\perp} \dots \int_{S^{n-1} \cap \eta_1^\perp \cap \dots \cap \eta_{n-1}^\perp} F(\eta_1, \dots, \eta_n) d\nu(\eta_n) \dots d\nu(\eta_1), \end{aligned}$$

where  $d\nu(\eta_i)$  is the Haar probability measure on  $S^{n-1} \cap \eta_1^\perp \cap \dots \cap \eta_{i-1}^\perp$ . Then, since

$$|T \circ U(X)|^2 = |\Lambda U_1 U X|^2 = \sum_{i=1}^n \langle \Lambda U_1 U X, e_i \rangle^2 = \sum_{i=1}^n \lambda_i^2 \langle X, \eta_i \rangle^2$$

$$\begin{aligned} \mathbb{E}_U \text{Var} |T \circ U(X)|^2 &= \sum_{i=1}^n \lambda_i^4 \mathbb{E}_U (\mathbb{E} \langle X, \eta_i \rangle^4 - (\mathbb{E} \langle X, \eta_i \rangle^2)^2) \\ &\quad + \sum_{i \neq j} \lambda_i^2 \lambda_j^2 \mathbb{E}_U (\mathbb{E} \langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E} \langle X, \eta_i \rangle^2 \mathbb{E} \langle X, \eta_j \rangle^2). \end{aligned}$$

Since for every  $i$

$$\begin{aligned} \mathbb{E}_U (\mathbb{E} \langle X, \eta_i \rangle^4 - (\mathbb{E} \langle X, \eta_i \rangle^2)^2) &\leq \mathbb{E}_U \mathbb{E} \langle X, \eta_i \rangle^4 = \mathbb{E} |X|^4 \int_{S^{n-1}} \langle e_1, \eta_1 \rangle^4 d\nu(\eta_1) \\ &= \frac{3}{n(n+2)} \mathbb{E} |X|^4, \end{aligned}$$

and for every  $i \neq j$ , denoting by  $Y$  an independent copy of  $X$ ,

$$\mathbb{E}_U (\mathbb{E} \langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E} \langle X, \eta_i \rangle^2 \mathbb{E} \langle X, \eta_j \rangle^2)$$

$$\begin{aligned}
&= \mathbb{E}|X|^4 \int_{S^{n-1}} \left\langle \frac{X}{|X|}, \eta_1 \right\rangle^2 \int_{S^{n-1} \cap \eta_1^\perp} \left\langle \frac{X}{|X|}, \eta_2 \right\rangle^2 d\nu(\eta_2) d\nu(\eta_1) \\
&\quad - \mathbb{E}|X|^2 |Y|^2 \int_{S^{n-1}} \left\langle \frac{X}{|X|}, \eta_1 \right\rangle^2 \int_{S^{n-1} \cap \eta_1^\perp} \left\langle \frac{Y}{|Y|}, \eta_2 \right\rangle^2 d\nu(\eta_2) d\nu(\eta_1) \\
&= \frac{\mathbb{E}|X|^4}{n-1} \left( \frac{1}{n} - \int_{S^{n-1}} \langle e_1, \eta_1 \rangle^4 d\nu(\eta_1) \right) \\
&\quad - \frac{\mathbb{E}|X|^2 |Y|^2}{n-1} \left( \frac{1}{n} - \int_{S^{n-1}} \left\langle \frac{X}{|X|}, \eta_1 \right\rangle^2 \left\langle \frac{Y}{|Y|}, \eta_1 \right\rangle^2 d\nu(\eta_1) \right) \\
&= \frac{\mathbb{E}|X|^4}{n-1} \left( \frac{1}{n} - \frac{3}{n(n+2)} \right) \\
&\quad - \frac{\mathbb{E}|X|^2 |Y|^2}{n-1} \left( \frac{1}{n} - \frac{1}{n(n+2)} - \frac{2}{n(n+2)} \left\langle \frac{X}{|X|}, \frac{Y}{|Y|} \right\rangle^2 \right)
\end{aligned}$$

we have that

$$\begin{aligned}
\mathbb{E}_U \text{Var} |T \circ U(X)|^2 &\leq \frac{3}{n(n+2)} \mathbb{E}|X|^4 \sum_{i=1}^n \lambda_i^4 \\
&\quad + \left( \frac{\mathbb{E}|X|^4 - (\mathbb{E}|X|^2)^2}{n(n+2)} - \frac{2\mathbb{E}|X|^2 |Y|^2}{(n-1)n(n+2)} \left( 1 - \left\langle \frac{X}{|X|}, \frac{Y}{|Y|} \right\rangle^2 \right) \right) \sum_{i \neq j} \lambda_i^2 \lambda_j^2 \\
&\leq \frac{3}{n(n+2)} \mathbb{E}|X|^4 \sum_{i=1}^n \lambda_i^4 + \frac{\text{Var} |X|^2}{n(n+2)} \sum_{i \neq j} \lambda_i^2 \lambda_j^2.
\end{aligned}$$

Now, since for every  $\theta \in S^{n-1}$ ,  $\mathbb{E}\langle X, \theta \rangle^2 = 1$  and  $X$  satisfies the variance conjecture with constant  $C_1$ , we have

$$\mathbb{E}|X|^4 = \text{Var}|X|^2 + (\mathbb{E}|X|^2)^2 \leq C_1 n + n^2 \leq CC_1 n^2.$$

and

$$\mathbb{E}_U \text{Var} |T \circ U(X)|^2 \leq CC_1 \sum_{i=1}^n \lambda_i^4 + \frac{C_1}{n} \sum_{i \neq j} \lambda_i^2 \lambda_j^2$$

