REARRANGEMENT OF HARDY-LITTLEWOOD MAXIMAL FUNCTIONS IN LORENTZ SPACES.

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Abstract. For the classical Hardy-Littlewood maximal function Mf, a well known and important estimate due to Herz and Stein, gives the equivalence $(Mf)^*(t) \sim f^{**}(t)$. In the present note, we study the validity of analogous estimates for maximal operators of the form

$$M_{p,q}f(x) = \sup_{x\in Q} \frac{\|f\chi_Q\|_{p,q}}{\|\chi_Q\|_{p,q}},$$

where $\|.\|_{p,q}$ denotes the Lorentz space $L(p,q)$-norm.

1. Introduction. The Hardy-Littlewood maximal function M plays a central role in classical harmonic analysis, differentiation theory and PDE's. It is well known that the maximal operator M is of weak type $(1,1)$ and strong type (∞,∞) from where it follows readily, for example using K-functionals (see [BS]), that there exists an absolute constant $C>0$ such that

$$(Mf)^*(t) \le Cf^{**}(t), \forall t>0, \forall f \in L^1_{loc}(\mathbb{R}^n), \tag{1}$$

where * denotes non-increasing rearrangement, and $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$.

Herz (cf. [H], and also [BS]) proved that the reverse inequality is also true, that is,

$$f^{**}(t) \le c(Mf)^*(t),\ t>0. \tag{2}$$

Inequalities (1) and (2) contain the basic information to study M, and the operators it controls, in rearrangement invariant function spaces. We refer to [AKMP] for a recent and exhaustive study of inequalities (1) and (2) where the underlying measure is more general than Lebesgue measure.

Recall that the maximal operator M is defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(t)|dt = \sup_{x\in Q} \frac{\|f\chi_Q\|_{L^1}}{\|\chi_Q\|_{L^1}}.$$

A commonly used variant of the maximal operator, $M_pf = (M|f|^p)^{1/p}$, is obtained by means of replacing L^1 averages with L^p-averages. More generally Stein [S],

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in order to obtain certain end point results in differentiation theory, introduced maximal operators associated with $L(p,q)$ averages as follows. Let $1 \leq p < +\infty$, $1 \leq q \leq +\infty$, then Stein defines

$$M_{p,q}f(x) = \sup_{x \in Q} \frac{\|f\chi_Q\|_{p,q}}{\|\chi_Q\|_{p,q}} = \sup_{x \in Q} \frac{\|f\chi_Q\|_{p,q}}{|Q|^{1/p}}.$$

These operators have been also considered by other authors, for instance see [N], [LN] and [P].

It is a basic fact of real interpolation (see [BS]) that $f^{**}(t)$ can be obtained in terms of the K-functional for the pair (L^1, L^∞) as

$$f^{**}(t) = \frac{K(t, f; L^1, L^\infty)}{t},$$

where for a compatible pair of Banach spaces (X, Y), $f \in X + Y, t > 0$,

$$K(t, f; X, Y) = \inf\{\|x\|_X + t\|y\|_Y\},$$

and the inf runs over all possible decompositions $f = x + y$ with $x \in X, y \in Y$.

From the definition of M_p given above it follows readily, using (1), (2) and the reiteration theorem, that

$$(M_p f)^*(t) \approx t^{-1/p} K(t^{1/p}, f; L^p, L^\infty).$$

Therefore one is led to ask if a similar relationship exists between $(M_{p,q}f)^*(t)$ and the corresponding K-functional for the pair $(L(p,q), L^\infty)$, which is given by

$$\frac{K(t^{1/p}, f; L(p,q), L^\infty)}{t^{1/p}} \approx \frac{1}{t^{1/p}} \left(\int_0^t f^*(s)^q s^{q/p-1} ds \right)^{1/q} = \frac{c}{t^{1/p}} \|f^* \chi_{(0,t)}\|_{p,q}.$$

