REVERSE HÖLDER INEQUALITIES AND INTERPOLATION

J. BASTERO, M. MILMAN, AND F. J. RUIZ

ABSTRACT. We present new methods to derive end point versions of Gehring's Lemma using interpolation theory. We connect reverse Hölder inequalities with Maurey-Pisier extrapolation and extrapolation theory.

1. Introduction

It has been known for a long time that there exists a strong connection between the theory of weighted norm inequalities for classical operators and interpolation theory. However, one feels that there are still many basic questions that remain open. In recent work we have been exploring the interaction between weighted norm inequalities and interpolation theory (cf. [1], [2], [3], [16]). In our work we have found that ideas and methods from one field often lead to new ideas and results in the other. In particular in [1] we have shown a version of the extrapolation theorem of Rubio de Francia (an important result from the theory of weighted norm inequalities cf. [10]) in the context of the real method of interpolation, while at same time observing that the extrapolation methods of [15] yield extrapolation theorems in the theory of weighted norm inequalities. Moreover, these developments led to the study of certain classes of weights in connection with interpolation theory and function spaces (cf. [1], [2]). In [3] and [1] we develop new techniques to prove reverse type Hölder inequalities for certain types

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of weights and in [16] we show an extension of Gehring's Lemma via Holmstedt's formula and differential inequalities.

In this paper we continue to develop these connections. First we consider some limiting cases of reverse Hölder inequalities which are important in PDE's and have been recently studied by R. Fefferman [7] and R. Fefferman, C. Kenig and J. Pipher [8]. More precisely we give new approaches to work by R. Fefferman [7] on a limiting case of Gehring's Lemma. Our method is based on the idea of using maximal functions and rearrangement inequalities to reformulate the problem as an inverse reiteration theorem (cf. [16]). Once the problem has been reformulated in this fashion we can also prove Fefferman's result through the use the iteration method developed in [3] and [1]. Indeed, using this method we obtain a somewhat sharper result in as much as we get a more precise estimate of the improvement obtained in the Gehring type Lemma. (We note, however, that neither of our methods can be expected to give sharp estimates on the improvement in the index of integrability due to the constants that we accumulate to reformulate the problem.)

A second problem we treat in this paper is motivated both by the applications of weighted norm inequalities to PDE's and some problems and methods from operator theory. In the applications to PDE's it is important to consider reverse Hölder inequalities where the cubes involved in the estimates maybe dilations one of the other. We show that if we reinterpret these conditions in terms of probability measures we are in a situation that is also considered in functional analysis. In fact in this fashion we connect reverse Hölder conditions, the extrapolation method of Maurey-Pisier [19] and extrapolation theory in the sense of [15], [17]. Indeed we show that the Maurey-Pisier method can be incorporated to the general theory of extrapolation of [15] through the introduction of a suitable extrapolation functor. Conversely, the method also applies in the realm of PDE's and "cubes" (cf. [13]).

It has not been our purpose here to prove the most general results but to illustrate the new methods arising from interactions between interpolation theory and the theory of weighted norm inequalities. We shall consider other applications and interactions elsewhere.

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2. Reverse Hölder's Inequalities and Reiteration

Let us start recalling some well known facts from the theory of weighted norm inequalities. As a general reference we use [10]. Let Q_0 be a fixed cube with sides parallel to the coordinate axes and let w be a positive measurable function defined on Q_0. We shall say that w satisfies a Gehring condition (or a reverse Hölder inequality) if there exists $p > 1$, and a constant $c > 0$, such that for every cube $Q \subset Q_0$, with sides parallel to the coordinate axes, we have

$$\left\{ \frac{1}{|Q|} \int_Q w^p(x)dx \right\}^{1/p} \leq c \frac{1}{|Q|} \int_Q w(x)dx. \tag{2.1}$$