Hence, given any  $u \in O(n)$ , let  $\{\nu_i\}_{i=1}^n$  be the orthonormal basis defined by  $\nu_i = U_1 \circ u(e_i)$ , for all  $i$ . Then we have

$$\begin{aligned}
\lambda_{T \circ u(X)}^2 &= \sup_{\theta \in S^{n-1}} \mathbb{E}\langle T \circ u(X), \theta \rangle^2 = \sup_{\theta \in S^{n-1}} \mathbb{E}\langle \Lambda U_1 u X, \theta \rangle^2 \\
&= \sup_{\theta \in S^{n-1}} \mathbb{E} \left( \sum_{i=1}^n \lambda_i \langle X, \nu_i \rangle \langle e_i, \theta \rangle \right)^2 \\
&= \sup_{\theta \in S^{n-1}} \sum_{i=1}^n \lambda_i^2 \langle e_i, \theta \rangle^2 = \max_{1 \leq i \leq n} \lambda_i^2 = \|T\|_{op}^2
\end{aligned}$$

and

$$\mathbb{E}|T \circ u(X)|^2 = \sum_{i=1}^n \lambda_i^2 \mathbb{E}\langle X, \nu_i \rangle^2 = \sum_{i=1}^n \lambda_i^2 = \|T\|_{HS}^2$$

Thus

$$\begin{aligned}
\mathbb{E}_U \text{Var} |T \circ U(X)|^2 &\leq CC_1 \|T\|_{op}^2 \|T\|_{HS}^2 + \frac{C_1}{n} \|T\|_{op}^2 \sum_{i \neq j} \lambda_j^2 \\
&\leq CC_1 \|T\|_{op}^2 \|T\|_{HS}^2 + C_1 \|T\|_{op}^2 \|T\|_{HS}^2 \\
&= CC_1 \|T\|_{op}^2 \|T\|_{HS}^2 = CC_1 \lambda_{T \circ u(X)}^2 \mathbb{E} |T \circ u(X)|^2
\end{aligned}$$

□

**Remark.** *The same proof as before can be applied when  $X$  is not necessarily isotropic. In this case*

$$\mathbb{E}_U \text{Var} |T \circ U(X)|^2 \leq CC_1 \lambda_{T \circ u(X)}^2 \mathbb{E} |T \circ u(X)|^2$$

for any  $u \in O(n)$ , where  $B$  is the spectral condition number of its covariance matrix i.e.

$$B^2 = \frac{\max_{\theta \in S^{n-1}} \mathbb{E} \langle X, \theta \rangle^2}{\min_{\theta \in S^{n-1}} \mathbb{E} \langle X, \theta \rangle^2}.$$

As a consequence of Markov's inequality we obtain the following

**Corollary 2.2.** *Let  $X$  be an isotropic, log-concave random vector in  $\mathbb{R}^n$  verifying the variance conjecture with constant  $C_1$ . There exists an absolute constant  $C$  such that the measure of the set of orthogonal operators  $U$  for which the random vector  $T \circ U(X)$  verifies the variance conjecture with constant  $CC_1$  is greater than  $\frac{1}{2}$ .*

In [GM] it was shown that every log-concave isotropic random vector verifies the thin-shell width conjecture with constant  $C_1 = Cn^{\frac{1}{3}}$ . Also, an estimate for  $\sigma_X$  was given when  $X$  is not isotropic.

The following proposition is well known for the experts. However we include here for the sake of completeness. As a consequence and, by using the result in [GM], we will obtain that every centered log-concave vector, isotropic or not, verifies the variance conjecture with constant  $Cn^{\frac{2}{3}}$  rather than  $Cn^{\frac{2}{3}}(\log n)^2$ .

**Proposition 2.3.** *Let  $X$  be an isotropic log-concave random vector,  $T$  a linear map and  $\sigma_{TX}$  the thin-shell width of the random vector  $TX$  i.e.*

$$\sigma_{TX}^2 = \mathbb{E} \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2.$$

Then

$$\sigma_{TX}^2 \leq \frac{\text{Var} |TX|^2}{\mathbb{E}|TX|^2} \leq C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \lambda_{TX}^2.$$

*Proof.* The first inequality is clear, since

$$\begin{aligned}
\sigma_{TX}^2 &= \mathbb{E} \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \leq \mathbb{E} \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \frac{\left| |TX| + (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2}{\mathbb{E}|TX|^2} \\
&= \frac{\text{Var} |TX|^2}{\mathbb{E}|TX|^2}.
\end{aligned}$$

Let us now show the second inequality. Let  $B > 0$  to be chosen later.

$$\text{Var} |TX|^2 = \mathbb{E} \left| |TX|^2 - \mathbb{E}|TX|^2 \right|^2 \chi_{\{|TX| \leq B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}}$$

$$+ \mathbb{E} \left| |TX|^2 - \mathbb{E}|TX|^2 \right|^2 \chi_{\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}}.$$

The first term equals

$$\begin{aligned} & \mathbb{E} \left| |TX| + (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \chi_{\{|TX| \leq B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}} \\ & \leq (1+B)^2 \sigma_{TX}^2 \mathbb{E}|TX|^2. \end{aligned}$$

