DUAL QUERMASSINTEGRALS, EXTREMAL POSITIONS AND ISOTROPIC MEASURES

JESÚS BASTERO, JULIO BERNUÉS, AND MIGUEL ROMANCE

Abstract. We consider the optimization of functionals of the form $S \to f(SK)$ where $K \subseteq \mathbb{R}^n$ is a convex body and S is an linear transformation, under different constraints. The functionals involve dual mixed volumes. We review some of the latest results obtained by the authors which include necessary and suficient conditions for a convex body to be in MM^* position, John, Gauss-John positions and others. Such extremal positions are characterized in terms of isotropic properties of measures. We also look at the role of translations and obtain some new results for the affine setting.

1. Introduction and notation

In 2000, [5], the authors introduced a general approach in order to produce isometric descriptions for many relevant positions of convex bodies. They showed how, in many situations, they can be characterized as the solution of a suitable defined optimization program. Those positions have played a significant role in classical convex geometry and in the local theory of Banach spaces and we will recall some of them throughout the paper.

Our first example refers to the classical concept of isotropic position of a (symmetric) convex body. In 1989, [11], it was proved that a symmetric convex body K is in isotropic position (that is, $\int_K \langle x, \theta\rangle^2\, dx$ is independent of $\theta \in S^{n-1}$) if and only if $\int_K |x|^2\, dx \le \int_{SK} |x|^2\, dx, \forall\, S \in SL(n)$.

A second result by Petty, [12] and Giannopoulos and Papadimitrakis, [7], stated: a convex body K has *minimal surface area* in the family $\{SK, S \in SL(n)\}$ if and only if the area measure of K is isotropic (see definitions at the end of the section).

The problem of minimizing the quermassintegrals $\{W_i(SK), S \in SL(n)\}$, $i = 1 \dots n$ was considered in [5]. As particular cases, $i = 1$ corresponds to Petty's problem and $i = n-1$ ($W_{n-1}(K) = c \int_{S^{n-1}} h_K(u)\, d\sigma(u)$, h_K is the support function) to minimizing the mean width. More precisely, it is proved that a "smooth enough" convex body K has minimal mean width (among all volume preserving linear transformations) if and only if the measure $h_K(u)\, d\sigma(u)$ is isotropic. In the same paper [5], optimization

2000 *Mathematics Subject Classification.* 52A40, 46B20, 52A39.

Key words and phrases. dual mixed volumes, optimization, isotropic measures.

The three authors were partially supported by MCYT Grant (Spain) MTM2004-03036, FEDER funds and FP6 Marie Courie Actions, PHD (EU) and by DGA E-12/25.

of the parameter $M(SK) = \int_{S^{n-1}} \rho_{SK}^{-1}(u)\, d\sigma(u), S \in SL(n),\ \rho_K$ the radial function, was also investigated. (When K is centrally symmetric $M(S)$ is the average of the norm defined by K since $\rho_K^{-1}(\cdot) = \|\cdot\|_K$).

A common feature of the cited examples is that the extremum for those functionals is attained at only an isotropic position (for the suitable measure on S^{n-1}).

Lutwak in 1975, [10], introduced the "dual quermassintegrals" $\tilde{W}_i(K), i \in \mathbb{R}$. Their role in the dual Brunn-Minkowski theory mimic that of the quermassintegrals in the classical Brunn-Minkowski theory. While the latter theory is a natural framework for studying projections (shadows) of convex bodies, the former is useful in the study of sections.

Optimization of $\{\tilde{W}_i(SK), S \in SL(n)\}$ was investigated in 2004, [2], which, in particular, includes the case when K is in isotropic position $(i = -2)$ and when it minimizes the mean diameter $(i = n-1)$ or $M(SK)$ $(i = n+1)$.

A related question, also started in [5], is the study of the MM^* position of a (symmetric) convex body that is, the extreme of $M(SK)M((SK)^\circ), S \in GL(n)$, which has important applications in the local theory of Banach spaces. The necessary condition obtained in [5] was proved to be sufficient in 2004, [3], even in a more general context, namely $\tilde{W}_i(SK)\tilde{W}_i((SK)^\circ), S \in GL(n)$, which was also solved for a range of indexes (including $i = n+1$).

We finish this set of examples with the well known John's position. A convex body $K \subset D_n$ (D_n the euclidian ball) is in John's position if it maximizes the function vol SK with constraint $SK \subset D_n, S \in GL(n)$. That is, if it is in a maximal volume position inside D_n. Observe that vol $SK = \tilde{W}_0(SK)$. A related position, introduced in 2000, [6], is the so called Gauss-John position which corresponds to minimizing $\tilde{W}_{n+1}(SK)$ with constraint $SK \subset D_n, S \in GL(n)$. Necessary and sufficient conditions were obtained in [1] for a range of indexes.