In this case we shall write $w \in RH_p$. A well known result obtained by Gehring [11] states that if $w \in RH_p$ then w satisfies a better integrability condition, namely for sufficiently small $\varepsilon > 0$, and $q = p + \varepsilon$, we have for every cube $Q \subset Q_0$,

$$\left\{ \frac{1}{|Q|} \int_Q w^q(x)dx \right\}^{1/q} \leq c \left\{ \frac{1}{|Q|} \int_Q w^p(x)dx \right\}^{1/p}.$$

In other words, Gehring's Lemma states that

$$w \in RH_p \Longrightarrow \exists \varepsilon > 0 \text{ s.t. } w \in RH_{p+\varepsilon}.$$

It is not difficult to extend Gehring's Lemma by means of replacing "dx" by a measure of the form $d\mu(x) = h(x)dx$, as long as this measure satisfies a doubling condition. In such case we should, of course, also consider averages with respect to this measure, and replace $|Q|$ by $\mu(Q)$. Let us denote the corresponding classes $RH_p(d\mu)$.

Let us also recall the definition of the Muckenhoupt A_p classes, which control the weighted L^p norm inequalities for the maximal operator of Hardy and Littlewood, and play a central role in the theory of weights. For $1 < p < \infty$, we say that $w \in A_p$ if we have

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty.$$

Note that $p' = p/(p-1), 1 - p' = -1/(p-1)$, therefore we see that $w \in A_p$ implies that for all $Q \subset Q_0$,

$$|Q|^{-p'} \left(\int_Q w(x)dx \right)^{p'-1} \int_Q w(x)^{1-p'}dx \leq c.$$

Consequently, if we write $|Q| = \int_Q w(x)w(x)^{-1}dx, d\mu = w(x)dx$, we get

$$\left(\frac{1}{\mu(Q)}\int_Q \left(w(x)^{-1}\right)^{p'} d\mu\right)^{1/p'} \leq c\frac{1}{\mu(Q)}\int_Q w^{-1}(x)d\mu.$$

We have shown that: $w \in A_p \Rightarrow w^{-1} \in RH_{p'}(d\mu)$. Therefore, by the weighted version of Gehring's Lemma, we can find $\varepsilon > 0$ such that

$$w^{-1} \in RH_{p'+\varepsilon}(d\mu),$$

and translating back in terms of A_p conditions we readily get

$$w \in A_p \Longrightarrow \exists\varepsilon > 0 \text{ s.t. } w \in A_{p-\varepsilon}.$$

This last property plays a central role in the theory of A_p weights. Let us also define the A_∞ condition by

$$A_\infty = \bigcup_{p>1} A_p.$$

There is a detailed study of A_∞ and its relationship to RH_p conditions in the literature. In particular it is known that (cf. [6])

$$A_\infty = \bigcup_{p>1} RH_p.$$

The usual proofs of Gehring's Lemma involve the use of Calderón-Zygmund decompositions and the scale structure of L^p spaces. However, only recently we observed in [16] an explicit connection of Gehring's Lemma to interpolation theory. More precisely, it was shown in [16] that Gehring's Lemma can be interpreted as an inverse type of reiteration theorem valid in the general context of real interpolation spaces. In particular a new proof of Gehring's Lemma was then derived via Holmstedt's formula! Here is the statement of the result

Theorem 2.1. *Let (A_0, A_1) be an ordered pair of Banach spaces (i.e. $A_1 \subset A_0$) and suppose that $f \in A_0$ is such that for some constant $c > 1$, $\theta_0 \in (0,1)$, $1 \leq p < \infty$, we have for all $t \in (0,1)$,*

$$K(t, f; A_{\theta_0,p;K}, A_1) \leq ct\frac{K(t^{1/(1-\theta_0)}, f; A_0, A_1)}{t^{1/(1-\theta_0)}}. \tag{2.2}$$

Then, there exists $\theta_1 > \theta_0$, such that for $q \geq p$, $0 < t < 1$, we have

$$K(t, f; A_{\theta_1,q;K}, A_1) \approx t\frac{K(t^{1/(1-\theta_1)}, f; A_0, A_1)}{t^{1/(1-\theta_1)}}. \tag{2.3}$$