If  $B \geq \frac{1}{\sqrt{2}}$ , the second term verifies

$$\mathbb{E} \left| |TX|^2 - \mathbb{E}|TX|^2 \right|^2 \chi_{\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}} \leq \mathbb{E}|TX|^4 \chi_{\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}}$$

By Paouris' strong estimate for log-concave isotropic probabilities (see [Pa]) there exists an absolute constant  $c$  such that

$$\mathbb{P}\{|TX| > ct(\mathbb{E}|TX|^2)^{\frac{1}{2}}\} \leq e^{-t \frac{\|T\|_{HS}}{\|T\|_{op}}} \quad \forall t \geq 1.$$

Choosing  $B = \max\left\{c, \frac{1}{\sqrt{2}}\right\}$  we have that the second term is bounded from above by

$$\begin{aligned} & \mathbb{E}|TX|^4 \chi_{\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}} = B^4 (\mathbb{E}|TX|^2)^2 \mathbb{P}\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\} \\ & + B^4 (\mathbb{E}|TX|^2)^2 \int_1^\infty 4t^3 \mathbb{P}\{|TX| > Bt(\mathbb{E}|TX|^2)^{\frac{1}{2}}\} dt \\ & \leq B^4 \|T\|_{op}^4 \frac{\|T\|_{HS}^4}{\|T\|_{op}^4} e^{-\frac{\|T\|_{HS}}{\|T\|_{op}}} + B^4 \|T\|_{HS}^4 \int_1^\infty 4t^3 e^{-t \frac{\|T\|_{HS}}{\|T\|_{op}}} dt \\ & \leq C_2 \|T\|_{op}^4. \end{aligned}$$

Hence, we achieve

$$\begin{aligned} \frac{\text{Var } |TX|^2}{\mathbb{E}|TX|^2} & \leq \sigma_{TX}^2 (1+B)^2 + C_2 \frac{\|T\|_{op}^4}{\|T\|_{HS}^2} \\ & \leq C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^4}{\|T\|_{HS}^2} \\ & = C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \lambda_{TX}^2. \end{aligned}$$

□

As a consequence of this proposition we obtain that Conjecture 1.4 and Conjecture 1.5 are equivalent. Combining it with the estimate of  $\sigma_{TX}$  given in [GM] we obtain the following

**Corollary 2.4.** *There exists an absolute constant  $C$  such that for every log-concave isotropic random vector  $X$  and any linear map  $T \in GL(n)$  we have*

$$\sigma_{TX} \leq C_1 \lambda_{TX} \quad \implies \quad \text{Var } |TX|^2 \leq C_1 \mathbb{E}|TX|^2$$

and

$$\text{Var } |TX|^2 \leq C_2 \mathbb{E}|TX|^2 \quad \implies \quad \sigma_{TX} \leq C C_2 \lambda_{TX}.$$

Moreover, both inequalities are true with  $C_2 = Cn^{2/3}$ .

*Proof.* The two implications are a direct consequence of the previous proposition and the fact that  $\|T\|_{op} \leq \|T\|_{HS}$ . It was proved in [GM] that

$$\sigma_{TX} \leq C \|T\|_{op}^{\frac{1}{3}} \|T\|_{HS}^{\frac{2}{3}}.$$

Thus, by the previous proposition

$$\begin{aligned} \frac{\text{Var } |TX|^2}{\mathbb{E}|TX|^2} &\leq C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \lambda_{TX}^2 \\ &\leq C \lambda_{TX}^2 \left( \frac{\|T\|_{HS}^{\frac{4}{3}}}{\|T\|_{op}^{\frac{4}{3}}} + \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \right) \\ &\leq C \lambda_{TX}^2 \frac{\|T\|_{HS}^{\frac{4}{3}}}{\|T\|_{op}^{\frac{4}{3}}} \leq C n^{\frac{2}{3}} \lambda_{TX}^2, \end{aligned}$$

since  $\|T\|_{op} \leq \|T\|_{HS} \leq \sqrt{n} \|T\|_{op}$ .  $\square$

The square negative correlation property appeared in [ABP] in the context of the central limit problem for convex bodies.

**Definition 2.5.** Let  $X$  be a centered log-concave random vector in  $\mathbb{R}^n$  and  $\{\eta_i\}_{i=1}^n$  an orthonormal basis of  $\mathbb{R}^n$ . We say that  $X$  satisfies the square negative correlation property with respect to  $\{\eta_i\}_{i=1}^n$  if for every  $i \neq j$

$$\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 \leq \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2.$$

In [ABP], the authors showed that a random vector uniformly distributed on  $B_p^n$  satisfies the square negative correlation property with respect to the canonical basis of  $\mathbb{R}^n$ . The same property was proved for random vectors uniformly distributed on generalized Orlicz balls in [W], where it was also shown that this property does not hold in general, even in the class of random vectors uniformly distributed on 1-symmetric convex bodies.

It is easy to see that if a random centered log-concave vector  $X$  satisfies the square negative correlation property with respect to some orthonormal basis, then it also satisfies the Conjecture 1.4. Furthermore, the following proposition shows that, in such case, also some class of linear perturbations of  $X$  verify the Conjecture 1.4.

**Proposition 2.6.** Let  $X$  be a centered log-concave random vector in  $\mathbb{R}^n$  satisfying the square negative correlation property with respect to any orthonormal basis, then the Conjecture 1.4 holds for any linear image  $T \in GL(n)$ .

