A characterization of the ℓ-position of a convex body in terms of covariance matrices

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Abstract

We characterize the position of a convex body K such that minimizes $M(TK)M^\star(TK)$ (the ℓ-position) in terms of properties of the measures $\|\cdot\|_K d\sigma(\cdot)$ and $\|\cdot\|_{K^\circ} d\sigma(\cdot)$, answering a question posted by A. Giannopoulos and V. Milman. The techniques used allow us to study other extremal problems in the context of dual Brunn-Minkowski theory.

1 Introduction and Notation

In [GM] A. Giannopoulos and V. Milman characterized extremal positions of convex bodies by the existence of some isotropic measures associated to them and they showed that there are deep relations between the solutions of different extremal problems involving convex bodies and the existence of some measures with isotropic type properties. Following these ideas, the authors in [BR] considered similar problems for extremal positions of convex bodies but in the framework of the dual Brunn-Minkowski theory and they realized that there also strong relations between the solutions of extremal problems and properties of isotropic type of some Borel measures. The aim of this work is to study around these ideas and answer

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a question stated by A. Giannopoulos and V. Milman in [GM] about positions of convex bodies minimizing $M(TK)M^\star(TK)$. If $K \subseteq \mathbb{R}^n$ is a convex body, $M(K)$ is defined by

$$M(K) = \frac{1}{n|D_n|}\int_{S^{n-1}} \|x\|_K \, d\sigma(x),$$

where D_n denotes the the euclidean ball in $\mathbb{R}^n$, $|\cdot|$ is the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and $\|\cdot\|_K$ is the gauge of K. In the same way it is defined $M^\star(K)$ as $M^\star(K) = M(K^\circ)$, where K° is the polar of K given by

$$K^\circ = \{x \in \mathbb{R}^n \colon \langle x, y\rangle \leq 1 \ \ \forall y \in K\}.$$

It is a central topic in the context of local theory of Banach spaces to give upper estimates for $\min\{M(TK)M^\star(TK)\colon T \in GL(n)\}$, since they have many remarkable applications. In this line, T. Figiel, N. Tomczak-Jaegermann (see [FT]) and G. Pisier (see [Pi]) proved that for every centrally symmetric convex body $K \subseteq \mathbb{R}^n$ there exists a position TK (i.e. a regular transformation $T \in GL(n)$), called *ℓ-position*, such that

$$M(TK)M^\star(TK) \leq C \log n,$$

for some absolute constant $C > 0$. This upper estimate is known as the *$MM^\star$-estimate* of K. For a general convex body $K \subseteq \mathbb{R}^n$ a $MM^\star$-estimate was given by M. Rudelson (see [R]) who proved that there exists an affine position $t + TK$ of K (involving the Santaló point) such that

$$M(t+TK)M^\star(t+TK) \leq Cn^{1/3}\log^a(n).$$

In [GM], A. Giannopoulos and V. Milman tried to characterize when a convex body $K \subseteq \mathbb{R}^n$ verifies that

$$M(K)M^\star(K) = \min\{M(TK)M^\star(TK)\colon T \in GL(n)\} \tag{1.1}$$

in terms of the probability Borel measures on S^{n-1} defined by

$$d\mu_K(u) = \frac{\|u\|_K}{\int_{S^{n-1}}\|v\|_K d\sigma(v)} d\sigma(u),$$

where $d\sigma(\cdot)$ denotes the $(n-1)$-dimensional Hausdorff measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Actually they proved that a necessary condition for a symmetric convex body K to verify (1.1) is that $d\mu_K(\cdot)$ and $d\mu_{K^\circ}(\cdot)$ have the same covariance matrix. The main goal of this paper is to show that this kind of conditions are also sufficient conditions and we prove the following result:

Theorem 1.1 *A symmetric convex body K in $\mathbb{R}^n$ having the origin in its interior is in ℓ-position if and only if the probabilities μ_K and μ_{K^0} have the same covariance matrices.*

The techniques we use allow us to study this problem in a more general framework: the dual Brunn-Minkowski theory, obtaining results for convex bodies not centrally symmetric. If K is a convex body having the origin in its interior, the i-th dual quermassintegral of K (see [L]) is given by