In order to formulate Gehring's Lemma in this fashion we observe that if define the local maximal operators of Hardy-Littlewood by

$$M_q w(x) = \sup_{Q \subset Q_0, x \in Q} \left(\frac{1}{|Q|} \int_Q w(u)^q du \right)^{1/q},$$

where $q \in [1, \infty)$, then $w \in RH_q$ implies the existence of $c > 0$ such that

$$M_q w \leq c M w. \tag{2.4}$$

Taking rearrangements in (2.4), and using the well known fact (cf. [14], [4]) that

$$(Mw)^* (t) \approx \frac{1}{t} \int_0^t w^*(s) ds, \; 0 < t < |Q_0|, \tag{2.5}$$

we see that (2.4) implies the following rearrangement inequality

$$\left(\frac{1}{t} \int_0^t w^*(s)^q ds \right)^{1/q} \leq \frac{c}{t} \int_0^t w^*(s) ds, \; 0 < t < |Q_0|.$$

Next, a well known formula for the K functionals for the pairs (L^p, L^∞), $p \in (0, \infty)$, (cf. [5]) can be used to show that the previous inequality takes the form

$$K(t, w; A_{\theta_0, q; K}, A_1) \leq ct \frac{K(t^{1/(1-\theta_0)}, w; A_0, A_1)}{t^{1/(1-\theta_0)}},$$

with $A_0 = L^1, A_1 = L^\infty, \theta_0 = 1 - 1/q$. Gehring's Lemma can be readily derived from this estimate, in fact we shall consider in detail the mechanism involved in the proof in the next section where, moreover, we consider an end point version of these results.

The proof of the theorem is given in [16] and is based on Holmstedt's formula and some elementary differential inequalities associated with it.

3. An End Point Version of Gehring's Lemma

In this section we consider a limiting case of Gehring's Lemma due to R. Fefferman which plays an interesting role in some problems in PDE's (cf. [7] and [8] for more on this). Our approach is entirely different and based on interpolation theory. In fact our final result will a limiting version of Theorem 2.1.

We would like to let $q \to 1$ in the assumptions of Gehring's Lemma: this leads us to define the classes RH_{LLogL} as follows. We shall say that

$w \in RH_{LLogL}$ if there exists a constant $c > 0$, such that for every cube $Q \subset Q_0$, with sides parallel to the coordinate axes, we have

$$\|w\|_{L(LogL)(Q,\frac{dx}{|Q|})} \le c\frac{1}{|Q|}\int_Q w(x)dx, \tag{3.1}$$

where

$$\|w\|_{L(LogL)(Q,\frac{dx}{|Q|})} = \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q \frac{w(x)}{\lambda}(1+\log^+ \frac{w(x)}{\lambda})dx \le 1\}.$$

It follows readily that

$$A_\infty = \bigcup RH_p \subset RH_{LLogL}.$$

We now show (cf. [7]) that we actually have

$$A_\infty = RH_{LLogL}.$$

At this point we need to introduce a maximal function associated with $L(LogL)$. Define,

$$M_{L(LogL)}w(x) = \sup_{Q\subset Q_0, x\in Q} \|w\|_{L(LogL)(Q,\frac{dx}{|Q|})}.$$

It is known, and not difficult to see (cf. [20]), that

$$M_{L(LogL)}w \approx M(Mw). \tag{3.2}$$

Therefore, if $w \in RH_{LLogL}$, we see that

$$M(Mw)(x) \le cMw(x). \tag{3.3}$$

We shall show that (3.3) implies the existence of $q > 1$ such that

$$\left(\frac{1}{|Q_0|}\int_{Q_0} w^q(x)dx\right)^{1/q} \le c\frac{1}{|Q_0|}\int_{Q_0} w(x)dx. \tag{3.4}$$

Since this argument can be applied to any fixed subcube $Q \subset Q_0$ simply by considering localized maximal functions with respect to Q, we see that

$$w \in RH_{L(LogL)} \Rightarrow w \in \bigcup_{p>1} RH_p.$$

We start by rearranging the inequality (3.3), then, using (2.5), we get

$$(M(Mw))^*(t) \le c\,(Mw)^*(t)$$

$$\frac{1}{t}\int_0^t (Mw)^*(s)ds \le c\,(Mw)^*(t)$$

$$P^{(2)}w^*(t) \le cPw^*(t), \tag{3.5}$$

where $Pf(t) = \frac{1}{t}\int_0^t f(s)ds$, and $P^{(2)}f(t) = P(Pf)(t) = \frac{1}{t}\int_0^t f(s)\log\frac{t}{s}ds$.