As it is mentioned in the abstract, our aim is to review the latest results by the authors [2], [3] and [1] and sketch the main ideas needed in their proof. In the linear case, whenever we include detailed proofs, they correspond to new (not previously published) approaches, otherwise we just refer to those three papers. As we shall see, the necessary conditions depend upon variational arguments which work for every index $i \in \mathbb{R}$. The sufficient conditions depend on the convexity properties of the functional. The arguments vary from simpler Hölder's inequality to more delicate computations involving the Laplace-Beltrami operator. In all cases, a range of indexes remains not completely understood.

The quermassintegrals are translation invariant parameters. But dual quermassintegrals are not, a fact that in this context has already been studied, for instance, in the case of classical John's theorem, or the MM^* position (by Rudelson, [13]). In this paper we also consider the role of translations within this optimization program and obtain new results in the affine setting.

Throughout the paper $K \subset \mathbb{R}^n$ will always denote a convex body with the origin as an interior point. $\langle \cdot, \cdot \rangle$ will denote the standard scalar product, $|\cdot|$ the Euclidean norm in the appropiate dimension, D_n the Euclidean ball and $|\cdot|_n$ the Lebesgue measure on $\mathbb{R}^n$. We recall the following well known notions,

- $h_K(x) = \max\{\langle x, y\rangle, y \in K\}$ is the support function of K at $x \in \mathbb{R}^n$.
- $\rho_K(x) = \max\{\lambda \geq 0, \lambda x \in K\}$ is the radial function of K at $x \in \mathbb{R}^n \setminus \{0\}$.
- The polar body of K is $K^\circ = \{x \in \mathbb{R}^n, \langle x, y\rangle \leq 1, \forall\, y \in K\}$.

We denote by $GL(n)$ the set of $n \times n$ regular real matrices and by $SL(n)$ those of determinant 1.

The following properties will be useful in the sequel. For all $T \in GL(n)$,

- $h_{TK}(x) = h(T^*x)$, where T^* is the transposed matrix of T.
- $\rho_{TK}(x) = \rho_K(T^{-1}x)$.
- $h_K(x) = \rho_{K^\circ}^{-1}(x)$ and $h_{K^\circ}(x) = \rho_K^{-1}(x)$.
- $(TK)^\circ = (T^{-1})^*(K^\circ)$.

Let $i \in \mathbb{R}$ and $K \subset \mathbb{R}^n$. The i-th dual mixed volume (or i-th dual quermassintegral) of K, $\tilde{W}_i(K)$ is defined as

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)d\sigma(u).$$

By using polar coordinates it is easy to see the following alternative formulas,

$$\tilde{W}_i(SK) = \frac{|n-i||\det(S)|}{n}\int_{K(i)} \frac{dx}{|Sx|^i} = \frac{|\det(S)|}{n}\int_{S^{n-1}} \frac{\rho_K^{n-i}(u)}{|Su|^i}\, d\sigma(u),$$

where $K(i) = K$ if $i < n$ and $K(i) = \mathbb{R}^n \setminus K$ if $i > n$. Observe that the case $i = n$ is trivial (and from now on excluded in our results), and $i = 0$ corresponds to the volume.

A (finite) positive Borel measure μ on S^{n-1} is *isotropic* if there exists $c > 0$ such that $\int_{S^{n-1}} \langle u, \theta\rangle^2\, d\mu(u) = c,\ \forall\, \theta \in S^{n-1}$.

For example, K is in isotropic position if and only if $\rho_K^{n+2}(u)\, d\sigma(u)$ is isotropic.

We will make use of the following equivalent definition of isotropic measures: μ is isotropic if for some constant $C > 0$ and all matrices $T \in GL(n)$,

$$\int_{S^{n-1}} \langle u, Tu\rangle\, d\mu(u) = C\,\mathrm{tr}\, T$$

with $\mathrm{tr}\, T$ being the trace of T.

2. Optimization of $\tilde{W}_i(SK)$ for volume preserving linear transformations

The natural questions to study are to maximize $\left\{\tilde{W}_i(SK); S \in SL(n)\right\}$, if $i \in (0,n)$ and to minimize it for $i \notin [0,n]$.

2.1. **Necessary conditions.**

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ such that h_{K° is continuously differentiable. If K is in extremal position for the problem $\{\tilde{W}_i(SK), S \in SL(n)\}$, then*

$$\begin{aligned} \frac{\operatorname{tr} T}{n}\tilde{W}_i(K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u) \end{aligned}$$

for all $T \in GL(n)$.