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)d\sigma(u),$$

where ρ_K is the radial function of K defined by $\rho_K(x) = \max\{\lambda > 0\colon \lambda x \in K\}$. Note that $\rho_K(x) = \frac{1}{\|x\|_K} = \frac{1}{h_{K^\circ}(x)}$ where $h_K(\cdot)$ is the support function of K. Since

$$M(K) = \frac{1}{|D_n|}\tilde{W}_{n+1}(K),$$

we can extend the extremal problem (1.1) to the context of dual mixed volumes as

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ\colon T \in GL(n)\right\}, \tag{1.2}$$

for all $i \in \mathbb{R}$. In fact, it is known that

$$\lim_{\substack{T\in SL(n)\\ \|T\|\to\infty}} \tilde{W}_i(TK) = \begin{cases} 0 & \text{if } i \in (0,n), \\ +\infty & \text{if } \; i \in (-\infty,0)\cup(n,+\infty). \end{cases}$$

(see [BR], lemma 2.5) which makes that

$$\lim_{\substack{T\in GL(n)\\ \|T\|\to\infty}} \tilde{W}_i(TK)\tilde{W}_i(TK)^\circ = \begin{cases} 0 & \text{if } i \in (0,n), \\ +\infty & \text{if } \; i \in (-\infty,0)\cup(n,+\infty), \end{cases}$$

and therefore the extremal problem (1.2) has solution if $i \in (-\infty,0)\cup(n,+\infty)$ and it must be replaced by the extremal problem

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\left\{\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ\colon T \in GL(n)\right\}, \tag{1.3}$$

for $i \in (0,n)$. In the section 2 we characterize the solutions of (1.2) in terms of covariance matrices of some probabilities whenever $i \in (-\infty,0)\cup$

$[n+1,+\infty)$ while in section 3 we will show that this kind of conditions characterizes the solutions of some related extremal problems when $i \in (0,n) \cup (n,n+1)$. The methods we use there lead us to find some related extremal problems concerning another situations, such as extremal quermassintegrals or dual quermassintegrals. We will use essentially the same notation that appears in [Ga] and [Sc].

2 Main results

If $K \subseteq \mathbb{R}^n$ is a convex body and $i \in \mathbb{R}$, we study the extremal values of $W_i(TK)\tilde{W}_i(TK)^\circ$, where T runs over all regular transformation $T \in GL(n)$. As we noticed in the introduction, we can wonder for necessary and sufficient conditions for a convex body K and $i \in \mathbb{R}$ to verify

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ : T \in GL(n)\right\},$$

if $i \in (-\infty,0) \cup (n,+\infty)$ or

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\left\{\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ : T \in GL(n)\right\},$$

if $i \in (0,n)$. The following result gives a characterization of the solution of this problem when $i \in (-\infty,0) \cup [n+1,+\infty)$.

Theorem 2.1 *Let $i \in (-\infty,0) \cup [n+1,\infty)$, $n \in \mathbb{N}$ and let $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body (i.e. $h_K(\cdot)$ and $h_{K^\circ}(\cdot)$ are twice continuously differentiable) having the origin in its interior. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) : T \in GL(n)\right\}$.

(ii) For every $T \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), T^\star u\rangle \, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), Tu\rangle \, d\sigma(u).$$

(iii) For every $T \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ symmetric

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), Tu\rangle \, d\sigma(u).$$

(iv) For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u)$$
$$= \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u).$$

(v) For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ *symmetric*

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u)$$
$$= \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u).$$

(vi) $\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\right\}$ *and the minimum is unique up to orthogonal transformation.*

Proof:
(i) $\Longrightarrow$ *(ii).* For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we can define $T_\varepsilon = I_n + \varepsilon T \in GL(n)$. Since for every $\varepsilon\|T\| < \frac{1}{2}$

$$\begin{aligned}(I_n + \varepsilon T)^{-1}u &= u - \varepsilon Tu + O(\varepsilon^2),\\ \rho_K((I_n + \varepsilon T)^{-1}u) &= \frac{1}{h_{K^\circ}(u) - \varepsilon\langle \nabla h_{K^\circ}(u), Tu\rangle + O(\varepsilon^2)},\end{aligned}$$

we get that

$$\tilde{W}_i(T_\varepsilon K)$$
$$= \tilde{W}_i(K) - \frac{i-n}{n}\varepsilon \int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle Tu, \nabla h_{K^\circ}(u)\rangle \, d\sigma(u) + O(\varepsilon^2),$$
$$\tilde{W}_i\left((T_\varepsilon K)^\circ\right)$$
$$= \tilde{W}_i(K^\circ) + \frac{i-n}{n}\varepsilon \int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle T^\star u, \nabla h_K(u)\rangle \, d\sigma(u) + O(\varepsilon^2).$$

By hypothesis $\tilde{W}_i(K)\tilde{W}_i(K^\circ) \leq \tilde{W}_i(T_\varepsilon K)\tilde{W}_i(T_\varepsilon K)^\circ$, therefore if we let $\varepsilon \longrightarrow 0^+$, by using the last expressions for $\tilde{W}_i(T_\varepsilon K)$ and $\tilde{W}_i(T_\varepsilon K)^\circ$ we get that for every $T \in GL(n)$

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u)$$
$$\geq \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i+1}(u)\langle \nabla h_K(u), T^\star u\rangle \, d\sigma(u),$$

but if we replace T by $-T$ in the last expression we obtain *(ii)*.