*Proof.* Let  $T = V\Lambda U$ , with  $U, V \in O(n)$  and  $\Lambda = [\lambda_1, \dots, \lambda_n]$  ( $\lambda_i > 0$ ) a diagonal map. Let  $\{\eta_i\}_i$  the orthonormal basis defined by  $U\eta_i = e_i$  for all  $i$ . By the square negative correlation property

$$\begin{aligned} \text{Var } |TX|^2 &= \sum_{i=1}^n \lambda_i^4 (\mathbb{E}\langle X, \eta_i \rangle^4 - (\mathbb{E}\langle X, \eta_i \rangle^2)^2) \\ &\quad + \sum_{\substack{i \neq j \\ i, j=1 \\ i=1}}^n \lambda_i^2 \lambda_j^2 (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2) \\ &\leq \sum_{i=1}^n \lambda_i^4 (\mathbb{E}\langle X, \eta_i \rangle^4 - (\mathbb{E}\langle X, \eta_i \rangle^2)^2) \end{aligned}$$



By Borell's lemma (see, for instance, [Bor], Lemma 3.1 or [MS], Appendix III)

$$\mathbb{E}\langle X, \eta_i \rangle^4 \leq C(\mathbb{E}\langle X, \eta_i \rangle^2)^2.$$

Thus

$$\begin{aligned} \text{Var } |TX|^2 &\leq C \sum_{i=1}^n \lambda_i^4 (\mathbb{E}\langle X, \eta_i \rangle^2)^2 \leq C \lambda_{TX}^2 \sum_{i=1}^n \lambda_i^2 \mathbb{E}\langle X, \eta_i \rangle^2 \\ &= C \lambda_{TX}^2 \mathbb{E}|TX|^2. \end{aligned}$$

□

**Remark.** Notice that if  $X$  satisfies the square negative correlation property with respect to one orthonormal basis  $\{\eta_i\}_{i=1}^n$  and  $U$  is the orthogonal map such that  $U(\eta_i) = e_i$ , the same proof gives that  $\Lambda UX$  verifies Conjecture 1.4 for any linear image  $\Lambda = [\lambda_1, \dots, \lambda_n]$  ( $\lambda_i > 0$ ).

Even though verifying the variance conjecture is not equivalent to satisfy a square negative correlation property, the following lemma shows that it is equivalent to satisfy a “weak averaged square negative correlation” property with respect to one and every orthonormal basis.

**Lemma 2.7.** *Let  $X$  be a centered log concave random vector in  $\mathbb{R}^n$ . The following are equivalent*

- i)  $X$  verifies the variance conjecture with constant  $C_1$

$$\text{Var } |X|^2 \leq C_1 \lambda_X^2 \mathbb{E}|X|^2.$$

- ii)  $X$  satisfies the following “weak averaged square negative correlation” property with respect to some orthonormal basis  $\{\eta_i\}_{i=1}^n$  with constant  $C_2$

$$\sum_{i \neq j} (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2) \leq C_2 \lambda_X^2 \mathbb{E}|X|^2.$$

- iii)  $X$  satisfies the following “weak averaged square negative correlation” property with respect to every orthonormal basis  $\{\eta_i\}_{i=1}^n$  with constant  $C_3$

$$\sum_{i \neq j} (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2) \leq C_3 \lambda_X^2 \mathbb{E}|X|^2,$$

where

$$C_2 \leq C_1 \leq C_2 + C \quad C_3 \leq C_1 \leq C_3 + C$$

with  $C$  an absolute constant.

*Proof.* For any orthonormal basis  $\{\eta_i\}_{i=1}^n$  we have

$$\begin{aligned} \text{Var } |X|^2 &= \sum_{i=1}^n (\mathbb{E}\langle X, \eta_i \rangle^4 - (\mathbb{E}\langle X, \eta_i \rangle^2)^2) \\ &\quad + \sum_{i \neq j} (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2). \end{aligned}$$

Denoting by  $A(\eta)$  the second term we have, using Borell's lemma, that

$$A(\eta) \leq \text{Var } |X|^2 \leq C \lambda_X^2 \mathbb{E}|X|^2 + A(\eta),$$

since  $\sum_{i=1}^n \mathbb{E}\langle X, \eta_i \rangle^4 \leq C \sup_i \mathbb{E}\langle X, \eta_i \rangle^2 \sum_{i=1}^n \mathbb{E}\langle X, \eta_i \rangle^2 = C \lambda_X^2 \mathbb{E}|X|^2$ . □

## 3. HYPERPLANE PROJECTIONS OF THE CROSS-POLYTOPE AND THE CUBE

In this section we are going to give some new examples of random vectors verifying the variance conjecture. We will consider the family of random vectors uniformly distributed on a hyperplane projection of some symmetric isotropic convex body  $K_0$ . These random vectors will not necessarily be isotropic. However, as we will see in the next proposition, they will be almost isotropic. *i.e.* the spectral condition number  $B$  of their covariance matrix verifies  $1 \leq B \leq C$  for some absolute constant  $C$ .

**Proposition 3.1.** *Let  $K_0 \subset \mathbb{R}^n$  be a symmetric isotropic convex body, and let  $H = \theta^\perp$  be any hyperplane. Let  $K = P_H(K_0)$  and  $X$  a random vector uniformly distributed on  $K$ . Then, for any  $\xi \in S_H = \{x \in H; |x| = 1\}$*

$$\mathbb{E}\langle X, \xi \rangle^2 \sim \frac{1}{\text{Vol}(K)^{1+\frac{2}{n-1}}} \int_K \langle x, \xi \rangle^2 dx \sim L_{K_0}^2.$$

Consequently  $\lambda_X \sim L_{K_0}$  and  $B(X) \sim 1$ .