Proof. We identify every $S \in GL(n)$ with an element in R^{n^2}. Now our hypothesis states that the function $S \to W_i(SK) = \displaystyle\int_{S^{n-1}} \frac{\rho_K^{n-i}(u)}{|Su|^i}\, d\sigma(u)$, under the constraint $\det(S) - 1 = 0$ has an extreme point at the identity $I \in GL(n)$. By using Lagrange's multipliers, the gradient of the function $W_i(SK)$ at I must be proportional to I (the gradient of the determinant function at I) that is,

$$\nabla(\tilde{W}_i(SK))(I) = -i\int_{S^{n-1}} \rho_K^{n-i}(u)(u\otimes u)\, d\sigma(u) = \lambda I, \quad \text{for some} \quad \lambda \in \mathbb{R}$$

where $u \otimes u$ is the matrix $(u_i u_j)$ and $u = (u_1, \dots u_n)$. Take traces to compute λ ($\lambda = -i\tilde{W}_i(K)$) and deduce that the measure $\rho_K^{n-i}(u)\, d\sigma(u)$ is isotropic. This yields the second equality in the statement of the theorem. For the first one, proceed similarly with the formula $W_i(SK) = \displaystyle\frac{1}{n}\int_{S^{n-1}} h_{K^\circ}^{i-n}(S^{-1}(u))\, d\sigma(u)$.

□

Remark 2.2. The result is proved in [2] by using the Taylor expansion of $W_i(SK)$ at I (and h_{K° is needed to be twice continuously differentiable. Similar type of arguments are found in [5].

The two necessary conditions stated above come from the use of variational arguments under the hypothesis of K being in extremal position. But as we shall now see, those two conditions are related *independently* of the extremal problem considered. This fact is important in the analysis of the sufficient conditions. In order to see this, we need some previous results.

We will suppose all functions below to be twice continuously differentiable. Let $f\colon S^{n-1} \to \mathbb{R}$ and let $F\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be its 0-homogeneous extension defined by $F(x) = f(\frac{x}{|x|})$. We denote by $\nabla f = \nabla F|_{S^{n-1}}$, that is, the restriction to S^{n-1} of the gradient of F. In the same way, we define the Laplace-Beltrami operator as $\Delta f = \Delta F|_{S^{n-1}}$, the restriction of the Laplacian.

Let $f, g\colon S^{n-1} \to \mathbb{R}$ and $F, G\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ their 0-homogeneous extensions. By using Green's formula for the Beltrami operator (see for instance [8], pp. 7), we get that

$$\int_{S^{n-1}} G(u)\Delta F(u)\, d\sigma(u) = -\int_{S^{n-1}} \langle \nabla F(u), \nabla G(u)\rangle\, d\sigma(u).$$

Theorem 2.3. *Let $i \in \mathbb{R}$ and $K \subset \mathbb{R}^n$ such that h_{K° is twice continuous differentiable. The following assertions are equivalent,*

(i) For every symmetric matrix $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(ii) For every symmetric matrix $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

Proof. Since T is symmetric, it can be represented by $T = \sum_{j=1}^n \lambda_j \theta_j \otimes \theta_j$ with $\theta_j \in S^{n-1}$ and so it is enough to prove the equivalence for matrices of the form $\theta \otimes \theta$ with $\theta \in S^{n-1}$.

Now, by considering the functions $f(u) = \langle u, \theta\rangle^2/2$ and $g(u) = h_{K^\circ}^{i-n}(u)$ and applying the previous formula we obtain,

$$(n-i)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), (\theta\otimes\theta)(u)\rangle\, d\sigma(u) =$$

$$= \quad n\tilde{W}_i(K) - i\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, (\theta\otimes\theta)(u)\rangle\, d\sigma(u)$$

which yields the result. □

2.2. **Sufficient conditions.** For the indexes $i \in (-\infty, 0)$ and $i \in [n+1, \infty)$ we give a complete characterization.

Theorem 2.4. *Let $K \subset \mathbb{R}^n$ such that h_{K° is twice continuously differentiable. Let $i \in (-\infty, 0)$ or $i \in [n+1, \infty)$. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K) = \min\left\{\tilde{W}_i(SK); S \in SL(n)\right\}$.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iii) For every $T \in GL(n)$ symmetric,

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iv) The measure given by $\rho_K^{n-i}(\cdot)\, d\sigma(\cdot)$ is isotropic in S^{n-1}.

Moreover, K is the unique position, up to orthogonal transformation, that minimizes $\tilde{W}_i(SK)$.

Proof.
The case $i<0$.
(iii)⇒(iv) By Theorem 2.3 we have that the isotropic condition holds for symmetric matrices. Now, for any $T\in GL(n)$, consider the symmetric matrix $(T+T^*)/2$. Straightforward computations yield the result.
(iv)⇒(i) is a consequence of the following fact (see [2]) applied to the measure $d\mu(\cdot)=\rho_K^{n-i}(\cdot)\,d\sigma(\cdot)$:

$$\mu \text{ is isotropic} \iff \int_{S^{n-1}}\frac{d\mu(u)}{|Su|^i}\geq\int_{S^{n-1}}d\mu(u), \forall S\in SL(n) \text{ and } i<0.$$

The case $i\geq n+1$.