(ii) $\Longrightarrow$ *(iii)* and *(iv)* $\Longleftrightarrow$ *(v)* are trivial.

In order to prove *(iii)* $\Longleftrightarrow$ *(iv)* it is enough to check that the following assertions are equivalent:

(iii') For every $\theta \in S^{n-1}$

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), \theta\rangle \langle u, \theta\rangle \, d\sigma(u)$$
$$= \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle \nabla h_K(u), \theta\rangle \langle u, \theta\rangle \, d\sigma(u).$$

(iv') For every $\theta \in S^{n-1}$

$$\tilde{W}_i(K^\circ) \int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, \theta\rangle^2 \, d\sigma(u) = \tilde{W}_i(K) \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, \theta\rangle^2 \, d\sigma(u).$$

It was proved in [BR] (by using Laplace-Beltrami operator techniques) that for every *"smooth enough"* convex bodies $L, M \subseteq \mathbb{R}^n$ with 0 in their interior

$$(n-i) \int_{S^{n-1}} \rho_L^{n-i+1}(u)\rho_M^i(u)\langle \nabla h_{L^\circ}(u), \theta\rangle \langle u, \theta\rangle \, d\sigma(u)$$
$$= \int_{S^{n-1}} \rho_L^{n-i}(u)\rho_M^i(u) d\sigma(u)$$
$$- i \int_{S^{n-1}} \rho_L^{n-i}(u)\rho_M^{i+1}(u)\langle \nabla h_{M^\circ}(u), \theta\rangle \langle u, \theta\rangle \, d\sigma(u),$$

for all $\theta \in S^{n-1}$. Therefore, if we state $L = K$, $M = D_n$ and $L = K^\circ$, $M = D_n$ in the last expression we obtain that *(iv')* $\Longleftrightarrow$ *(iii')*.

The final part of the proof of the theorem is different depending on the index i and we prove *(v)* $\Rightarrow$ *(vi)* for $i < 0$ and *(iii)* $\Rightarrow$ *(vi)* for $i \geq n+1$.

(v) $\Rightarrow$ *(vi)* ($i < 0$). It is easy to check that we only have to consider diagonal operators $T \in SL(n)$ with diagonal elements $d_1, \ldots, d_n > 0$. If $i \leq -1$, by using Hölder's inequality it follows that

$$\begin{aligned}\tilde{W}_i(TK) &= \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \, |Tu|^{-i} \, d\sigma(u) \\ &\geq \tilde{W}_i(K)^{i+1} \left(\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \, |Tu| \, d\sigma(u) \right)^{-i}\end{aligned}$$

and since $0 \leq \langle u, Tu \rangle \leq |Tu|$ we get that

$$\tilde{W}_i(TK) \geq \tilde{W}_i(K) \left(\frac{\tilde{W}_i(K)}{\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, Tu \rangle \, d\sigma(u)} \right)^i .$$

If we use the same philosophy with $\tilde{W}_i(TK)^\circ$, we obtain that

$$\begin{aligned} &\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ) \cdot \\ &\cdot \left(\frac{\tilde{W}_i(K)\tilde{W}_i(K^\circ)}{\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, Tu \rangle \, d\sigma(u) \frac{1}{n} \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u) \langle u, T^{-1}u \rangle \, d\sigma(u)} \right)^i . \end{aligned}$$

By using the hypothesis, we get that

$$\frac{\tilde{W}_i(K)}{\int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, T^{-1}u \rangle \, d\sigma(u)} = \frac{\tilde{W}_i(K^\circ)}{\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u) \langle u, T^{-1}u \rangle \, d\sigma(u)},$$

hence, since $i < 0$, it is enough to prove that

$$\tilde{W}_i(K)^2 \leq \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, Tu \rangle \, d\sigma(u) \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, T^{-1}u \rangle \, d\sigma(u). \tag{2.4}$$

For every $u \in S^{n-1}$

$$\begin{aligned} \langle u, Tu \rangle \, \langle u, T^{-1}u \rangle &= \left(\sum_{j=1}^n d_j u_j^2 \right) \left(\sum_{j=1}^n d_j^{-1} u_j^2 \right) \\ &\geq \left(\prod_{j=1}^n d_j^{u_j^2} \right) \left(\prod_{j=1}^n d_j^{-u_j^2} \right) = 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{W}_i(K) &\leq \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) (\langle u, Tu \rangle)^{1/2} (\langle u, T^{-1}u \rangle)^{1/2} \, d\sigma(u) \\ &\leq \left(\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, Tu \rangle \, d\sigma(u) \right)^{1/2} \\ &\qquad \left(\frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \langle u, T^{-1}u \rangle \, d\sigma(u) \right)^{1/2}, \end{aligned}$$

which makes that $\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ)$.