As we shall show below the somewhat surprising answer to this question is: no! The two cases we need to consider $p < q$ and $q < p$ turn out to be very different from each other. In fact for $q < p$, the $L(p,q)$ version of inequality (1) is known to hold as can be readily seen since $M_{p,q}$ is bounded from $L(p,q)$ into $L(p,\infty)$ and from L^∞ into L^∞ (cf. [S] and also [LN] for a different proof). For $p < q$, the validity of the corresponding $L(p,q)$ version of inequality (1) must be ruled out since, as it is well known, $M_{p,q}$ is not bounded from $L(p,q)$ into $L(p,\infty)$ (cf. [S]).

Our purpose in this note is to complete these results by showing in section 2 that the $L(p,q)$ version of inequality (2) is true when $q > p$ and false when $q < p$. In view of these negative results it is natural to ask: what is the appropriate maximal operator associated with the $K-$functional for the pair $(L(p,q), L^\infty)$ so that the corresponding version of Herz's theorem holds? In section 3 we provide an answer by means of finding an improvement on the operator $(M_{p,q}f)^*$.

It will be convenient for us to work in the more general context of r.i. spaces. Indeed the added generality does not complicate the proofs and helps one to see better how the geometrical properties of the $L(p,q)$ spaces intervene in the analysis of the cases $q > p$ or $p > q$.

2. As usual, a Banach space $(X, \|.\|_X)$ of real-valued, locally integrable, Lebesgue measurable functions on $\mathbb{R}^n$ is said to be a r.i. space if it satisfies the following conditions:

i) If $g^* \leq f^*$ and $f \in X$, then $g \in X$ with $\|g\|_X \leq \|f\|_X$, (f^* denotes the non increasing rearrangement of the function f).
ii) If A is a Lebesgue measurable set of finite measure, then $\chi_A \in X$.
iii) $0 \leq f_n \uparrow$, $\sup_{n\in\mathbb{N}} \|f_n\|_X \leq M$, imply that $f = \sup f_n \in X$ and $\|f\|_X = \sup_{n\in\mathbb{N}} \|f_n\|_X$.

For each r.i. space X on $\mathbb{R}^n$, a r.i. space $\overline{X}$ on $I = (0, +\infty)$ is associated such that $f \in X$ if and only if $f^* \in \overline{X}$ and $\|f\|_X = \|f^*\|_{\overline{X}}$ (see [BS]).

The fundamental function of a r.i. Banach space X is defined by

$$\Phi(t) = \Phi_X(t) = \|\chi_{[0,t)}\|_{\overline{X}},\ t > 0.$$

We will denote by $M^*(X)$ the space of all measurable functions for which

$$\|f\|_{M^*(X)} = \sup_{t\in I} \Phi_X(t) f^*(t) < \infty.$$

The function $\|.\|_{M^*(X)}$ is a quasinorm on $M^*(X)$.

For any measurable function f such that $f\chi_Q \in X$, we define the maximal operator

$$M_X f(x) = \sup_{x\in Q} \frac{\|f\chi_Q\|_X}{\|\chi_Q\|_X},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ which contain x with sides parallel to the coordinate axes.

Since a conditional expectation operator is a norm one projection in any r.i. space, it is clear that for any cube Q we have

$$\| \left(\frac{1}{|Q|} \int_Q f \right) \chi_Q \|_X \leq \|f\chi_Q\|_X,$$

and therefore, $Mf \leq M_X f$, where as usual Mf is the classical Hardy-Littlewood maximal function.

Definition. *Let X be a r.i. space and let $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ an increasing bijection. X is said to satisfy an* **upper ϕ-estimate** *(resp.* **lower ϕ-estimate***) if there exists a constant $M < +\infty$ such that for every choice $\{f_i\}_{i=1}^n$ of functions in X with disjoint supports,*

$$\| \sum_{i=1}^n f_i \|_X \leq M\phi \left(\sum_{i=1}^n \phi^{-1}(\|f_i\|_X) \right),$$

respectively

$$\| \sum_{i=1}^n f_i \|_X \geq M\phi \left(\sum_{i=1}^n \phi^{-1}(\|f_i\|_X) \right).$$

In the special case when $\phi(t) = t^{1/p}$ we recover the well known notions of lower and upper p-estimates (see [LT]).