We extend the arguments used in [2] for a centrally symmetric K, to a general convex body.
(iii)⇒(i) Let $S\in SL(n)$ which we can assume symmetric and positive definite.

$$\tilde{W}_i(SK)=\frac{1}{n}\int_{S^{n-1}}\rho_{SK}^{n-i}(u)\,d\sigma(u)=\frac{1}{n}\int_{S^{n-1}}h^{i-n}_{S^{-1}(K^\circ)}(u)\,d\sigma(u).$$

By using Hölder's inequality with indexes $p=i-n$ and $q=\frac{i-n}{i-n-1}$ we get

$$\begin{aligned}(\tilde{W}_i(SK))^{\frac{1}{i-n}} \quad &\geq \quad \left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\,d\sigma(u)\right)^{\frac{n-i+1}{i-n}}\\ &\qquad \times\left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i+1}(u)h_{S^{-1}(K^\circ)}(u)d\sigma(u)\right).\end{aligned}$$

Finally, since $\langle\nabla h_{K^\circ}(u),S^{-1}u\rangle\leq h_{(S)^{-1}(K^\circ)}(u)$ for all $u\in S^{n-1}$ (see [14], pp. 40) and $S\in SL(n)$ is positive definite, we get that

$$\begin{aligned}(\tilde{W}_i(SK))^{\frac{1}{i-n}} \quad &\geq \quad \left(\tilde{W}_i(K)\right)^{\frac{n-i+1}{i-n}}\left(\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i+1}(u)\langle\nabla h_{K^\circ}(u),S^{-1}u\rangle\,d\sigma(u)\right)\\ &= \quad \left(\tilde{W}_i(K)\right)^{\frac{n-i+1}{i-n}}\left(\frac{\operatorname{tr}S^{-1}}{n}\tilde{W}_i(K)\right)\\ &\geq \quad \left(\det S^{-1}\right)^{1/n}(\tilde{W}_i(K))^{\frac{1}{i-n}}=(\tilde{W}_i(K))^{\frac{1}{i-n}}\end{aligned}$$

so we obtain the result.
The uniqueness, up to orthogonal transformations, is a consequence of having equality in the inequalities involved.

□

2.3. **The role of translations.**

Observe that the case $\tilde{W}_i(K)$ are not translation invariants (except in the case $i=0$) and so it is natural to ask about optimizing $\{\tilde{W}_i(a+SK), S\in SL(n), a\in\mathbb{R}^n\}$ (of course, we implicitly assume 0 to be interior point of $a+SK$).

We start with the case when K is (centrally) symmetric and prove that the solutions to the affine and the linear problem are the same as it can be easily deduced from the following proposition.

Proposition 2.5. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and $a \in \mathbb{R}^n$. Suppose the origin is an interior point of $a + K$. Then,*

(i) $\tilde{W}_i(a+K) \geq \tilde{W}_i(K)$, *if* $i \leq 0$.
(ii) $\tilde{W}_i(a+K) \leq \tilde{W}_i(K)$, *if* $0 < i < n$.
(iii) $\tilde{W}_i(a+K) \geq \tilde{W}_i(K)$, *if* $n \leq i$.

Proof. We will only proof *(ii)*, as the other cases are similar.

$$\begin{aligned}\tilde{W}_i(a+K) &= \int_K \frac{dx}{|x-a|^i} = \int_0^\infty \left|\{x \in K, \frac{1}{|x-a|^i} > t\}\right|_n dt \\ &= \int_0^\infty \left|K \cap (a + B(0, t^{-1/i}))\right|_n dt = \int_0^\infty \mu_K(a + B(0, t^{-1/i}))dt,\end{aligned}$$

where μ_K is the measure on $\mathbb{R}^n$ with density $\mathcal{X}_K(x)$ and $B(0,t)$ is the euclidean ball of radius t centered in the origin. Brunn-Minkowski inequality states that the measure μ_K is $1/n$-concave. That implies that for any $A \subset \mathbb{R}^n$ symmetric convex set and $a \in \mathbb{R}^n$,

$$\mu_K^{\frac{1}{n}}(A) = \mu_K^{\frac{1}{n}}\left(\frac{(a+A)+(-a+A)}{2}\right) \geq \frac{\mu_K^{\frac{1}{n}}(a+A) + \mu_K^{\frac{1}{n}}(-a+A)}{2} = \mu_K^{\frac{1}{n}}(a+A)$$

and the result follows. □

Remark 2.6. If $A \subset \mathbb{R}^n$ is a symmetric convex set and $a \in \mathbb{R}^n$, the inequality $\mu(A) \geq \mu(a+A)$ holds for any quasi-concave symmetric measure μ on $\mathbb{R}^n$.