Now, if $-1 < i < 0$ and we take a diagonal operator $T \in SL(n)$ with diagonal elements $d_1, \ldots, d_n > 0$, since $f(x) = x^{-i/2}$ is concave in $[0, +\infty)$ we get that

$$\begin{aligned}
\tilde{W}_i(TK) &= \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) \, |Tu|^{-i} \, d\sigma(u) \\
&= \frac{1}{n} \int_{S^{n-1}} \left(\sum_{j=1}^n d_j^2 u_j^2 \right)^{-i/2} \rho_K^{n-i}(u) \, d\sigma(u) \\
&\geq \frac{1}{n} \int_{S^{n-1}} \sum_{j=1}^n d_j^{-i} u_j^2 \rho_K^{n-i}(u) \, d\sigma(u) \\
&\geq \frac{1}{n} \int_{S^{n-1}} \prod_{j=1}^n d_j^{-iu_j^2} \rho_K^{n-i}(u) \, d\sigma(u).
\end{aligned}$$

On the other hand, by hypothesis we can ensure that

$$\begin{aligned}
\tilde{W}_i(TK)^\circ &= \frac{1}{n} \int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u) \left| T^{-1} u \right|^{-i} \, d\sigma(u) \\
&\geq \frac{1}{n} \int_{S^{n-1}} \sum_{j=1}^n d_j^i u_j^2 \rho_{K^\circ}^{n-i}(u) \, d\sigma(u) \\
&= \frac{\tilde{W}_i(K^\circ)}{\tilde{W}_i(K)} \frac{1}{n} \int_{S^{n-1}} \sum_{j=1}^n d_j^i u_j^2 \rho_K^{n-i}(u) \, d\sigma(u) \\
&\geq \frac{\tilde{W}_i(K^\circ)}{\tilde{W}_i(K)} \frac{1}{n} \int_{S^{n-1}} \prod_{j=1}^n d_j^{iu_j^2} \rho_K^{n-i}(u) \, d\sigma(u).
\end{aligned}$$

Now, by combining the last two expresions and by using Cauchy-Schwartz inequality we obtain that

$$\begin{aligned}
&\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ \frac{\tilde{W}_i(K)}{\tilde{W}_i(K^\circ)} \\
&\quad \geq \left(\frac{1}{n} \int_{S^{n-1}} \prod_{j=1}^n d_j^{-iu_j^2/2} \prod_{j=1}^n d_j^{iu_j^2/2} \rho_K^{n-i}(u) \, d\sigma(u) \right)^2 = \tilde{W}_i(K)^2,
\end{aligned}$$

which proves that $\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ \geq \tilde{W}_i(K)\tilde{W}_i(K^\circ)$.

The uniqueness of the extremal position up to orthogonal transformation is a straightforward consequence of the fact that the equality in the AGM-inequality only happens for $d_1 = \cdots = d_n$, what means that $T = I_n$.

(iii) $\Longrightarrow$ *(vi)* $(i \geq n+1)$. If $T \in GL(n)$, there exist orthogonal transformations $U, V \in O(n)$ and diagonal transformation $T \in GL(n)$ such that $T = VDU$. It is easy to check that if $K_1 = UK$, then $\tilde{W}_i(K) = \tilde{W}_i(K_1)$, $W_i(K^\circ) = \tilde{W}_i(K_1^\circ)$ and if K verifies *(iii)* then K_1 also verifies *(iii)*.