Now if we take into account the fact that $K(s, f, L^1, L^\infty)/s = Pf^*(s)$, we can also rewrite the previous estimate as

$$\int_0^t K(s, w; L^1, L^\infty)\frac{ds}{s} \leq cK(t, w; L^1, L^\infty). \tag{3.6}$$

Observe that (3.6) implies that there exists $\gamma \in (0, 1)$ such that

$$\frac{d}{dt}\log(\int_0^t K(s, w; L^1, L^\infty)\frac{ds}{s}) = \frac{K(t, w; L^1, L^\infty)/t}{\int_0^t K(s, w; L^1, L^\infty)\frac{ds}{s}} \geq \frac{\gamma}{t}.$$

Therefore, if $x < y$, we see, integrating from x to y, that

$$\log\left(\frac{\int_0^y K(s, w; L^1, L^\infty)\frac{ds}{s}}{\int_0^x K(s, w; L^1, L^\infty)\frac{ds}{s}}\right) \geq \log(\frac{y}{x})^\gamma,$$

which leads to

$$\left(\frac{\int_0^y K(s, w; L^1, L^\infty)\frac{ds}{s}}{\int_0^x K(s, w; L^1, L^\infty)\frac{ds}{s}}\right) \geq (\frac{y}{x})^\gamma.$$

Consequently we obtain

$$y^{-\gamma}\int_0^y K(s, w; L^1, L^\infty)\frac{ds}{s} \geq x^{-\gamma}\int_0^x K(s, w; L^1, L^\infty)\frac{ds}{s}.$$

Note that by assumption

$$y^{-\gamma}\int_0^y K(s, w; L^1, L^\infty)\frac{ds}{s} \leq cy^{-\gamma}K(y, w; L^1, L^\infty),$$

moreover, since $K(s, f; L^1, L^\infty))/s$ decreases, we have

$$x^{-\gamma}\int_0^x K(s, w; L^1, L^\infty)\frac{ds}{s} \geq x^{-\gamma}K(x, w; L^1, L^\infty).$$

Thus, the function $x \to x^{-\gamma}K(x, w; L^1, L^\infty)$ is essentially increasing. We now claim that for some $\theta \in (0, 1)$ we have

$$\int_0^t s^{-\theta}K(s, w; L^1, L^\infty)\frac{ds}{s} \leq ct^{-\theta}K(t, w; L^1, L^\infty). \tag{3.7}$$

Assuming the validity of our claim and combining (3.7) with the following form of Holmstedt's formula

$$K(t, f; \bar{A}_{\theta,1;K}, A_1) \approx \int_0^{t^{1/(1-\theta)}} s^{-\theta}K(s, f; A_0, A_1)\frac{ds}{s}, \tag{3.8}$$

we see that the assumptions of Theorem 2.1 are verified, and consequently we deduce the existence $\theta_1 \in (0, 1)$ such that for all $q \geq 1$ we have

$$K(t,w;(L^1,L^\infty)_{\theta_1,q;K},L^\infty)\approx t\frac{K(t^{1/(1-\theta_1)},w;L^1,L^\infty)}{t^{1/(1-\theta_1)}}.$$

Selecting $1/q=1-\theta_1$, and translating back, we obtain

$$\left(\frac{1}{t}\int_0^t w^*(s)^q ds\right)^{1/q}\leq c\frac{1}{t}\int_0^t w^*(s)ds,\ 0<t<|Q_0|\,. \tag{3.9}$$

The inequality (3.9) applied to $t=|Q_0|$ gives

$$\begin{aligned}\left(\frac{1}{|Q_0|}\int_{Q_0} w(x)^q dx\right)^{1/q} &= \left(\frac{1}{|Q_0|}\int_0^{|Q_0|} w^*(s)^q ds\right)^{1/q}\\ &\leq \frac{c}{|Q_0|}\int_0^{|Q_0|} w^*(s)ds\\ &= \frac{c}{|Q_0|}\int_{Q_0} w(x)dx\end{aligned}$$

as desired.