For the general case we use the same type of variational arguments as in Theorem 2.1 to produce a necessary condition for K to be in the optimal position. Now, further restrictions on the indexes appear due to differentiability assumptions. More precisely, the result is,

Theorem 2.7. *Let $i \notin [n-1, n]$. Let $K \subset \mathbb{R}^n$ such that h_{K° is continuously differentiable. If K is in extremal position for the problem $\{\tilde{W}_i(a+SK), S \in SL(n), a \in R^n\}$, then*

$$\begin{aligned}\frac{\operatorname{tr} T}{n}\tilde{W}_i(K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\rho_L^i(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u), \quad \forall T \in GL(n)\end{aligned}$$

and $\displaystyle\int_{S^{n-1}} u\ \rho_K^{n-i-1}(u)\, d\sigma(u) = 0.$

Proof. Use the representation $\tilde{W}_i(a+SK) = \frac{|n-i|}{n}\int_{K(i)} \frac{dx}{|a+Sx|^i}$ to differentiate it with respect to the variable $a \in \mathbb{R}^n$ and use the hypothesis that there is a extreme point at $a=0$ (and $S=I$). □

In the particular case $i=-2$ the extra condition reads (after using polar coordinates) $\int_K x dx = 0$, that is, the origin must be the centroid of K.

3. Optimization of $\tilde{W}_i(SK)$ inside the Euclidian ball

The natural questions are to minimize $\{\tilde{W}_i(SK), SK \subset D_n, S \in GL(n)\}$, if $i > n$ and to maximize it for $i < n$. In order to obtain the necessary condition, a variational argument is now more delicate since we have an infinite (although compact) number of constraints, namely $|Sx| \leq 1, \forall x \in K$. This difficulty may be overcome with the help of John's variational theorem, [9], which can be seen as an extension of the Lagrange multiplier theorem:

Theorem 3.1 (John)**.** *Let $\Omega \subset \mathbb{R}^m, \Omega_1 \subset \mathbb{R}^l$ be (non empty) open sets and $S \subset \Omega_1$ compact. Let $F\colon \Omega \to \mathbb{R}$ and $G\colon \Omega \times \Omega_1 \to \mathbb{R}$ be $\mathcal{C}^{1)}$ functions. Let $A = \{x \in \Omega \mid G(x,y) \geq 0, \forall y \in S\}$. If F attains its minimum value at $x_0 \in A$, then there exist $y_1,\dots,y_s \in S$ and $\lambda_0,\lambda_1,\dots,\lambda_s \in \mathbb{R}$ such that*

- $0 \leq s \leq m$ *and* $\lambda_0 \geq 0$, $\lambda_1,\dots,\lambda_s > 0$.
- $G(x_0,y_1) = \cdots = G(x_0,y_s) = 0$.
- *The function* $\Phi(x) = \lambda_0 F(x) - \sum_{j=1}^s \lambda_j G(x,y_j)$ *verifies* $\nabla\Phi(x_0) = 0$.

Theorem 3.2. [1]. *Let $K \subset D_n$ and let i be a real number. If K is in extremal position for the problem $\{\tilde{W}_i(SK), SK \subseteq D_n, S \in GL(n)\}$ then, there exist contact points $w_1,\dots,w_N \in \partial K \cap S^{n-1}$ with $N \leq n(n+1)/2$ and $\lambda_1,\dots,\lambda_N > 0$ with $\sum_{j=1}^N \lambda_j = 1$, such that*

$$\operatorname{tr} T = i\int_{S^{n-1}} \langle Tu,u\rangle\, d\mu_i(u) + (n-i)\sum_{j=1}^N \lambda_j \langle Tw_j, w_j\rangle \tag{3.1}$$

for all $T \in GL(n)$, where $d\mu_i(u)$ is the probability on S^{n-1} with normalized density

$$d\mu_i(u) = \rho_K^{n-i}(u)d\sigma(u)/\int_{S^{n-1}} \rho_K^{n-i}(u)d\sigma(u).$$

Remark 3.3. If we define the real measure $i\mu_i + (n-i)\nu_i$, where ν_i is the suitable discrete measure on S^{n-1} with support on contact points, then we can describe formula 3.1 simply as such a (real) measure being isotropic.

The proof of the sufficient condition uses the ideas of section 2.

Theorem 3.4. [1]. *Assume $K \subseteq D_n$ satisfies the condition (3.1). Then,*

(i) If $i \in [-2,0] \cup [n+1,+\infty)$, K is in extremal position.