If $i = n+1$, by hypothesis we can choose $V_1 \in O(n)$ and diagonal transformation D_1 with diagonal elements $d_1, \ldots, d_n$ such that $T = V_1 D_1 U$ and for every $j = 1, \ldots, n$

$$
\begin{aligned}
d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\, d\sigma(u) &\geq 0, \\
d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1^\circ}}{\partial u_j}(u)\, d\sigma(u) &\geq 0.
\end{aligned}
$$

Now, by using the hypothesis for K_1 and this decomposition of T we get that

$$
\begin{aligned}
\tilde{W}_{n+1}(TK) &= \tilde{W}_{n+1}(D_1 K_1) \\
&= \frac{1}{n} \int_{S^{n-1}} h_{K_1^\circ}(D_1^{-1}u)\, d\sigma(u) \\
&\geq \frac{1}{n} \int_{S^{n-1}} \langle \nabla h_{K_1^\circ}(u), D_1^{-1}u \rangle\, d\sigma(u) \\
&= \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)} \frac{1}{n} \int_{S^{n-1}} \langle \nabla h_{K_1}(u), D_1^{-1}u \rangle\, d\sigma(u) \\
&= \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)} \frac{1}{n} \sum_{j=1}^{n} d_j^{-1} \int_{S^{n-1}} \frac{\partial h_{K_1}}{\partial u_j}(u)\, d\sigma(u) \geq 0.
\end{aligned}
$$

In the same way

$$
\begin{aligned}
\tilde{W}_{n+1}(TK)^\circ &= \tilde{W}_{n+1}(D_1^{-1} K_1)^\circ \\
&= \frac{1}{n} \int_{S^{n-1}} h_{K_1}(D_1 u)\, d\sigma(u) \\
&\geq \frac{1}{n} \int_{S^{n-1}} \langle \nabla h_{K_1}(u), D_1 u \rangle\, d\sigma(u) \\
&= \frac{1}{n} \sum_{j=1}^{n} d_j \int_{S^{n-1}} \frac{\partial h_{K_1}}{\partial u_j}(u)\, d\sigma(u) \geq 0.
\end{aligned}
$$

Therefore, by combining these two expressions and Cauchy-Schwarz inequality we get that

$$
\begin{aligned}
&\tilde{W}_{n+1}(TK)\tilde{W}_{n+1}(TK)^\circ \\
&\qquad \geq \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)} \left(\frac{1}{n} \sum_{j=1}^{n} d_j^{-1} \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\, d\sigma(u) \right) \\
&\qquad\qquad \cdot \left(\frac{1}{n} \sum_{j=1}^{n} d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\, d\sigma(u) \right) \\
&\qquad \geq \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)} \left(\frac{1}{n} \sum_{j=1}^{n} |d_j|^{1/2} |d_j^{-1}|^{1/2} \left| \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\, d\sigma(u) \right| \right)^2 \\
&\qquad \geq \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)} \left(\frac{1}{n} \int_{S^{n-1}} \langle \nabla h_{K_1}(u), u \rangle \, d\sigma(u) \right)^2 .
\end{aligned}
$$

Now, by the homogeneity of $h_{K_1}(\cdot)$ we obtain that

$$
\begin{aligned}
\tilde{W}_{n+1}(TK)\tilde{W}_{n+1}(TK)^\circ &\geq \frac{\tilde{W}_{n+1}(K_1)}{\tilde{W}_{n+1}(K_1^\circ)} \left(\frac{1}{n} \int_{S^{n-1}} h_{K_1}(u)\, d\sigma(u) \right)^2 \\
&= \tilde{W}_{n+1}(K_1)\tilde{W}_{n+1}(K_1^\circ) = \tilde{W}_{n+1}(K)\tilde{W}_{n+1}(K^\circ).
\end{aligned}
$$

If $i > n+1$ the proof can be completed by using the same ideas, since for every $T \in GL(n)$ symmetric we can find $V_1, U \in O(n)$ and diagonal transformation D_1 with diagonal elements $d_1, \dots, d_n$ such that $T = V_1 D_1 U$ and for every $j = 1, \dots, n$

$$
\begin{aligned}
d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u) \rho_{K_1^\circ}^{n-i+1}(u) d\sigma(u) \geq 0, \\
d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1^\circ}}{\partial u_j}(u) \rho_{K_1}^{n-i+1}(u) d\sigma(u) \geq 0,
\end{aligned}
$$

where $K_1 = UK$. Hence, by using Hölder inequality ($p = i-n$, $q = \frac{i-n}{i-n-1}$) we get that

$$
\begin{aligned}
\tilde{W}_i(TK) &= \tilde{W}_i(D_1 K_1) \\
&\geq \tilde{W}_i(K_1)^{n-i+1} \left(\frac{1}{n} \int_{S^{n-1}} h_{D_1 K_1^\circ}(u) \rho_{K_1}^{n-i+1}(u)\, d\sigma(u) \right)^{i-n} .
\end{aligned}
$$