It remains to establish (3.7). We simply pick $\theta\in(0,\gamma)$, then using the fact that $x^{-\gamma}K(x,w;L^1,L^\infty)$ is essentially increasing we find that

$$\begin{aligned}\int_0^t s^{-\theta}K(s,w;L^1,L^\infty)\frac{ds}{s} &= \int_0^t s^{-\theta+\gamma}s^{-\gamma}K(s,w;L^1,L^\infty)\frac{ds}{s}\\ &\leq ct^{-\gamma}K(t,w;L^1,L^\infty)\int_0^t s^{-\theta+\gamma}\frac{ds}{s}\\ &= ct^{-\gamma}K(t,w;L^1,L^\infty)t^{\gamma-\theta}\\ &= ct^{-\theta}K(t,w;L^1,L^\infty),\end{aligned}$$

as we wished to show.

The only part of the argument where we used specific information about the pair (L^1,L^∞) was in order to translate (back and forwards) the original problem. We have thus established an end point version of Theorem 2.1 which corresponds to $\theta_0=0$.

Theorem 3.1. *Let (A_0,A_1) be an ordered pair of Banach spaces (i.e. $A_1\subset A_0$) and suppose that $f\in A_0$ is such that for some constant $c>1$ we have $\forall t\in(0,1)$,*

$$\int_0^t K(s,f;A_0,A_1)\frac{ds}{s}\leq cK(t,f;A_0,A_1).$$

Then there exists $\theta \in (0,1)$*, such that for* $q \geq 1$*,* $0 < t < 1$*, we have*

$$K(t, f; A_{\theta,q;K}, A_1) \approx t\frac{K(t^{1/(1-\theta)}, f; A_0, A_1)}{t^{1/(1-\theta)}}.$$

It is interesting to note, in comparing the assumptions in the previous theorem with the ones in theorem 2.1, that the right formulation was obtained by replacing the hypothesis of theorem 2.1 using Holmstedt's formula $K(t, f; \bar{A}_{\theta_0,1;K}, A_1) \approx \int_0^{t^{1/(1-\theta_0)}} s^{-\theta_0} K(s, f; A_0, A_1)\frac{ds}{s}$, and then letting formally $\theta_0 \to 0$ to derive the new hypothesis. We note that in [12] there is also an extension of Holmstedt's formula for the case when $\theta = 0$ which in our context is related to a different way of defining maximal functions using Orlicz functions. For more on the relationship between reiteration formulae for extrapolation spaces and generalized Reversed Hölder inequalities see [18].

4. The Iteration Method

In this section we rederive the end point version of Gehring's Lemma discussed in the previous section using a completely different method. The method we employ here was developed by the authors to attack some problems connected with the application of weighted norm inequalities in interpolation theory in [3] and [1]. It is based on the iteration of inequalities and it probably has its roots in the early work of Gagliardo [9].

Let us then assume again that w is such that for some constant c we have

$$M(Mw) \leq cMw.$$

Rearranging, we may take as our starting point the estimate (cf. (3.5) above)

$$P^{(2)}w^*(t) \leq cPw^*(t).$$

Now we iterate applying repeatedly the operator P to both sides of the inequality and get

$$P^{(n)}w^*(t) \leq c^{n-1}Pw^*(t), n = 2, 3, ...$$

Pick $\varepsilon > 0$ such that $\varepsilon c < 1$ and multiply the corresponding n^{th} inequality by $\varepsilon^{n-1}, n = 2, 3, ...$ Summing we get

$$\sum_{n>1} \varepsilon^n P^{(n)} w^*(t) \leq \left(\sum_{n=1} \varepsilon^n c^n\right) Pw^*(t). \tag{4.1}$$