*Proof.* The two first expressions are equivalent, since  $\text{Vol}(K)^{\frac{1}{n-1}} \sim 1$ . Indeed, using Hensley's result [He] and the best general known upper bound for the isotropy constant of an  $n$ -dimensional convex body [K1], we have

$$\text{Vol}(K)^{\frac{1}{n-1}} \geq \text{Vol}(K_0 \cap H)^{\frac{1}{n-1}} \geq \left(\frac{c}{L_{K_0}}\right)^{\frac{1}{n-1}} \geq \left(\frac{c}{n^{\frac{1}{4}}}\right)^{\frac{1}{n-1}} \geq c.$$

On the other hand, since (see [RS] for a proof)

$$\frac{1}{n} \text{Vol}(K) \text{Vol}(K_0 \cap H^\perp) \leq \text{Vol}(K_0) = 1$$

we have

$$\text{Vol}(K)^{\frac{1}{n-1}} \leq \left(\frac{n}{2r(K_0)}\right)^{\frac{1}{n-1}} \leq \left(\frac{n}{2L_{K_0}}\right)^{\frac{1}{n-1}} \leq (cn)^{\frac{1}{n-1}} \leq c,$$

where we have used that  $r(K_0) = \sup\{r : rB_2^n \subseteq K_0\} \geq cL_{K_0}$ , see [KLS].

Let us prove the last estimate. Let  $S(K_0)$  be the Steiner symmetrization of  $K_0$  with respect to the hyperplane  $H$  and let  $S_1$  be its isotropic position. It is known (see [B] or [MP]) that for any isotropic  $n$ -dimensional convex body  $L$  and any linear subspace  $E$  of codimension  $k$

$$\text{Vol}(L \cap E)^{\frac{1}{k}} \sim \frac{L_C}{L_L},$$

where  $C$  is a convex body in  $E^\perp$ . In particular, we have that

$$\text{Vol}(S_1 \cap H) \sim \frac{1}{L_{S(K_0)}}$$

and

$$\text{Vol}(S_1 \cap H \cap \xi^\perp) \sim \frac{1}{L_{S(K_0)}^2}.$$

Since  $K_0$  is symmetric,  $S_1 \cap H$  is symmetric and thus centered. Then, by Hensley's result [He]

$$\begin{aligned} L_{S(K_0)}^2 &\sim \frac{\text{Vol}(S_1 \cap H)^2}{\text{Vol}(S_1 \cap H \cap \xi^\perp)^2} \sim \frac{1}{\text{Vol}(S_1 \cap H)} \int_{S_1 \cap H} \langle x, \xi \rangle^2 dx \\ &\sim \frac{1}{\text{Vol}(S_1 \cap H)^{1+\frac{2}{n-1}}} \int_{S_1 \cap H} \langle x, \xi \rangle^2 dx, \end{aligned}$$

because  $\text{Vol}(S_1 \cap H) \sim \frac{1}{L_{S(K_0)}}$  and so  $\text{Vol}(S_1 \cap H)^{\frac{1}{n-1}} \sim c$ .

But now,  $\widetilde{S_1 \cap H} = \text{Vol}(S_1 \cap H)^{-\frac{1}{n-1}} (S_1 \cap H) = S(\widetilde{K_0}) \cap H = \widetilde{K}$ , because, even though  $S(K_0)$  is not isotropic,  $S_1$  is obtained from  $S(K_0)$  multiplying it by some  $\lambda$  in  $H$  and by  $\frac{1}{\lambda^{n-1}}$  in  $H^\perp$ . Thus,

$$L_{S(K_0)}^2 \sim \int_{\widetilde{S_1 \cap H}} \langle x, \xi \rangle^2 dx = \int_{\widetilde{K}} \langle x, \xi \rangle^2 dx = \frac{1}{\text{Vol}(K)^{1+\frac{2}{n-1}}} \int_K \langle x, \xi \rangle^2 dx$$

and since  $L_{S(K_0)} \sim L_{K_0}$  we obtain the result.  $\square$

The first examples we consider are the random vectors uniformly distributed on hyperplane projections of the cube. We will see that these random vectors satisfy the negative square correlation property with respect to any orthonormal basis. Consequently, by Proposition 2.6, any linear image of these random vectors will verify the variance conjecture with an absolute constant.

**Theorem 3.2.** *Let  $\theta \in S^{n-1}$  and let  $K = P_H B_\infty^n$  be the projection of  $B_\infty^n$  on the hyperplane  $H = \theta^\perp$ . If  $X$  is a random vector uniformly distributed on  $K$  then, for any two orthonormal vectors  $\eta_1, \eta_2 \in H$ , we have*

$$\mathbb{E}\langle X, \eta_1 \rangle^2 \langle X, \eta_2 \rangle^2 \leq \mathbb{E}\langle X, \eta_1 \rangle^2 \mathbb{E}\langle X, \eta_2 \rangle^2.$$

Consequently,  $X$  satisfies the negative square correlation property with respect to any orthonormal basis in  $H$ .

*Proof.* Let  $F_i$  denote the facet  $F_i = \{y \in B_\infty^n; y_{|i|} = \text{sgn } i\}$ ,  $i \in \{\pm 1, \dots, \pm n\}$ . From Cauchy's formula, it is clear that for any function  $f$

$$\mathbb{E}f(X) = \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E}(f(P_H Y^i))$$

where  $Y^i$  is a random vector uniformly distributed on the facet  $F_i$ .