(ii) If $i \in (-\infty,-2) \cup (0,n)$ and the measure $d\mu_i$ is isotropic, K is in extremal position.

Moreover, the position of K is unique up to orthogonal transformations.

Remark 3.5. For instance, in the case $i=0$, our results read: For $K \subset D_n$ the following are equivalent:

(i) K is in maximal volume position inside the euclidean ball with respect to linear transformations (that is $|SK| \leq |K|, \forall S \in GL(n)$ such that $SK \subset D_n$).

(ii) ν_0 is isotropic, that is, there exist contact points $w_1, \ldots, w_N \in \partial K \cap S^{n-1}$ with $N \leq n(n+1)/2$ and $\lambda_1, \ldots, \lambda_N > 0$ with $\sum_{j=1}^N \lambda_j = 1$, such that $1 = n \sum_{j=1}^N \lambda_j \langle \theta, w_j \rangle^2, \forall \theta \in S^{n-1}$.

Now we look at the affine setting for centrally symmetric K and state the corresponding result (see [1]) in the spirit of Proposition 2.5. Observe it is stated for non negative indexes (a similar result for negative indexes cannot hold). From this result it is easy to deduce that the solution to the affine and the linear problems are the same.

Proposition 3.6. [1]. *If $K \subseteq \mathbb{R}^n$ is a centrally symmetric convex body then,*

(i) If $i > n$,

$$\min\{\tilde{W}_i(SK);\ SK \subseteq D_n\} = \min\{\tilde{W}_i(a+SK);\ 0 \in a+SK \subseteq D_n\}.$$

(ii) If $0 \leq i < n$,

$$\max\{\tilde{W}_i(SK);\ SK \subseteq D_n\} = \max\{\tilde{W}_i(a+SK);\ 0 \in a+SK \subseteq D_n\}.$$

Finally, as in the previous section, we can use variational arguments (John's theorem) to produce a necessary condition for K to be the extremal solution for $\{\tilde{W}_i(a+SK);\ 0 \in a+SK \subseteq D_n\}$, without any extra symmetry assumptions. Here, we present another approach that make use of separation theorems:

Theorem 3.7. *Let $K \subseteq D_n$ and let $i \notin [n-1, n]$. If K is in extremal position for the problem $\{\tilde{W}_i(a+SK), 0 \in a+SK \subseteq D_n, S \in GL(n), a \in \mathbb{R}\}$ then, there exist contact points $w_1, \ldots, w_N \in \partial K \cap S^{n-1}$ with $N \leq n(n+1)/2$ and $\lambda_1, \ldots, \lambda_N > 0$ with $\sum_{j=1}^N \lambda_j = 1$, such that*

$$\operatorname{tr} T = i \int_{S^{n-1}} \langle Tu, u \rangle \, d\mu_i(u) + (n-i) \sum_{j=1}^N \lambda_j \langle Tw_j, w_j \rangle \tag{3.2}$$

for all $T \in GL(n)$ and $\displaystyle \sum_{j=1}^N \lambda_j w_j = \frac{-i}{|n-i-1|} \int_{S^{n-1}} \frac{u}{\rho_k(u)} d\mu_i(u).$

Proof.

We will prove it only for $i < n-1$.

Write $W = \partial K \cap S^{n-1}$. Let $A = \dfrac{-i}{n-i} \displaystyle\int_{S^{n-1}} u \otimes u \, d\mu_i(u) + \dfrac{I}{n-i}$ and $v = \dfrac{-i}{n-i-1} \displaystyle\int_{S^{n-1}} \dfrac{u}{\rho_K(u)} \, d\mu_i(u)$.

What we want to prove is that $(A, v) \in conv\,\{(w \otimes w, w); w \in W\} \subseteq GL(n) \times \mathbb{R}^n$. Assume that $(A, v) \notin conv\,\{(w \otimes w, w); w \in W\}$. Then by separation theorems we would have that for some matrix $H \in GL(n)$ and a vector a in $\mathbb{R}^n$,

$$\langle A, H \rangle_{\mathrm{tr}} + \langle v, a \rangle > \langle w \otimes w, H \rangle_{\mathrm{tr}} + \langle w, a \rangle$$

for all $w \in W$ (we use trace duality for matrices). Let M be a real number such that

$$\langle Hw, w \rangle + \langle w, a \rangle - \frac{\mathrm{tr} H}{n-i} < M < \frac{-i}{n-i} \int_{S^{n-1}} \langle Hu, u \rangle d\mu_i(u) + \langle v, a \rangle$$

for all $w \in W$. We define $\tilde{H} = H - \left(\dfrac{\mathrm{tr} H}{n-i} + M \right) I$.