Since

$$P^{(n)}w^*(t) = \frac{1}{(n-1)!}\frac{1}{t}\int_0^t w^*(s)\left(\log\frac{t}{s}\right)^{n-1} ds,$$

we see that

$$\sum_{n>1}\varepsilon^n P^{(n)}w^*(t) = \sum_{n>1}\varepsilon^n \frac{1}{(n-1)!}\frac{1}{t}\int_0^t w^*(s)\left(\log\frac{t}{s}\right)^{n-1} ds$$

$$= \frac{1}{t}\int_0^t w^*(s)\varepsilon\sum_{n>1}\frac{1}{(n-1)!}\left(\log\left(\frac{t}{s}\right)^{\varepsilon}\right)^{n-1} ds$$

$$= \frac{\varepsilon}{t}\int_0^t w^*(s)[\left(\frac{t}{s}\right)^{\varepsilon} - 1]ds.$$

Inserting this back in (4.1) we obtain, for a suitable constant C,

$$\frac{1}{t}\int_0^t w^*(s)\left(\frac{t}{s}\right)^{\varepsilon} ds \leq CPw^*(t).$$

Combining the last estimate with the elementary inequality (cf. [5])

$$\left(\frac{1}{t}\int_0^t w^*(s)^{1/(1-\varepsilon)}ds\right)^{1-\varepsilon} \leq t^{1-\varepsilon}\int_0^t w^*(s)s^{-\varepsilon}ds,$$

we obtain, with $q = 1/(1-\varepsilon)$,

$$\left(\frac{1}{t}\int_0^t w^*(s)^q ds\right)^{1/q} \leq cPw^*(t).$$

The argument we gave in the previous section (right at the point where (3.9) was established) can be now used to obtain the desired result:

$$\left(\frac{1}{|Q_0|}\int_{Q_0} w(x)^q dx\right)^{1/q} \leq \frac{c}{|Q_0|}\int_{Q_0} w(x)dx.$$

We note that the argument just presented is contained in [1] in connection with reversed Hölder type conditions for weights in the M^1 class, which are precisely the weights that satisfy the condition (3.5).

5. Gehring's lemma and Maurey-Pisier Extrapolation

It is also natural to ask what happens if in (2.1) we fix $p > 1$ and consider the improvement to this inequality that would result from lowering the exponent on the right hand side. Namely, we consider if the validity of (2.1) implies that for some $r < 1$, the following inequality holds

$$\left\{ \frac{1}{|Q|} \int_Q w^p(x)dx \right\}^{1/p} \le c \left\{ \frac{1}{|Q|} \int_Q w^r(x)dx \right\}^{1/r}. \tag{5.1}$$

It turns out that this is true, and well known, and in fact the proof is an immediate consequence of Hölder's inequality.

Lemma 5.1. *Suppose that w satisfies the condition (2.1) then $\forall\ r \in (0,1)$, w satisfies (5.1).*

Proof. Given $r \in (0,1)$, choose $\theta \in (0,1)$ such that

$$1 = \frac{1-\theta}{p} + \frac{\theta}{r}.$$

Then, by Hölder's inequality, we have

$$\left\{ \frac{1}{|Q|} \int_Q w^p(x)dx \right\}^{1/p} \le c \frac{1}{|Q|} \int_Q w(x)dx$$

$$\le c \left\{ \frac{1}{|Q|} \int_Q w(x)^p dx \right\}^{(1-\theta)/p} \left\{ \frac{1}{|Q|} \int_Q w(x)^r dx \right\}^{\theta/r}.$$