Remark that

$$\text{Vol}(P_H(F_i)) = |\langle \theta, e_{|i|} \rangle| \text{Vol}(F_i) = 2^{n-1} |\theta_{|i|}|$$

for  $i = \pm 1, \dots, \pm n$  and  $\text{Vol}(K) = 2^{n-1} \|\theta\|_1$ .

For any unit vector  $\eta \in H$ , we have by isotropicity of the facets of  $B_\infty^n$ ,

$$\begin{aligned} \mathbb{E}\langle X, \eta \rangle^2 &= \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E}\langle Y^i, \eta \rangle^2 = \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E} \sum_{j=1}^n \eta_j^2 Y_j^{i^2} \\ &= \frac{1}{2} \sum_{j=1}^n \eta_j^2 \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{\|\theta\|_1} \mathbb{E} Y_j^{i^2} = \sum_{j=1}^n \eta_j^2 \left( \frac{|\theta_j|}{\|\theta\|_1} + \frac{1}{3} \sum_{i \neq j} \frac{|\theta_i|}{\|\theta\|_1} \right) \\ &= \sum_{j=1}^n \eta_j^2 \left( \frac{2|\theta_j|}{3\|\theta\|_1} + \frac{1}{3} \right) = \frac{1}{3} + \frac{2}{3} \sum_{j=1}^n \eta_j^2 \frac{|\theta_j|}{\|\theta\|_1}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathbb{E}\langle X, \eta_1 \rangle^2 \mathbb{E}\langle X, \eta_2 \rangle^2 \\ &= \frac{1}{9} + \frac{2}{9} \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} (\eta_1(i)^2 + \eta_2(i)^2) + \frac{4}{9} \sum_{i_1, i_2=1}^n \frac{|\theta_{i_1}| |\theta_{i_2}|}{\|\theta\|_1^2} \eta_1(i_1)^2 \eta_2(i_2)^2 \\ &\geq \frac{1}{9} + \frac{2}{9} \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} (\eta_1(i)^2 + \eta_2(i)^2). \end{aligned}$$

On the other hand, by symmetry,

$$\begin{aligned} \mathbb{E}\langle X, \eta_1 \rangle^2 \langle X, \eta_2 \rangle^2 &= \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E}\langle Y^i, \eta_1 \rangle^2 \langle Y^i, \eta_2 \rangle^2 \\ &= \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} (\langle y, P_{e_{|i|}} \eta_1 \rangle + \text{sgn}(i) \eta_1(i))^2 (\langle y, P_{e_{|i|}} \eta_2 \rangle + \text{sgn}(i) \eta_2(i))^2 dy \\ &= \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \left( \langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 + \eta_2(i)^2 \langle y, P_{e_i^\perp} \eta_1 \rangle^2 + \right. \\ &\quad \left. + \eta_1(i)^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 + \eta_1(i)^2 \eta_2(i)^2 + 4\eta_1(i) \eta_2(i) \langle y, P_{e_i^\perp} \eta_1 \rangle \langle y, P_{e_i^\perp} \eta_2 \rangle \right) dy \\ &= \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} \left( \frac{1}{3} \eta_1(i)^2 |P_{e_i^\perp} \eta_2|^2 + \frac{1}{3} \eta_2(i)^2 |P_{e_i^\perp} \eta_1|^2 + \eta_1(i)^2 \eta_2(i)^2 + \right. \\ &\quad \left. + 4\eta_1(i) \eta_2(i) \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle \langle y, P_{e_i^\perp} \eta_2 \rangle dy \right. \\ &\quad \left. + \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 dy \right) \\ &= \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} \left( \frac{1}{3} \eta_1(i)^2 + \frac{1}{3} \eta_2(i)^2 + \frac{1}{3} \eta_1(i)^2 \eta_2(i)^2 + \right. \\ &\quad \left. + 4\eta_1(i) \eta_2(i) \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle \langle y, P_{e_i^\perp} \eta_2 \rangle dy \right. \\ &\quad \left. + \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 dy \right) \end{aligned}$$

Since

$$\frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle \langle y, P_{e_i^\perp} \eta_2 \rangle dy = \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \left( \sum_{l_1, l_2 \neq i} y_{l_1} y_{l_2} \eta_1(l_1) \eta_2(l_2) \right) dy$$

$$\begin{aligned}
&= \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \left( \sum_{l \neq i} y_l^2 \eta_1(l) \eta_2(l) \right) dy \\
&= \frac{1}{3} \sum_{l \neq i} \eta_1(l) \eta_2(l) \\
&= \frac{1}{3} (\langle \eta_1, \eta_2 \rangle - \eta_1(i) \eta_2(i)) \\
&= -\frac{1}{3} \eta_1(i) \eta_2(i)
\end{aligned}$$

the previous sum equals

$$\sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} \left( \frac{1}{3} \eta_1(i)^2 + \frac{1}{3} \eta_2(i)^2 - \eta_1(i)^2 \eta_2(i)^2 + \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 dy \right).$$