Then it is easy to see that $\langle \tilde{H} w, w \rangle + \langle w, a \rangle < 0$, for all $w \in W$ and

$$\frac{-i}{n-i} \int_{S^{n-1}} \langle \tilde{H} u, u \rangle d\mu_i(u) + \langle v, a \rangle + \frac{\mathrm{tr} \tilde{H}}{n-i} > 0. \tag{3.3}$$

We consider the affine transformation $S_\delta = I + \delta \tilde{H} + \delta a$ for $0 \leq \delta \leq \delta_0$. Let us see that $S_\delta(K) \subseteq D_n$ for δ small enough. If $x \in K$ then $S_\delta(x)^2 = |x|^2 + \delta(f(x) + \delta g(x))$, where

$$f(x) = 2 \Big(\langle x, \tilde{H} x \rangle + \langle x, a \rangle \Big) \qquad g(x) = |\tilde{H} x|^2 + |a|^2 + 2 \langle \tilde{H} x, a \rangle.$$

Since $f(w) < 0$ for all $w \in W$ and $g(x) \geq 0$ for all $x \in K$, by using some topological argument we arrive at $S_\delta(K) \leq D_n$ for δ small enough. Consider the function φ defined by

$$\begin{aligned} \varphi(\delta) &= \tilde{W}_i(S_\delta(K)) \\ &= \frac{n-i}{n} \, |\det(I + \delta \tilde{H})| \int_K \frac{dx}{|\delta a + x + \delta \tilde{H} x|^i} \, . \end{aligned}$$

It is easy to compute that

$$\begin{aligned}\varphi'(0^+) &= \frac{n-i}{n}\operatorname{tr}\tilde{H}\int_K \frac{dx}{|x|^i} \\ &- i\frac{n-i}{n}\int_K \frac{\langle a,x\rangle + \langle x,\tilde{H}x\rangle}{|x|^{i+2}}dx \\ &= \frac{1}{n}\int_{S^{n-1}}\left(\operatorname{tr}\tilde{H} - i\langle u,\tilde{H}u\rangle\right)\rho_K^{n-i}(u)d\sigma(u) \\ &- \frac{i(n-i)}{n(n-i-1)}\int_{S^{n-1}}\langle a,u\rangle\rho_K^{n-i-1}(u)d\sigma(u) \\ &= (n-i)\tilde{W}_i(K)\left(\frac{\operatorname{tr}\tilde{H}}{n-i} - \frac{i}{n-i}\int_{S^{n-1}}\langle u,\tilde{H}u\rangle d\mu_i(u) + \langle v,a\rangle\right).\end{aligned}$$

By (3.3), $\varphi'(0^+) > 0$ which contradicts K being in extremal (maximum) position.

□

4. Optimization of $\{\tilde{W}_i(SK)\tilde{W}_i((SK)^\circ) \mid S \in GL(n)\}$

If $K \subseteq \mathbb{R}^n$ and $i \in \mathbb{R}$, we study the extremal values of $W_i(SK)\tilde{W}_i(SK)^\circ$, where S runs over all regular transformation $S \in GL(n)$. In this case, we wonder for necessary and sufficient conditions for K and $i \in \mathbb{R}$ to verify

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\left\{\tilde{W}_i(SK)\tilde{W}_i((SK)^\circ) \colon S \in GL(n)\right\}, \quad i \in (0,n)$$

or

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(SK)\tilde{W}_i((SK)^\circ) \colon S \in GL(n)\right\}, \quad i \notin (0,n).$$

The proof of the necessary condition once again rests upon a variational argument similar to that in 2.1.

Theorem 4.1. [3]. *Let $i \in \mathbb{R}$ and let $K \subseteq \mathbb{R}^n$ such that $h_K(\cdot)$ and $h_{K^\circ}(\cdot)$ are continuously differentiable. If K is in extremal position, then*

(i) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}}\rho_K^{n-i+1}(u)\langle\nabla h_{K^\circ}(u), T^\star u\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}}\rho_{K^\circ}^{n-i+1}(u)\langle\nabla h_K(u), Tu\rangle\, d\sigma(u).$$

(ii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}}\rho_K^{n-i}(u)\langle u,Tu\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}}\rho_{K^\circ}^{n-i}(u)\langle u,Tu\rangle\, d\sigma(u).$$

In [3], sufficient conditions are obtained $i \in (-\infty, 0) \cup [n+1, +\infty)$. The main difficulty is to adapt the tools explained in section 2 to this situation.

Theorem 4.2. [3]. *Let $i \in (-\infty, 0) \cup [n+1, \infty)$ and let $K \subseteq \mathbb{R}^n$ such that $h_K(\cdot)$ and $h_{K^\circ}(\cdot)$ are twice continuously differentiable. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(SK)\tilde{W}_i((SK)^\circ) \colon S \in GL(n)\right\}$.