Therefore, dividing, we find

$$\left\{ \frac{1}{|Q|} \int_Q w^p(x)dx \right\}^{\theta/p} \le c \left\{ \frac{1}{|Q|} \int_Q (w(x))^r \, dx \right\}^{\theta/r},$$

and the result follows. ∎

Conditions of the form (2.1) and (5.1) appear in other contexts in analysis. In these applications it is important to extrapolate, as indicated in Lemma 5.1, even though the cubes appearing on both sides of the hypothesized inequalities may be different. This is, for example, the case that appears commonly in problems in PDE's where cubes on each side of (2.1) are dilations one of the other. For a treatment of problems in this direction we refer to [13]. However, to connect this extrapolation process with other problems in functional analysis it is important to reformulate the problem in a more general context. In fact in the context of (2.1) one could view the conditions arising from PDE applications and consider conditions of the form: $\forall\ \lambda$ probability measure in Ω there exists a probability measure μ such that

$$\left\{ \int_\Omega |w(x)|^p \, d\lambda \right\}^{1/p} \le c \int_\Omega |w(x)| \, d\mu. \tag{5.2}$$

The argument in the previous Lemma would work if we had the *same* measure on both sides of the inequality. In order to arrange to have such a situation we use a method developed by Maurey and Pisier which essentially allows one to replace (5.2) with suitable norms so that the method of Lemma 5.1 can be applied. The process involved is via iteration and could be considered as a "fixed point theorem" for functional spaces.

Let us then consider the Maurey-Pisier method slightly rephrased so that it can be incorporated as a more general argument in the extrapolation theory of Jawerth-Milman. The appropriate extrapolation functor here is the Δ_1 functor. Recall that if $\{X_\gamma\}_{\gamma\in I}$ is a compatible family of Banach spaces we formally let

$$\Delta_1\left(\{X_\gamma\}_{\gamma\in I}\right) = \{f : \sum_{\gamma\in I} \|f\|_{X_\gamma} < \infty\},$$

with

$$\|f\|_{\Delta_1\left(\{X_\gamma\}_{\gamma\in I}\right)} = \sum_{\gamma\in I} \|f\|_{X_\gamma}\,.$$

Here is a typical example of such a construction. Let $\{(X_0^j, X_1^j)\}_{j=1}^\infty$ be a family of Banach pairs, and let us consider the corresponding interpolation spaces $X_\theta^j = F_\theta(X_0^j, X_1^j)$, where $\{F_\theta\}_{\theta\in(0,1)}$ is a family of interpolation functors. Let

$$X(\theta) = \ \Delta_1\left(\{2^{-n} X_\theta^n\}_{n=1}^\infty\right), \quad \theta \in [0,1].$$

A natural assumption on the family of functors $\{F_\theta\}_{\theta\in(0,1)}$ is that for $\theta \in (0,1)$, we have

$$\|f\|_{X_\theta} \le \|f\|_{X_0}^{1-\theta} \|f\|_{X_1}^{\theta}, \tag{5.3}$$

or in other words that the functors are exact of type θ. Then (5.3) persists at the level of the extrapolation spaces:

$$\begin{aligned}\|f\|_{X(\theta)} &= \sum_{n=1}^\infty 2^{-n} \|f\|_{X_\theta} \le \sum_{n=1}^\infty 2^{-n(1-\theta)} \|f\|_{X_0^n}^{1-\theta} 2^{-n\theta} \|f\|_{X_1^n}^{\theta} \\ &\le \left(\sum_{n=1}^\infty 2^{-n} \|f\|_{X_0^n}\right)^{1-\theta} \left(\sum_{n=1}^\infty 2^{-n} \|f\|_{X_1^n}\right)^{\theta} \\ &= \|f\|_{X(0)}^{1-\theta} \|f\|_{X(1)}^{\theta}\,.\end{aligned}$$

This construction allows us to consider the argument of Maurey-Pisier from an interpolation-extrapolation view point. In fact note that

$$\|f\|_{\Delta_1\left(\left\{2^{-n}X_\theta^n\right\}_{n=2}^\infty\right)} \leq 2\,\|f\|_{\Delta_1\left(\left\{2^{-n}X_\theta^n\right\}_{n=1}^\infty\right)}.$$