Now,

$$\begin{aligned}
&\frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 dy \\
&= \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \left( \sum_{l_1, l_2, l_3, l_4} y_{l_1} y_{l_2} y_{l_3} y_{l_4} \eta_1(l_1) \eta_1(l_2) \eta_2(l_3) \eta_2(l_4) \right) dy \\
&= \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} y_l^4 dy \\
&+ \sum_{l_1 \neq l_2 (\neq i)} \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} y_{l_1}^2 y_{l_2}^2 dy (\eta_1(l_1)^2 \eta_2(l_2)^2 + \eta_1(l_1) \eta_1(l_2) \eta_2(l_1) \eta_2(l_2)) \\
&= \frac{1}{5} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 + \frac{1}{9} \sum_{l_1 \neq l_2 (\neq i)} (\eta_1(l_1)^2 \eta_2(l_2)^2 + \eta_1(l_1) \eta_1(l_2) \eta_2(l_1) \eta_2(l_2)) \\
&= \frac{1}{5} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 \\
&+ \frac{1}{9} \left[ \sum_{l \neq i} (\eta_1(l)^2 (1 - \eta_2(l)^2 - \eta_2(i)^2) + \eta_1(l) \eta_2(l) (\langle \eta_1, \eta_2 \rangle - \eta_1(l) \eta_2(l) - \eta_1(i) \eta_2(i))) \right] \\
&= \frac{1}{5} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 \\
&+ \frac{1}{9} \left( 1 - \eta_1(i)^2 - \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 - \eta_2(i)^2 + \eta_1(i)^2 \eta_2(i)^2 \right. \\
&\quad \left. - \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 + \eta_1(i)^2 \eta_2(i)^2 \right) \\
&= \frac{1}{9} - \frac{1}{9} \eta_1(i)^2 - \frac{1}{9} \eta_2(i)^2 + \frac{2}{9} \eta_1(i)^2 \eta_2(i)^2 - \frac{1}{45} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2.
\end{aligned}$$

Consequently

$$\mathbb{E} \langle X, \eta_1 \rangle^2 \langle X, \eta_2 \rangle^2 = \frac{1}{9} + \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} \left( \frac{2}{9} \eta_1(i)^2 + \frac{2}{9} \eta_2(i)^2 - \frac{7}{9} \eta_1(i)^2 \eta_2(i)^2 \right)$$

$$\begin{aligned}
& - \frac{1}{45} \sum_{l \neq i} \eta_1(i)^2 \eta_2(l)^2 \Big) \\
& \leq \frac{1}{9} + \frac{2}{9} \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} (\eta_1(i)^2 + \eta_2(i)^2)
\end{aligned}$$

which concludes the proof.  $\square$

By Proposition 2.6 we obtain the following

**Corollary 3.3.** *There exists an absolute constant  $C$  such that for every hyperplane  $H$  and any linear map  $T$ , if  $X$  is a random vector uniformly distributed on  $P_H B_\infty^n$ , then  $TX$  verifies the variance conjecture with constant  $C$ , i.e.*

$$\text{Var } |TX|^2 \leq C \lambda_{TX}^2 \mathbb{E}|TX|^2$$

The next examples we consider are random vectors uniformly distributed on projections of  $B_1^n$ . Even though in this case we are not able to prove that these vectors satisfy a square negative correlation property, we are still able to show that they verify the variance conjecture with some absolute constant.

**Theorem 3.4.** *There exists an absolute constant  $C$  such that for every hyperplane  $H$ , if  $X$  is a random vector uniformly distributed on  $P_H B_1^n$ ,  $X$  verifies the variance conjecture with constant  $C$ , i.e.*

$$\text{Var } |X|^2 \leq C \lambda_X^2 \mathbb{E}|X|^2.$$

*Proof.* First of all, notice that by Proposition 3.1 we have that for every  $\xi \in S^{n-1} \cap H$

$$\mathbb{E} \langle \text{Vol}(B_1^n)^{-\frac{1}{n}} X, \xi \rangle^2 \sim L_{B_1^n}^2 \sim 1$$

and so

$$\lambda_X^2 \sim \frac{1}{n^2} \quad \text{and} \quad \mathbb{E}|X|^2 \sim \frac{1}{n}.$$

Thus, we have to prove that  $\text{Var } |X|^2 \leq \frac{C}{n^3}$ .

By Cauchy formula, denoting by  $\theta$  the unit vector orthogonal to  $H$ ,  $Y$  a random vector uniformly distributed on  $\Delta_{n-1} = \{y \in \mathbb{R}^n : y_i \geq 0, \sum_{i=1}^n y_i = 1\}$ ,  $\varepsilon$  a random vector, independent of  $Y$ , in  $\{-1, 1\}^n$  distributed according to

$$\mathbb{P}(\varepsilon = \varepsilon_0) = \frac{|\langle \varepsilon_0, \theta \rangle|}{\sum_{\varepsilon \in \{-1, 1\}^n} |\langle \varepsilon, \theta \rangle|} = \frac{\text{Vol}_{n-1}(\Delta_{n-1}) |\langle \varepsilon_0, \theta \rangle|}{2\sqrt{n} \text{Vol}_{n-1}(P_H(B_1^n))}$$

and

$$\varepsilon x = (\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$$

we have that

$$\begin{aligned}
\text{Var } |X|^2 &= \mathbb{E}|X|^4 - (\mathbb{E}|X|^2)^2 = \mathbb{E}|P_H(\varepsilon Y)|^4 - (\mathbb{E}|P_H(\varepsilon Y)|^2)^2 \\
&= \mathbb{E}(|\varepsilon Y|^2 - \langle \varepsilon Y, \theta \rangle^2)^2 - (\mathbb{E}(|\varepsilon Y|^2 - \langle \varepsilon Y, \theta \rangle^2))^2 \\
&= \mathbb{E}(|Y|^2 - \langle Y, \varepsilon \theta \rangle^2)^2 - (\mathbb{E}(|Y|^2 - \langle Y, \varepsilon \theta \rangle^2))^2 \\
&\leq \mathbb{E}|Y|^4 + \mathbb{E}\langle \varepsilon Y, \theta \rangle^4 - (\mathbb{E}|Y|^2 - \mathbb{E}\langle \varepsilon Y, \theta \rangle^2)^2.
\end{aligned}$$