Theorem 1. *Let X be a r.i. space with fundamental function Φ. If X satisfies a* **lower Φ-estimate**, *then $M_X : X \longrightarrow M^*(X)$ is a bounded operator. In other words, there exists $C > 0$ such that for all $f \in X$ we have*

$$\sup_t \Phi(t)(M_X f)^*(t) \leq C\|f\|_X.$$

As a consequence,

$$(M_X f)^*(t) \leq \frac{C}{\Phi(t)}\|f^*\chi_{(0,t)}\|_X, \quad \forall t > 0.$$

Proof. Let $f \in X$, in terms distribution functions we have to prove that

$$\Phi(|\{x : M_X f(x) > \lambda\}|) \leq \frac{C}{\lambda}\|f\|_X, \ \lambda > 0.$$

Let $\Omega = \{x : M_X f(x) > \lambda\}$. Using the definition of M_X and a standard covering lemma (cf. [BS], pg. 118), it is possible to choose a countable family $\mathbf{F}$ of cubes $\{Q\}_{i\in J}$ with pairwise disjoint interiors and such that

$$|\Omega| \leq 4^n \sum_{i\in J}|Q_i|, \quad \frac{\|f\chi_{Q_i}\|_X}{\|\chi_{Q_i}\|_X} > \lambda, \forall i \in J.$$

Moreover, if $Q \in\mathbf{F}$ then,

$$\Phi(|Q|) < \|(\frac{f}{\lambda})\chi_Q\|_X, \quad |Q| < \Phi^{-1}(\|(\frac{f}{\lambda})\chi_Q\|_X).$$

Therefore, using that $\Phi(5t) \leq 5\Phi(t)$ and the lower Φ-estimate, we get

$$\Phi(|\Omega|) \leq 5\Phi\left(\sum_{i\in J}\Phi^{-1}(\|(\frac{f}{\lambda})\chi_{Q_i}\|_X)\right) \leq 5M\|\sum_{i\in J}(\frac{f}{\lambda})\chi_{Q_i}\|_X \leq \frac{5M}{\lambda}\|f\|_X.$$

This proves the first part of the theorem. The proof of the second part is a routine argument in interpolation theory. Indeed, since M_X is a bounded operator on L^∞, we have that, $\forall t > 0$,

$$K(\Phi(t), M_X f, M^*(X), L^\infty) \leq CK(\Phi(t), f, X, L^\infty).$$

Now, we recall that the left hand side of this inequality is equivalent to

$$\sup_{s\leq t}(M_X f)^*(s)\Phi(s),$$

while the right hand side is equivalent to

$$\|f^*\chi_{(0,t)}\|_X,$$

(see [BR]), and the result follows.

Theorem 2. *Let X be a r.i. space with fundamental function Φ. If X satisfies an* **upper Φ-estimate***, then there exists and absolute constant $C > 0$ such that $\forall f \in X, t > 0$ we have*

$$(M_X f)^*(t) \geq \frac{C}{\Phi(t)} \|f^* \chi_{(0,t)}\|_X.$$

Proof. Fix $f \in X, t > 0$. Let $\alpha = (M_X f)^*(t)$ and $\Omega = \{x : M_X f(x) > \alpha\}$. Following [BS], pgs. 122-123, we can choose a sequence of dyadic cubes $\{Q\}_{i\in J}$ with pairwise disjoint interiors, which covers Ω, and such that

$$\sum_{i\in J} |Q_i| \leq C|\Omega| \leq Ct, \quad \frac{\|f\chi_{Q_i}\|_X}{\|\chi_{Q_i}\|_X} \leq \alpha, \forall i \in J.$$