But, by hypothesis

$$\begin{aligned}\int_{S^{n-1}} h_{D_1 K_1^\circ}(u)\rho_{K_1}^{n-i+1}(u)\,d\sigma(u) &\geq \int_{S^{n-1}} \langle \nabla h_{K_1^\circ}(u), D_1^{-1}u\rangle \rho_{K_1}^{n-i+1}(u)\,d\sigma(u)\\ &= \frac{\tilde{W}_i(K_1)}{\tilde{W}_i(K_1^\circ)} \int_{S^{n-1}} \langle \nabla h_{K_1}(u), D_1^{-1}u\rangle \rho_{K_1^\circ}^{n-i+1}(u)\,d\sigma(u)\\ &= \frac{\tilde{W}_i(K_1)}{\tilde{W}_i(K_1^\circ)} \sum_{j=1}^{n} d_j^{-1} \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)\,d\sigma(u) \geq 0,\end{aligned}$$

hence

$$\tilde{W}_i(TK) \geq \frac{\tilde{W}_i(K_1)}{\tilde{W}_i(K_1^\circ)^{i-n}} \left(\frac{1}{n}\sum_{j=1}^{n} d_j^{-1} \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)\,d\sigma(u)\right)^{i-n}.$$

If we use the same technique with $\tilde{W}_i(TK)^\circ$ and we combine both expressions we get that

$$\begin{aligned}&\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ\\ &\geq \frac{\tilde{W}_i(K_1)}{(\tilde{W}_i(K_1^\circ))^{2(i-n)-1}} \left(\frac{1}{n}\sum_{j=1}^{n} d_j^{-1} \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)\,d\sigma(u)\right)^{i-n}\\ &\qquad\cdot \left(\frac{1}{n}\sum_{j=1}^{n} d_j \int_{S^{n-1}} u_j \frac{\partial h_{K_1}}{\partial u_j}(u)\rho_{K_1^\circ}^{n-i+1}(u)\,d\sigma(u)\right)^{i-n}.\end{aligned}$$

Finally, by using again the Cauchy-Schwarz inequality we conclude the result. The uniqueness of the solution can be proved by using the same ideas than in the case $i < 0$.

□

Corollary 2.2 *Let $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body having the origin in its interior. Then the following assertions are equivalent:*

(i) $M(K)M^\star(K) = \min\{M(TK)M^\star(TK) \colon T \in GL(n)\}$.

(ii) For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$

$$\begin{aligned}M^\star(K) &\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), T^\star u\rangle\,d\sigma(u)\\ &= M(K)\int_{S^{n-1}} \langle \nabla h_K(u), Tu\rangle\,d\sigma(u).\end{aligned}$$

(iii) For every $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$

$$M^\star(K) \int_{S^{n-1}} \|u\|_K \langle u, Tu \rangle \, d\sigma(u) = M(K) \int_{S^{n-1}} \|u\|_{K^\circ} \langle u, Tu \rangle \, d\sigma(u).$$

(iv) $M(K)M^\star(K) = \min\{M(TK)M^\star(TK) \colon T \in GL(n)\}$ *and the minimum is unique up to orthogonal transformation.*

Remark 2.3 If $i \in [0, n+1)$ the assertions *(ii)* and *(iv)* (and therefore *(iii)* and *(v)*) in theorem 2.1 are necessary conditions for a convex body $K \subseteq \mathbb{R}^n$ *"smooth enough"* and such that the origin it is in its interior to verify that

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \max\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\right\}$$

if $i \in (0, n)$ or

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) = \min\left\{\tilde{W}_i(TK)\tilde{W}_i((TK)^\circ) \colon T \in GL(n)\right\}$$

if $i \in (n, n+1)$. In both cases it can be proved that *(iii)* and *(iv)* are equivalent conditions.

Remark 2.4 If $K \subseteq \mathbb{R}^n$ is a centrally symmetric convex body then the last corollary implies theorem 1.1, since the probability μ_K given by

$$d\mu_K(u) = \frac{\|u\|_K}{\int_{S^{n-1}} \|v\|_K d\sigma(v)} d\sigma(u)$$

has mean 0.

3 Related Extremal Problems

The previous section proves that if $i \in (-\infty, 0) \cap [n+1, +\infty)$ then we can characterize when a convex body K is in the position that minimizes $\tilde{W}_i(TK)\tilde{W}_i(TK)^\circ$ in terms of properties of measures μ_K and μ_{K°. A natural question lead us to wonder if a similar situation occurs if $i \in (0, n) \cap (n, n+1)$. As we said before, the same conditions for μ_K and μ_{K° that appear in theorem 2.1 are necessary conditions for a convex body to be the solution of certain extremal problems (see remark2.3).