Since for every  $a, b \in \mathbb{N}$  with  $a + b = 4$  we have

$$\mathbb{E}Y_1^a Y_2^b = \frac{a!b!}{(n+3)(n+2)(n+1)n}$$

we have

$$\begin{aligned}
\mathbb{E}|Y|^4 &= n\mathbb{E}Y_1^4 + n(n-1)\mathbb{E}Y_1^2Y_2^2 \\
&= \frac{4!}{(n+3)(n+2)(n+1)} + \frac{4(n-1)}{(n+3)(n+2)(n+1)} \\
&= \frac{4}{n^2} + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

Denoting by  $\epsilon$  a radom vector uniforly distributed on  $\{-1, 1\}^n$  we have, by Khintchine inequality,

$$\begin{aligned}
\mathbb{E}\langle \epsilon Y, \theta \rangle^4 &= \frac{\text{Vol}_{n-1}(\Delta_{n-1})}{2\sqrt{n}\text{Vol}_{n-1}(P_H(B_1^n))} \mathbb{E}_Y \sum_{\epsilon \in \{-1, 1\}^n} |\langle \epsilon, \theta \rangle| |\langle \epsilon Y, \theta \rangle|^4 \\
&= \frac{2^n \text{Vol}_{n-1}(\Delta_{n-1})}{2\sqrt{n}\text{Vol}_{n-1}(P_H(B_1^n))} \mathbb{E}_Y \mathbb{E}_\epsilon |\langle \epsilon, \theta \rangle| |\langle \epsilon Y, \theta \rangle|^4 \\
&\leq C \mathbb{E}_Y (\mathbb{E}_\epsilon \langle \epsilon, \theta \rangle^2)^{\frac{1}{2}} (\mathbb{E}_\epsilon \langle \epsilon Y, \theta \rangle^8)^{\frac{1}{2}} \\
&\leq C \mathbb{E}_Y \left( \sum_{i=1}^n Y_i^2 \theta_i^2 \right)^2 \\
&= C \left( \mathbb{E}Y_1^4 \sum_{i=1}^n \theta_i^4 + \mathbb{E}Y_1^2 Y_2^2 \sum_{i \neq j} \theta_i^2 \theta_j^2 \right) \\
&= \frac{C}{(n+3)(n+2)(n+1)n} \left( 24 \sum_{i=1}^n \theta_i^4 + 4 \sum_{i,j=1}^n \theta_i^2 \theta_j^2 \right) \\
&\leq \frac{C}{n^4}
\end{aligned}$$

since  $\sum_{i=1}^n \theta_i^4 \leq \sum_{i=1}^n \theta_i^2 = 1$ .

On the other hand, since

$$\mathbb{E}Y_1^2 = \frac{2}{(n+1)n} \quad \text{and} \quad \mathbb{E}Y_1 Y_2 = \frac{1}{(n+1)n}$$

we have

$$\mathbb{E}|Y|^2 = n\mathbb{E}Y_1^2 = \frac{2}{n+1}$$

and

$$\begin{aligned}
\mathbb{E}\langle \epsilon Y, \theta \rangle^2 &= \mathbb{E} \left( \sum_{i=1}^n Y_i^2 \theta_i^2 + \sum_{i \neq j} \epsilon_i \epsilon_j Y_i Y_j \theta_i \theta_j \right) \\
&= \mathbb{E}Y_1^2 + \sum_{i \neq j} \theta_i \theta_j \mathbb{E}_\epsilon \epsilon_i \epsilon_j \mathbb{E}_Y Y_1 Y_2 \\
&= \frac{2}{(n+1)n} + \frac{1}{(n+1)n} (\mathbb{E}_\epsilon \langle \epsilon, \theta \rangle^2 - 1) \\
&= \frac{1}{(n+1)n} (1 + \mathbb{E}_\epsilon \langle \epsilon, \theta \rangle^2) \sim \frac{1}{n^2},
\end{aligned}$$

since, by Khintchine inequality,

$$\mathbb{E}_\epsilon \langle \epsilon, \theta \rangle^2 = \frac{\text{Vol}_{n-1}(\Delta_{n-1})}{2\sqrt{n}\text{Vol}_{n-1}(P_H(B_1^n))} \sum_{\epsilon \in \{-1, 1\}^n} |\langle \epsilon, \theta \rangle|^2$$

$$\leq C\mathbb{E}_\varepsilon|\langle\varepsilon, \theta\rangle|^3 \sim C$$

Thus

$$(\mathbb{E}|Y|^2 - \mathbb{E}\langle\varepsilon Y, \theta\rangle^2)^2 = \frac{4}{n^2} + O\left(\frac{1}{n^3}\right)$$

and so

$$\text{Var } |X|^2 \leq \frac{C}{n^3}.$$

□

*Acknowledgements.* Part of this work was done while the first named author was a postdoctoral fellow at the Department of Mathematical and Statistical Sciences at University of Alberta. He would like to thank the department for providing such good environment and working conditions.

The authors also want to thank the referee for providing several comments which improved the presentation of the paper a lot.

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