Then, we decompose

$$f = \sum_{i\in J} f\chi_{Q_i} + h = g + h.$$

Using the upper Φ-estimate, we get

$$\begin{aligned}\|\frac{g}{\alpha}\|_X &\leq M\Phi\left(\sum_{i\in J} \Phi^{-1}(\|\frac{f}{\alpha}\chi_{Q_i}\|_X)\right) \\ &\leq M\Phi\left(\sum_{i\in J} \Phi^{-1}(\Phi(|Q_i|))\right) \leq C\Phi(t).\end{aligned}$$

Thus,

$$\|g\|_X \leq C(M_X f)^*(t)\Phi(t).$$

On the other hand, since $f \leq Mf \leq M_X f$ a.e., we have

$$\|h\|_\infty \leq (M_X f)^*(t).$$

Then using [BR] and the definition of the K-functional we obtain

$$\frac{1}{\Phi(t)}\|f^*\chi_{(0,t)}\|_X = \frac{1}{\Phi(t)} K(\Phi(t), f, X, L^\infty) \leq C(M_X f)^*(t).$$

We now turn to study the case when $X = L(p,q)$ is a Lorentz space, in this case the fundamental function of X is $\Phi(t) = t^{1/p}$. We shall need the following

Lemma. *Let $p \in (1, +\infty)$, then*

i) *If $1 < p \leq q \leq +\infty$ then $L(p,q)$ satisfies an upper p-estimate.*

ii) *If $1 \leq q \leq p$, $L(p,q)$ satisfies a lower p-estimate.*

Proof. Recall that, in terms of the distribution function, an expression equivalent to the $L(p,q)$-norm can be given as follows

$$\|f\|_{p,q} = \left(\int_0^\infty (\lambda_f(s))^{q/p} s^{q-1} ds\right)^{1/q},$$

with the usual modification when $q = +\infty$. Given a sum $f = \sum f_i$ where the f_i have disjoint supports, it is clear than

$$\lambda_f(s) = \sum \lambda_{f_i}(s).$$

Therefore in the case $q < \infty$ the corresponding lower and upper p-estimates concern only the interchange of sums and integrals and can be obtained easily using Minkowski's vector-valued inequality, while the case $q = +\infty$ is even simpler.

Corollary. *Let $p \in (1, +\infty)$, then*

i) *If $1 < p \leq q \leq +\infty$, then there exists a constant $C > 0$ such that $\forall f \in X, t > 0$,*

$$(M_{p,q}f)^*(t) \geq \frac{C}{t^{1/p}} \left(\int_0^t f^*(s)^q s^{q/p-1} ds \right)^{1/q}$$

(with the usual modification if $q = +\infty$).

ii) *If $1 \leq q \leq p$, then there exists a constant $C > 0$ such that $\forall f \in X, t > 0$,*

$$(M_{p,q}f)^*(t) \leq \frac{C}{t^{1/p}} \left(\int_0^t f^*(s)^q s^{q/p-1} ds \right)^{1/q}.$$

We now focus on the validity of the reverse inequalities which correspond to those stated in the previous Corollary. We remark that, as we mentioned in the introduction, under the conditions of **i)**, the corresponding reverse inequality cannot be true. Our next completes this result by showing that the reverse inequality in **ii)** is not true either. Since there is no loss of generality we shall work in dimension one.

Theorem 3. *There exists a function f defined on $\mathbb{R}$ for which $\int_0^t f^*(x)x^{1/p-1}dx = \infty$ for all $t > 0$ while $(M_{p,1}f)^*(t) < \infty$, for all $t > 0$.*

Proof. Let

$$h(x) = \sum_{k=1}^{\infty} \frac{2^{k/p}}{k} \chi_{[2^{-k}, 2^{-k+1})}(x), \ x > 0.$$