In this final section we are going to see that the conditions for μ_K and μ_{K° that appear in theorem 2.1 characterize the solution of certain extremal problem that are slightly different from those of section 2. This idea of changing briefly a extremal problem to get another whose solutions can be characterized completely can be used in different situations and in the last part of this section we will use it to show that some *isotropic type* conditions that are necessary conditions for a convex body K to be the solution of extremal problems (see [GM] and [BR]) also characterize the solutions of some slightly modified problems.

Let $K_1, K_2, K_3 \subseteq \mathbb{R}^n$ be star-shaped bodies at 0 and $i_1, i_2, i_3 \in \mathbb{R}$. We denote by $\tilde{V}_{i_1,i_2,i_3}(K_1, K_2, K_3)$ the value

$$\tilde{V}_{i_1,i_2,i_3}(K_1, K_2, K_3) = \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}^{i_1}(u)\rho_{K_2}^{i_2}(u)\rho_{K_3}^{i_3}(u)\, d\sigma(u).$$

Following this notation we can state slightly different extremal problems from those presented in section 2 but in this new situation the conditions for μ_K and μ_{K° that appeared there will exactly characterize the solutions of these new extremal problem.

Proposition 3.1 *Let $i \in \mathbb{R}$ and $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body. Then the following assertions are equivalent:*

(i) For every $T \in GL(n)$

$$\tilde{W}_i(K)\tilde{W}_i(K^\circ) \leq \tilde{V}_{n-i,i+1,-1}(TK, TD_n, D_n)\tilde{V}_{n-i,i+1,-1}((TK)^\circ, (TD_n)^\circ, D_n).$$

(ii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle \nabla h_K(u), Tu\rangle\, d\sigma(u).$$

(iii) For every $T \in GL(n)$

$$\tilde{W}_i(K^\circ)\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u) = \tilde{W}_i(K)\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, Tu\rangle\, d\sigma(u).$$

(iv) $\tilde{W}_i(K)\tilde{W}_i(K^\circ)$ *is the unique solution for the extremal problem stated in (i) up to orthogonal transformation, i.e. if there exists* $T_0 \in SL(n)$ *such that for every* $T \in GL(n)$

$$\begin{aligned}&\tilde{V}_{n-i,i+1,-1}(T_0K, T_0D_n, D_n)\tilde{V}_{n-i,i+1,-1}((T_0K)^\circ, (T_0D_n)^\circ, D_n)\\&\quad\le \tilde{V}_{n-i,i+1,-1}(TK, TD_n, D_n)\tilde{V}_{n-i,i+1,-1}((TK)^\circ, (TD_n)^\circ, D_n),\end{aligned}$$

then $T_0 \in O(n)$.

Proof:
(i) $\Longrightarrow$ *(ii)* and *(ii)* $\Longrightarrow$ *(iii)* can be proved as theorem 2.1.

(iii) $\Longrightarrow$ *(iv)* It is enough to prove that for every $T \in SL(n)$ symmetric positive definite

$$\begin{aligned}&\tilde{W}_i(K)\tilde{W}_i(K^\circ)\\&\qquad\le \tilde{V}_{n-i,i+1,-1}(TK, TD_n, D_n)\tilde{V}_{n-i,i+1,-1}((TK)^\circ, (TD_n)^\circ, D_n).\end{aligned}$$

Since $T \in SL(n)$ is symmetric positive definite

$$\begin{aligned}\tilde{V}_{n-i,i+1,-1}(TK, TD_n, D_n) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)h_{TD_n}(u)\,d\sigma(u)\\&\ge \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle\,d\sigma(u) \ge 0\end{aligned}$$

and, in the same way,

$$\tilde{V}_{n-i,i+1,-1}((TK)^\circ, (TD_n)^\circ, D_n) \ge \frac{1}{n}\int_{S^{n-1}} \rho_{K^\circ}^{n-i}(u)\langle u, T^{-1}u\rangle\,d\sigma(u) \ge 0.$$

By combining these two expressions, following the same ideas than in theorem 2.1 we get that

$$\begin{aligned}&\tilde{V}_{n-i,i+1,-1}(TK, TD_n, D_n)\tilde{V}_{n-i,i+1,-1}((TK)^\circ, (TD_n)^\circ, D_n)\\&\qquad\qquad\qquad\qquad\ge \tilde{W}_i(K)\tilde{W}_i(K^\circ).\end{aligned}$$

□

The main idea of proposition 3.1 is that we can find extremal problems related to those in section 2 such that they can be characterized in terms of properties of μ_K and μ_K (actually the same conditions that appeared in theorem 2.1) for any index $i \in \mathbb{R}$. This kind of *transformation technique* can be also applied to other situations where it is known that some isotropic type conditions are necessary for a convex body to be solution of a certain extremal problem but it is not known if they are sufficient. In this final part we are going to show two situation where we can also find some related extremal problems which fit to some well known isotropic type conditions.