Therefore if we start with the assumption (5.2) and are given a probability measure λ we construct a sequence of probability measures $\{\lambda_n\}_{n=1}^\infty$, with $\lambda_1 = \lambda$, so that

$$\left\{\int_\Omega |w(x)|^p\,d\lambda_n\right\}^{1/p} \leq c\int_\Omega |w(x)|\,d\lambda_{n+1}, n = 1, 2, ...$$

Applying the Δ_1 functor we get

$$\|w\|_{\Delta_1(\{2^{-n}L^p(d\lambda_n)\}_{n=1}^\infty)} \leq c\,\|w\|_{\Delta_1(\{2^{-n}L^1(d\lambda_n)\}_{n=2}^\infty)}$$

$$\leq 2c\,\|w\|_{\Delta_1(\{2^{-n}L^1(d\lambda_n)\}_{n=1}^\infty)}.$$

At this stage the argument of Lemma 5.1 can be applied verbatim in view of our previous discussion. Indeed, for any $r < 1$, let us write $1 = \frac{1-\theta}{p} + \frac{\theta}{r}$, with $\theta \in (0, 1)$, then

$$\|w\|_{\Delta_1(\{2^{-n}L^1(d\lambda_n)\}_{n=1}^\infty)} \leq \left(\|w\|_{\Delta_1(\{2^{-n}L^p(d\lambda_n)\}_{n=1}^\infty)}\right)^{1-\theta}\left(\|w\|_{\Delta_1(\{2^{-n}L^r(d\lambda_n)\}_{n=1}^\infty)}\right)^{\theta},$$

and thus

$$\|w\|_{\Delta_1(\{2^{-n}L^p(d\lambda_n)\}_{n=1}^\infty)} \leq C\,\|w\|_{\Delta_1(\{2^{-n}L^r(d\lambda_n)\}_{n=1}^\infty)}.$$

We can rephrase this estimate in terms of the original assumptions if we observe that

$$\left\{\int_\Omega |w(x)|^p\,d\lambda\right\}^{1/p} \leq 2\,\|w\|_{\Delta_1(\{2^{-n}L^p(d\lambda_n)\}_{n=1}^\infty)},$$

and

$$\|w\|_{\Delta_1(\{2^{-n}L^r(d\lambda_n)\}_{n=1}^\infty)} = \sum_{n=1}^\infty 2^{-n}\left\{\int_\Omega |w(x)|^r\,d\lambda_n\right\}^{1/r}$$

$$\leq \left\{\sum_{n=1}^\infty 2^{-nr}\int_\Omega |w(x)|^r\,d\lambda_n\right\}^{1/r}$$

$$= \left(\sum_{n=1}^\infty 2^{-nr}\right)^{1/r}\left\{\int_\Omega |w(x)|^r\,\frac{\sum_{n=1}^\infty 2^{-nr}d\lambda_n}{\sum_{n=1}^\infty 2^{-nr}}\right\}^{1/r}$$

$$= c\left(\int_\Omega |w(x)|^r\,d\mu\right)^{1/r},$$

where $\mu = \frac{\sum_{n=1}^{\infty} 2^{-nr} d\lambda_n}{\sum_{n=1}^{\infty} 2^{-nr}}$ is a probability measure on Ω. We have thus obtained the following

Theorem 5.2. *(Maurey-Pisier extrapolation) Suppose w is such that for each probability measure λ on Ω there exists a probability measure μ such that (5.2) holds for a universal constant $c > 0$. Then $\forall r \in (0,1)$ there exists a constant $C > 0$ such that for each probability measure λ there exists a probability measure μ such that*

$$\left\{ \int_{\Omega} |w(x)|^p \, d\lambda \right\}^{1/p} \leq C \left(\int_{\Omega} |w(x)|^r \, d\mu \right)^{1/r}.$$

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Departamento de Matematica, Universidad de Zaragoza, Zaragoza 50009, Spain

E-mail address: jbastero@msf.unizar.es

Department of Mathematics, Florida Atlantic University, Boca Raton, Fl. 33431

E-mail address: milman@acc.fau.edu

Departmento de Matematica, Universidad de Zaragoza, Zaragoza 50009, Spain

E-mail address: fruiz@msf.unizar.es