Since

$$h(x) \geq C \frac{x^{-1/p}}{|\log x|}, \ 0 < x < 1/2,$$

it is clear that

$$\int_0^t h(x)x^{1/p-1}dx = +\infty, \forall t > 0.$$

Now, we want to distribute the values of $h(x)$ in a convenient form. Let $A_1 = [a_1, b_1] = [0, 2^{-1}]$, $A_2 = [a_2, b_2] = [b_1 + 2^2, b_1 + 2^2 + 2^{-2}]$ and, in general, $A_k = [a_k, b_k] = [b_{k-1} + 2^k, b_{k-1} + 2^k + 2^{-k}]$, $k \in \mathbb{N}$. Let

$$f = \sum_{k=1}^{\infty} \frac{2^{k/p}}{k} \chi_{A_k}.$$

It is clear that the rearrangement of f is the function h. In order to estimate its maximal function, we need the following facts which have straightforward proofs (in what follows Q stands for an interval in $\mathbb{R}$):

i) If $Q \cap A_i \neq \emptyset$ for $i = k, k+1, \dots l$, $l > k$, then

$$\frac{\|f\chi_Q\|_{p,1}}{|Q|^{1/p}} \leq \frac{2^{l/p}}{l} \frac{\|\chi_{A_k \cup \dots \cup A_l}\|_{p,1}}{2^{l/p}} \leq \frac{1}{l}.$$

ii) Let $b_k < x < a_{k+1}$. If $Q = [z, x]$ with $z \in A_k$, then

$$\frac{\|f\chi_Q\|_{p,1}}{|Q|^{1/p}} \leq \frac{1}{k|x - a_k|^{1/p}}.$$

iii) Let $b_k < x < a_{k+1}$. If $Q = [x, y]$ with $y \in A_{k+1}$, then

$$\frac{\|f\chi_Q\|_{p,1}}{|Q|^{1/p}} \le \frac{1}{(k+1)|b_{k+1} - x|^{1/p}}.$$

iv)

$$x - a_k < 1 \Longleftrightarrow \frac{1}{k|x - a_k|^{1/p}} > \frac{1}{k}.$$

$$x - a_k < \frac{(k+1)^p}{(k+1)^p + k^p}(b_{k+1} - a_k) \Longleftrightarrow \frac{1}{k|x - a_k|^{1/p}} > \frac{1}{(k+1)|b_{k+1} - x|^{1/p}}.$$

$$x - a_k < b_{k+1} - a_k - \frac{k^p}{2^k(k+1)^p} \Longleftrightarrow \frac{1}{(k+1)|b_{k+1} - x|^{1/p}} < \frac{1}{k}.$$

From facts i)-iv), it is easy to see that for k large enough, the following estimates hold

$$M_{p,1}f(x) = \frac{2^{k/p}}{k}, \ x \in A_k,$$

$$M_{p,1}f(x) = \frac{1}{k|x - a_k|^{1/p}}, \ b_k < x < a_k + 1,$$

$$M_{p,1}f(x) \le \frac{1}{k}, \ a_k + 1 < x < b_{k+1} - \frac{k^p}{2^k(k+1)^p},$$

$$M_{p,1}f(x) \le \frac{1}{(k+1)|b_{k+1} - x|^{1/p}}, \ b_{k+1} - \frac{k^p}{2^k(k+1)^p} < x < a_{k+1}.$$

Now, given $s > 0$, we choose k_0 such that $sk_0 > 1$. From above estimates, if $Mf(x) > s$ and $x > b_{k_0}$ we have that

$$x \in \bigcup_{k > k_0} \left(A_k \cup (b_k, b_k + \frac{1}{k^p s^p}) \cup (a_{k+1} - \frac{1}{(k+1)^p s^p}, a_{k+1}) \right).$$