A first situation where we can find related extremal problem is concerned with the characterization of the position of a convex body with extremal mixed quermassintegral. In [GM], A. Giannopoulos and V. Milman proved the following result:

Proposition 3.2 *Let $K \subseteq \mathbb{R}^n$ be a "smooth enough" convex body. If*

$$W_i(K) = \min \{W_i(TK) \colon T \in SL(n)\},$$

where $W_i(K)$ denotes the ith-quermassintegral, then:

(i) $dS_{i-1}(K)(\cdot)$ is isotropic on S^{n-1}.

(ii) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \langle \nabla h_K(u), Tu\rangle \, dS_i(K)(u) = \frac{\operatorname{tr} T}{n} W_i(K),$$

where $dS_j(K)(\cdot)$ denotes the jth-surface measure $dS_j(K, D_n)$.

It is stated in [GM] that it would be interesting to determine not only necessary but also sufficient conditions for the positions minimizing W_i. According to the idea of *finding related extremal problems* that we used before, we have realized that these necessary conditions that appear in proposition 3.2 are also sufficient for a slightly different extremal position involving mixed volumes and it can be proved the following result:

Proposition 3.3 *Let $K \subseteq \mathbb{R}^n$ be a convex body and take $0 < i < n$. The following assertions:*

(i) $W_i(K) = \min \left\{V(K, \overset{n-i)}{\ldots}, K, TD_n, D_n, \ldots, D_n) \colon T \in SL(n)\right\}$,

(ii) $dS_{i-1}(K)(\cdot)$ is isotropic,

(iii) $W_i(K) = \min \left\{V(T(K), K, \ldots, K, D_n, \overset{i)}{\ldots}, D_n) \colon T \in SL(n)\right\}$,

(iv) For every $T \in GL(n)$,

$$\frac{1}{n}\int_{S^{n-1}} \langle \nabla h_K(u), T(u)\rangle \, dS_{n-i-1}(K, u) = \frac{\operatorname{tr} T}{n} W_i(K).$$

verify that (i) $\iff$ (ii) and (iii) $\iff$ (iv).

The same philosophy can be applied to the analogous situation in dual Brunn-Minkoski theory. In [BR], the authors proved that if $K \subseteq \mathbb{R}^n$ is a convex body having 0 in its interior such that K° is *"smooth enough"* and $i \notin \{0, n\}$, then either

$$\tilde{W}_i(K) = \max\left\{\tilde{W}_i(TK) \colon T \in SL(n)\right\}$$

for $i \in (0, n)$ or

$$\tilde{W}_i(K) = \min\left\{\tilde{W}_i(TK) \colon T \in SL(n)\right\}$$

for $i \notin [0, n]$ imply that

$$\begin{aligned} \frac{\operatorname{tr} T}{n}\tilde{W}_i(K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i}(u)\langle u, Tu\rangle \, d\sigma(u) && (3.5)\\ &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u), && (3.6) \end{aligned}$$

for all $T \in GL(n)$.

It is known (see [BR]) that for some indexes i, the *isotropic type* conditions (3.5) and (3.6) are also sufficient conditions for a convex body K to be in extremal dual mixed volume position, but it is not known that these conditions are sufficient for all index i. The next result shows that we can find a related extremal problem involving dual mixed volumes that *fits* to these isotropic type conditions.

Proposition 3.4 *Let $K \subseteq \mathbb{R}^n$ be a convex body such that* 0 *belongs to its interior. Let $i \in \mathbb{R}$. Then the following assertions are equivalent:*

(i) $\tilde{W}_i(K) = \min\left\{\tilde{V}_{n-i,-1,i+1}(K, TD_n, D_n) \colon T \in SL(n)\right\}$.

(ii) $\rho_K^{n-i}(\cdot)d\sigma(\cdot)$ *is isotropic on* S^{n-1}.

(iii) For every $T \in GL(n)$ symmetric

$$\frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i+1}(u)\langle \nabla h_{K^\circ}(u), Tu\rangle \, d\sigma(u) = \frac{\operatorname{tr} T}{n}\tilde{W}_i(K).$$

(iii) $\tilde{W}_i(K)$ *is the only solution, up to orthogonal transformation, for the extremal problem stated in (i).*

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