(ii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), T^\star u\rangle \, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), Tu\rangle \, d\sigma(u).$$

(iii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u).$$

For the affine case, let us suppose K centrally symmetric. The following Proposition, plus Proposition 2.5, imply that for $i \in (-\infty, 0) \cup [n+1, \infty)$, the solutions to the affine and the linear problems are the same.

Proposition 4.3. *Let $K \subset \mathbb{R}^n$ be centrally symmetric. Then,*

(i) $\tilde{W}_i((a+K)^\circ) \le \tilde{W}_i(K^\circ)$, *if* $i \in (n, n+1]$.

(ii) $\tilde{W}_i((a+K)^\circ) \ge \tilde{W}_i(K^\circ)$, *if* $i \notin [n, n+1]$.

Proof. Observe that

$$\begin{aligned}\tilde{W}_i((a+K)^\circ) &= \frac{1}{n}\int_{S^{n-1}} \rho_{(a+K)^\circ}^{n-i}(u)\, d\sigma(u) = \frac{1}{n}\int_{S^{n-1}} h_{(a+K)}^{i-n}(u)\, d\sigma(u) \\ &= \frac{1}{n}\int_{S^{n-1}} (h_K(u) + \langle a, u\rangle)^{i-n}\, d\sigma(u).\end{aligned}$$

The case $i = n+1$ is trivial. For the remaining cases, by differenciating with respect to a, we have that the gradient of the function $a \to \tilde{W}_i((a+K)^\circ)$ is

$$\nabla \tilde{W}_i((a+K)^\circ) = \frac{i-n}{n}\int_{S^{n-1}} u\, \rho_{(a+K)^\circ}^{n-i+1}(u)\, d\sigma(u)$$

and its Hessian matrix is

$$\frac{(i-n)(i-n-1)}{n}\int_{S^{n-1}} (u \otimes u)\rho_{(a+K)^\circ}^{n-i+2}(u)\, d\sigma(u)$$

which is negative definite if $i \in (n, n+1)$ and positive definite otherwise.

Finally, for K symmetric, we have $(-a+K)^\circ = (-(a+K))^\circ = -(a+K)^\circ$. Therefore, the function $a \to \tilde{W}_i((a+K)^\circ)$ is even and the result easily follows.

□

References

[1] J. Bastero, J. Bernues and M. Romance, *From John to Gauss-John position via dual mixed volumes*, Preprint (2004).

[2] J. Bastero and M. Romance, *Positions of convex bodies associated to extremal problems and isotropic measures*, Adv. Math **184** (2004), pp. 64-88.

[3] J. Bastero and M. Romance, *A characterization of the MM^* position of a convex body in terms of covariance matrices*, Israel J. Math. **141** (2004), pp. 145-156.

[4] R.J. Gardner, *Geometric Tomography*, Encyclopedia of Math. and its applications **58**, Cambridge Univ. Press. Cambridge (1995).

[5] A. Giannopoulos and V. Milman, *Extremal problems and isotropic positions of convex bodies*, Israel J. Math. **117** (2000), pp. 29-60.

[6] A. Giannopoulos, V. Milman and M. Rudelson, *Convex bodies with minimal mean width*, in *Lecture Notes in Math. GAFA 1996-2000* Springer **1745** (2000), pp. 81-94.

[7] A. Giannopoulos and M. Papadimitrakis, *Isotropic surface area measures*, Mathematika **46** (1999), pp. 1-13.

[8] H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics*, Encyclopedia of Math. and its applications, **61**, Cambridge Univ. Press. Cambridge (1996).

[9] F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume. New York (1948),pp. 187-204

[10] E. Lutwak, *Dual Mixed Volumes*, Pacific Journal of Math. **58** (1975), pp. 531-538.

[11] V. Milman and A. Pajor, *Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, GAFA Seminar 87-89, Springer Lecture Notes in Math. **1376** (1989), pp. 64-104.

[12] C. M. Petty, *Surface area of a convex body under affine transformations*, Proc. Amer. Math. Soc. **12** (1961), pp. 824-828.

[13] M. Rudelson, *Distances between non-symmetric convex bodies and the MM^*-estimate*, Positivity **2** (2000), pp. 161-178.

[14] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Math. and its applications **44**, Cambridge Univ. Press. Cambridge (1993).

Dpto. de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address, (Jesús Bastero): `bastero@unizar.es`

E-mail address, (Julio Bernués): `bernues@unizar.es`

Dpto. de Matemática y Física Aplicadas, Universidad Rey Juan Carlos, C/Tulipán s/n, 28933 Móstoles (Madrid) Spain

E-mail address, (Miguel Romance): `mromance@escet.urjc.es`