On the other hand, if we take $y_0 < 0$ such that

$$\sum_{k=1}^{k_0} \frac{2^{k/p}}{k(|y_0| + 2^1 + \cdots + 2^k)^{1/p}} < s,$$

then $M_{p,1}f(x) \le M_{p,1}f(y_0) < \frac{1}{k} < s$, $\forall x < y_0$. In other words, we have obtained the estimate

$$\lambda_{M_{p,1}f}(s) \le |b_{k_0} - y_0| + \sum_{k_0+1}^{\infty} \left(\frac{1}{2^k} + \frac{1}{s^p k^p} + \frac{1}{s^p(k+1)^p} \right) < \infty.$$

Moreover, $\lim_{s\to\infty} \lambda_{M_{p,1}f}(s) = 0$, and therefore

$$(M_{p,1}f)^*(t) < \infty, \ \forall t > 0.$$

Remark. The counterexample given above can be easily extended to the case $1 \le q < p$ since we have

$$M_{p,q}f(x) = (M_{r,1}(f^q))^{1/q}, \quad r = p/q.$$

2. We recall that a mapping τ from $\mathbb{R}^n$ into $\mathbb{R}^n$ is said to be a *measure-preserving transformation* if, whenever A is a Borel subset of $\mathbb{R}^n$, the set $\tau^{-1}(A)$ is also Borel set, and $|\tau^{-1}(A)| = |A|$.

Theorem 3. *Let X be a r.i. space. Then, there exists an absolute constant $C > 0$ such that for any function $f \in X + L^\infty$,*

$$c \inf_\tau (M_X(f \circ \tau))^*(t) \leq \frac{1}{\phi(t)} \|f^* \chi_{[0,t]}\|_X \leq C \sup_\tau (M_X(f \circ \tau))^*(t)$$

for all $t > 0$, where τ runs through all measure-preserving transformations.

Proof. We begin by proving the second inequality. Fix t and f. We can suppose that $\alpha = \frac{1}{\Phi(t)} \|f^* \chi_{[0,t]}\|_X > 0$. Without loss of generality, we can also suppose that f is Borel measurable. Then, there exists a Borel set B in $\mathbb{R}^n$ such that $|B| = t$ and satisfying

$$\{x \in \mathbb{R}^n; |f(x)| > f^*(t)\} \subseteq B \subseteq \{x \in \mathbb{R}^n; |f(x)| \geq f^*(t)\}.$$

Now, given a cube Q_0 with measure t it is always possible to find a bijective measure-preserving transformation τ such that $\tau(Q_0) = B$ (except perhaps null measurable subsets of Q and B) (This statement can be readily proved using [R], pgs, 315, Th. 15.2, Th. 15.13, see also [W] for the one dimensional case.) That is, we can find a measure-preserving transformation τ (which, of course, depends on f and t) such that

$$(f \circ \tau)\chi_{Q_0} = f\chi_B \quad a.e.$$

Then,

$$\frac{\|(f \circ \tau)\chi_{Q_0}\|_X}{\|\chi_{Q_0}\|_X} = \frac{\|f\chi_B\|_X}{\Phi(t)} = \frac{\|f^*\chi_{[0,t]}\|_X}{\Phi(t)}.$$

If we take $Q_1 = 2Q_0$, the cube with equal center and double the side of Q_0, then since $\Phi(|Q_1|) \leq 2^n \Phi(t)$, we have

$$\frac{\|(f \circ \tau)\chi_{Q_1}\|_X}{\|\chi_{Q_1}\|_X} \geq \frac{\alpha}{2^n} > \frac{\alpha}{2.2^n}$$

Therefore, $M_X(f \circ \tau)(x) > \frac{\alpha}{2.2^n}$ for all $x \in Q_1$ and

$$|\{s \in (0, \infty); (M_x(f \circ \tau))^*(s) > \frac{\alpha}{4.2^n}\}| \geq |Q_1| = 2^n |Q_0| > t.$$

Thus,

$$(M_X(f \circ \tau))^*(t) > \frac{\alpha}{4.2^n},$$

and the second inequality is proved.

It remains to prove the first inequality. Let $t > 0$ and suppose, as we may, that $\alpha = \frac{1}{\Phi(t)} \|f^* \chi_{[0,t]}\|_X > 0$ (since otherwise $f = 0$). By repeating the arguments as before we find the same Q_0, Q_1 and τ.

We are going to estimate $M_X(f \circ \tau)(x)$ for $x \notin Q_1$. Let Q a cube such that $x \in Q$. If $Q \cap Q_0 = \emptyset$ then

$$\frac{\|(f \circ \tau)\chi_Q\|_X}{\|\chi_Q\|_X} \leq f^*(t) \leq \frac{\|f^*\chi_{[0,t]}\|_X}{\Phi(t)} = \alpha.$$

If $Q \cap Q_0 \neq \emptyset$ then there exists a cube $\bar{Q} \subseteq Q \cap Q_1$ such that $|Q_0| = |\bar{Q}|$. Thus

$$\begin{aligned}\frac{\|(f \circ \tau)\chi_Q\|_X}{\|\chi_Q\|_X} &\leq \frac{\|(f \circ \tau)\chi_{Q\setminus Q_0}\|_X}{\|\chi_Q\|_X} + \frac{\|(f \circ \tau)\chi_{Q\cap Q_0}\|_X}{\|\chi_Q\|_X} \\ &\leq f^*(t)\frac{\|\chi_{Q\setminus Q_0}\|_X}{\|\chi_Q\|_X} + \frac{\|(f \circ \tau)\chi_{Q_0}\|_X}{\|\chi_{Q_0}\|_X} \\ &\leq f^*(t) + \alpha \leq 2\alpha.\end{aligned}$$

Therefore,

$$|\{x \in \mathbb{R}^n; M(f \circ \tau)(x) > 3\alpha\}| \leq |Q_1| = 2^n t,$$

and so

$$(M(f \circ \tau))^*(2^n t) \leq 3\alpha.$$

In particular, if we apply the preceding argument to $t_0 = t/2^n < t$ we find τ_0 for which

$$\begin{aligned}(M(f \circ \tau_0))^*(t) = (M(f \circ \tau_0))^*(2^n t_0) &\leq 3\frac{1}{\Phi(t_0)}\|f^*\chi_{[0,t_0]}\|_X \\ &\leq 3\frac{2^n}{\Phi(t)}\|f^*\chi_{[0,t]}\|_X = 3.2^n\alpha,\end{aligned}$$

and the desired inequality follows by taking infimum.

Corollary 2. *Let $p \in (1, +\infty)$, then*

i) *If $1 \leq q \leq p$,*

$$\sup_\tau (M_{p,q}(f \circ \tau))^*(t) \sim \frac{1}{t^{1/p}}\left(\int_0^t f^*(s)^q s^{q/p-1} ds\right)^{1/q}$$

ii) *If $1 < p \leq q \leq \infty$,*

$$\inf_\tau (M_{p,q}(f \circ \tau))^*(t) \sim \frac{1}{t^{1/p}}\left(\int_0^t f^*(s)^q s^{q/p-1} ds\right)^{1/q}.$$

Proof.

i) One inequality follows directly from the Corollary to Theorem 2. To prove the reverse inequality we note that since f and $f \circ \tau$ have the same distribution $M_X(f \circ \tau)$ is a bounded operator from X into $M^*(X)$ and from L^∞ into L^∞. Therefore, (see proof of theorem 1)

$$(M_{p,q}(f \circ \tau))^*(t) \leq C\frac{1}{t^{1/p}}\left(\int_0^t f^*(s)^q s^{q/p-1} ds\right)^{1/q},$$

and the result follows taking the supremum over all τ.

ii) As in the previous proof one inequality comes again from Theorem 2, while the other follows from

$$(M_{p,q}(f \circ \tau))^*(t) \geq C\frac{1}{t^{1/p}}\left(\int_0^t (f \circ \tau)^*(s)^q s^{q/p-1} ds\right)^{1/q}$$

where the constant C is independent of τ (see proof of Theorem 2).

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